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Fixed points of a family of mappings and equivalent characterizations

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Abstract. In the present paper we prove a fixed point theorem for a one parameter family of contractive self-mappings, of a complete metric space or a complete b-metric space, each member of which has a unique fixed point that is also the unique common fixed point of the family; the mappings may be continuous or discontinuous at the fixed point. We also prove that under the assumption of a weaker form of continuity the fixed point property for mappings satisfying the contractive conditions employed by us implies completeness of the underlying space. The characterization of completeness obtained by us not only contains Subrahmanyam's theorem on characterization of completeness as a particular case but also extends it to b-metric spaces. Results on contractive mappings with discontinuity at the fixed point have found applications in neural networks with discontinuous activation function (e.g. Ozgur and Tas [19, 20]).

1. Introduction

Kannan [11, 12] proved that a self-mapping f of a complete metric space (X, d) satisfying the condition

$$d(fx, fy) \leq a[d(x, fx) + d(y, fy)], \text{ for all } x, y \text{ in } X,\ 0 \leq a < \frac{1}{2} \tag{1}$$

possesses a unique fixed point. Kannan's theorem is remarkable for two reasons: (a) it characterizes metric completeness [24], and (b) it was the genesis of the once open question on the existence of contractive mappings which are discontinuous at fixed point [23]. Several researchers have studied metric completeness (e.g. Kirk [13], Liu [14], Subrahmanayam [24], Suzuki [25]. Kirk [13] proved that Caristi's fixed theorem characterizes metric completeness. Subrahmanayam [24] proved that Kannan's theorem characterizes metric completeness. Suzuki [25] proved a fixed point theorem that generalizes the Banach contraction theorem and characterizes metric completeness. The Banach contraction mapping theorem [1] itself does not characterize metric completeness [4].

The problem of continuity of contractive mappings at fixed points was resolved by Pant [21] in 1999 by giving a contractive condition which ensures the existence of a fixed point but does not imply continuity at the fixed point. While continuity is a nice and desirable property of functions, discontinuities occur naturally in diverse biological, industrial and economic phenomena and many of these phenomena involve

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threshold operations which are discontinuous. For example, a neuron in a neural network either fires or does not fire depending on whether the input crosses a certain threshold or not. Various industrial censors, band pass filters and the diode also work in this manner. Cromme and Diener [5] and Cromme [6] have proved results on approximate fixed points for such functions and have given applications of their results to neural nets, economic equilibria and analysis. We show that many functions representing threshold operations satisfy weaker forms of continuity and various contractive conditions; and possess fixed point. Fixed point theorems for discontinuous mappings have found wide applications, for example application of such theorems in the study of neural networks with discontinuous activation functions is presently a very active area of research (e. g. Ding et al [8], Forti and Nistri [9], Nie and Zheng [16–18], Wu and Shan [26]). Recently Ozgur and Tas [19, 20] have obtained application of the results on discontinuity at the fixed point by Pant [21] and Bisht and Pant [2, 3] in neural networks with discontinuous activation functions. We now give some relevant definitions.

Definition 1.1 ([7, 10]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:*

(1). $d(x, y) = 0$ if and only if $x = y$
(2). $d(x, y) = d(y, x)$
(3). $d(x, z) \leq s[d(x, y) + d(y, z)]$. The triplet (X, d, s) is called a b-metric space.

Definition 1.2 ([7, 10]). *Let (X, d, s) be a b-metric space. The sequence $\{x_n\}$ in X is called convergent if and only if for all $\epsilon > 0$, there exists $k \in N$ such that $d(x_n, x) < \epsilon$ for all $n \geq k$. In this case, we write $\lim_{n\to\infty} d(x_n, x) = 0$.*

Definition 1.3 ([7, 10]). *Let (X, d, s) be a b-metric space. The sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\epsilon > 0$, there exists $k \in N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$.*

Definition 1.4 ([7, 10]). *The b-metric space (X, d, s) is said to be complete if and only if every Cauchy sequence converges to some x in X.*

In a recent work Pant and Pant [22] introduced the following weaker form of continuity for a mapping:

Definition 1.5. *A self-mapping f of a metric space X will be called k-continuous, $k = 1, 2, 3, \ldots$ if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \to t$.*

Example 1.6. *Let $X = [0, 2]$ equipped with the usual metric and $f : X \to X$ be defined by*

$$fx = 1 \text{ if } 0 \leq x \leq 1, \quad fx = 0 \text{ if } x > 1.$$

Then $fx_n \to t \implies f^2 x_n \to t$ since $fx_n \to t$ implies $t = 0$ or $t = 1$ and $f^2 x_n = 1$ for all n, that is, $f^2 x_n \to 1 = ft$. Hence f is 2-continuous. However f is discontinuous at $x = 1$.

Example 1.7. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = \frac{(1 + x)}{2} \text{ if } 0 \leq x \leq 1, \quad fx = 0 \text{ if } x > 1.$$

Then it can be verified that f is 2-continuous but not continuous. It is also easy to see that f^k is discontinuous for each positive integer k. Thus 2-continuity of f does not imply continuity of f^2. In general, k-continuity of f does not imply continuity of f^n. It can be shown that continuity of f^k and k-continuity of f are independent conditions when $k > 1$ (see [22]).

It is easy to see that 1- continuity is equivalent to continuity; and $continuity \implies 2-continuity \implies 3-continuity \implies \ldots$, but not conversely.

2. Results

Theorem 2.1. *Let $\{f_r : 0 \le r \le 1\}$ be a family self-mapping of a complete metric space (X, d) such that for any given mapping f_r the following conditions are satisfied:*

(i) $d(f_r x, f_r y) < max\{d(x, f_r x), d(y, f_r y)\}$, *whenever* $max\{d(x, f_r x), d(y, f_r y)\} > 0$,

(ii) given $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\epsilon < max\{d(x, f_r x), d(y, f_r y)\} \le \epsilon + \delta \implies d(f_r x, f_r y) \le \epsilon$.

If f_r is k-continuous or if f_r^k is continuous for some integer $k \ge 1$ then f_r has a unique fixed point, say t_r, and $\lim_{n\to\infty} f_r^n x_0 = t_r$ for each x_0 in X. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition

(iii) $d(f_r x, f_s y) \le max\{d(x, f_r x), d(y, f_s y)\}$,

then the mappings $\{f_r\}$ have a unique common fixed point which is also the unique fixed of each f_r.

Proof. Select any mapping f_r. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f_r x_{n-1}$. If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f_r. If $x_n \neq x_{n+1}$ for each n, then using (i) we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(f_r x_{n-1}, f_r x_n) &< max\{d(x_{n-1}, f_r x_{n-1}), d(x_n, f_r x_n)\} \\ &= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}$is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $l \ge 0$. Suppose $l > 0$. Then there exists a positive integer N such that

$$n \ge N \implies l < d(x_n, x_{n+1}) < l + \delta(l). \tag{2}$$

This yields $l < max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = max\{d(x_n, f_r x_n), d(x_{n+1}, f_r x_{n+1})\} < l + \delta(l)$ which by virtue of (ii) yields $d(f_r x_n, f_r x_{n+1}) = d(x_{n+1}, x_{n+2}) \le l$. This contradicts (2). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then

$$\begin{aligned} d(x_n, x_{n+p}) &= d(f_r x_{n-1}, f_r x_{n+p-1}) \\ &< max\{d(x_{n-1}, f_r x_{n-1}), d(x_{n+p-1}, f_r x_{n+p-1})\} \\ &= max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} = d(x_{n-1}, x_n). \end{aligned}$$

This implies that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists t_r in X such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f_r x_n = t_r. \tag{3}$$

Now suppose that f_r is k-continuous. Since $f_r^{k-1} x_n \to t_r$ for each $k \ge 1$, k-continuity of f_r implies that $f_r^k x_n \to f_r t_r$. Hence $t_r = f_r t_r$ as $f_r^k x_n \to t_r$. Therefore, t_r is fixed point of f_r.
Next suppose that f_r^k is continuous for some positive integer k. Then, $\lim_{n\to\infty} f_r^k x_n = f_r^k t_r$. This yields $f_r^k t_r = t_r$ as $f_r^k x_n \to t_r$. If $t_r \neq f_r t_r$ we get

$$\begin{aligned} d(t_r, f_r t_r) = d(f_r^k t_r, f_r^{k+1} t_r) &< max\{d(f_r^{k-1} t_r, f_r^k t_r), d(f_r^k t_r, f_r^{k+1} t_r)\} \\ &= d(f_r^{k-1} t_r, f_r^k t_r) < d(f_r^{k-2} t_r, f_r^{k-1} t_r) < \ldots < d(t_r, f_r t_r), \end{aligned}$$

a contradiction. Hence $t_r = f_r t_r$ and t_r is a fixed point of f_r. Uniqueness of the fixed point follows from (i). Moreover, if u_r and u_s are the fixed points of f_r and f_s respectively then by (iii) we get

$$d(u_r, u_s) = d(f_r u_r, f_s u_s) \le max\{d(u_r, f_r u_r), d(u_s, f_s u_s)\} = 0.$$

Hence $u_r = u_s$ and each mapping f_r has a unique fixed point which is also the unique common fixed point of the family of mappings. □

The next example illustrates the above theorem.

Example 2.2. *Let $X = [0,3]$ and d be the usual metric. Define $f_r : X \to X, 0 \le r \le 1$, by*

$$f_r x = 1 \text{ if } 0 \le x \le 1, \quad f_r x = r[3-x] \text{ if } 1 < x \le 3,$$

where $[a]$ denotes the greatest integer not greater than the nonnegative real number a.
Then the mappings f_r satisfy all the conditions of the above theorem and have a unique common fixed point $x = 1$ which is also the unique fixed point of each mapping. The mapping f_r is discontinuous at the fixed point if $r < 1$ while f_r is continuous at the fixed point if $r = 1$. However, f_r^2 is continuous for each r and f_r is 2-continuous for each r. To see that f_r is 2-continuous, consider a sequence $\{x_n\}$ such that $f_r x_n \to t$ for some t in X. Then $t = 0$ or $t = r$ or $t = 1$ and $f_r^2 x_n = 1 = f_r 0 = f_r r = f_r 1$. Therefore f_r is 2-continuous. It can be easily verified that if $0 < r < 1$ then

$$\begin{aligned}
d(f_r x, f_r y) &= 0, \quad 0 < max\{d(x, f_r x), d(y, f_r y)\} \le 1 \quad \text{if } x, y \le 1,\\
d(f_r x, f_r y) &= 1-r, \quad 1-r < max\{d(x, f_r x), d(y, f_r y)\} \le 2-r \quad \text{if } x \le 1, 1 < y \le 2,\\
d(f_r x, f_r y) &= 1, \quad 2 < max\{d(x, f_r x), d(y, f_r y)\} \le 3 \quad \text{if } x \le 1, 2 < y \le 3,\\
d(f_r x, f_r y) &= 0, \quad 1-r < max\{d(x, f_r x), d(y, f_r y)\} \le 2-r \quad \text{if } 1 < x, y \le 2,\\
d(f_r x, f_r y) &= r, \quad 2 < max\{d(x, f_r x), d(y, f_r y)\} \le 3 \quad \text{if } 1 < x \le 2, 2 < y \le 3,\\
d(f_r x, f_r y) &= 0, \quad 2 < max\{d(x, f_r x), d(y, f_r y)\} \le 3 \quad \text{if } 2 < x, y \le 3.
\end{aligned}$$

Therefore, f_r satisfies condition (ii) with $\delta(\epsilon) = 1 - r - \epsilon$ if $\epsilon < 1 - r$, $\delta(\epsilon) = 2 - \epsilon$ if $1 - r \le \epsilon < 2$ and $\delta(\epsilon) = 1$ for $\epsilon \ge 2$. For $r = 0$, the function f_0 satisfies (ii) with $\delta(\epsilon) = 1 - \epsilon$ if $\epsilon < 1$ and $\delta(\epsilon) = 1$ if $\epsilon \ge 1$. The function f_0 represents a threshold operation that can model the firing of a neuron, function of a diode, and also a low pass filter that allows low voltages to pass but not higher voltages (e.g. noise in music systems). It may also be seen that for $r < 1$ the functions f_r in this example do not satisfy the Meir – Keeler [15] type $(\epsilon - \delta)$ contractive condition:

$$\epsilon \le max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \implies d(fx, fy) < \epsilon.$$

Remark 2.3. *If we take $r = 1$ in the above example then the mapping f_1 is continuous at the fixed point and satisfies the stronger contractive condition*

$$d(f_1 x, f_1 y) \le \frac{1}{2} max\{d(x, f_1 x), d(y, f_1 y)\}.$$

We thus see that the mappings satisfying the assumptions of Theorem 2.1 may be continuous at fixed point or discontinuous functions including threshold functions. Moreover, if f is any self-mapping of X satisfying the conditions of Theorem 2.1 and if we denote:

$$m(x, y) = max\{d(x, fx), d(y, fy)\},$$

then f is continuous at its fixed point, say z, if and only if $m(x, z) \to 0$ as $x \to z$. If f is continuous at its fixed point z then $fx \to fz$ and $m(x, z) = max\{d(x, fx), d(z, fz)\} \to 0$ as $x \to z$. On the other hand, if $m(x, z) \to 0$ as $x \to z$ then $d(x, fx) \to 0$ as $x \to z$, that is, $fx \to z = fz$. Hence f is continuous at the fixed point z.
The next example shows that the above theorem does not hold if neither f_r is k-continuous nor f_r^k is continuous for some integer $k \ge 1$.

Example 2.4. *Let $X = [0,2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = \frac{(1+x)}{2} \text{ if } 0 \le x < 1, \quad fx = 0 \text{ if } 1 \le x \le 2.$$

Then X is complete and f satisfies the contractive conditions (i) and (ii) with $\delta(\epsilon) = \frac{(1-\epsilon)}{2}$ for $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \ge 1$ but f does not have a fixed point. If we consider a sequence $\{x_n\}$ given by $x_n = 1 - \frac{1}{2n}$ then $\lim_{n\to\infty} fx_n = 1$, $\lim_{n\to\infty} f^k x = 1 \ne f1$ for each integer $k \ge 1$. The mapping f is, therefore, neither k-continuous nor is f_r^k continuous for some $k \ge 1$.

We now extend Theorem 2.1 to b-metric spaces.

Theorem 2.5. *Let (X, d) be a complete b-metric space and $\{f_r : 0 \leq r \leq 1\}$ be a family self-mapping of X such that for any given mapping f_r the conditions (i) and (ii) are satisfied. If f_r is k-continuous or if f_r^k is continuous for some integer $k \geq 1$ then f_r has a unique fixed point. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition (iii) then the mappings have a unique common fixed point which is also the unique fixed of each f_r.*

Proof. The proof of this theorem is the same as that of Theorem 2.1 since the proof of Theorem 2.1 does not involve the use of triangle inequality. □

Theorem 2.6. *Let (X, d) be a complete metric space or a complete b-metric space and $\{f_r : 0 \leq r \leq 1\}$ be a family of asymptotically regular self-mappings of X satisfying*

(iv) $d(f_r x, f_r y) \leq \lambda max\{d(x, f_r x), d(y, f_r y)\}, \lambda > 0,$ *for each r.*

If f_r is k-continuous for some integer $k \geq 1$ then f_r has a unique fixed point. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition

(v) $d(f_r x, f_s y) \leq \lambda max\{d(x, f_r x), d(y, f_s y)\}, \lambda > 0,$

then the mappings have a unique common fixed point which is also the unique fixed of each f_r.

Proof. Select any mapping f_r. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f_r x_{n-1}$. If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f_r. If $x_n \neq x_{n+1}$ for each n, then using (iv), for each positive integer p we get

$$\begin{aligned} d(x_n, x_{n+p}) &= d(f_r x_{n-1}, f_r x_{n+p-1}) \\ &\leq \lambda max\{d(x_{n-1}, f_r x_{n-1}), d(x_{n+p-1}, f_r x_{n+p-1})\} \\ &= \lambda max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\}. \end{aligned}$$

Asymptotic regularity of f_r implies $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. This further implies that $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$, that is, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists t in X such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f_r^p x_n = t, p = 1, 2, 3, \ldots$$

Since f_r is k-continuous and $f_r^{k-1} x_n \to t$ for each $k \geq 1$, we get $f_r^k x_n \to ft$. Hence $t = ft$ as $f_r^k x_n \to t$. Therefore, t is fixed point of f_r. Uniqueness of the fixed point of f_r follows from (iv). Moreover, if u and v are the fixed points of f_r and f_s respectively then using (v) we get

$$d(u, v) = d(f_r u, f_s v) \leq \lambda max\{d(u, f_r u), d(v, f_s v)\} = 0.$$

Hence $u = v$ and the family of mappings $\{f_r\}$ has a unique common fixed point which is also the unique fixed point of each f_r. □

Remark 2.7. *Theorems 2.1and 2.5 provide a new type of solution to the once open problem on the continuity of a contractive mapping at the fixed point (see Rhoades ([23], p.242) in which either each member of a family of contractive mappings has discontinuity at its fixed point or some members may be continuous at the fixed point; and the fixed point is a unique common fixed point of the mappings. These results show that besides metric spaces the solution to the problem of continuity of contractive mappings at fixed point exists in b-metric spaces also.*

Remark 2.8. *In Theorem 2.6 if we assume f_r^k to be continuous for some $k > 1$ then we get $f_r^k t = t$, that is, t turns out to be a periodic point of f_r which may not be a fixed point unless (iv) is replaced by a contractive type condition. This shows that the notion of k-continuity is more useful than the notion of continuity of f_r^k in fixed point considerations. This difference in these weaker forms of continuity extends further and this will become evident in the following.*

If f is self-mapping of a metric space or a b-metric space (X, d) and satisfies (i) and (ii), Theorems 2.1 and 2.5 show that under the assumption of k-continuity of f or continuity of f^k completeness of X implies fixed point property for f. We now show that under the assumption of k-continuity fixed point property for every self-mapping of X satisfying conditions (i) and (ii) implies completeness of X. The same may not hold if f^k is assumed continuous.

There is, however, an essential difference between the next theorem and similar theorems (e. g. Kirk [13], Subrahmanyam [24], Suzuki [25]) giving characterization of completeness in terms of fixed point property for contractive type mappings. In [13], [24] and [25] the contractive condition implies continuity at the fixed point and completeness of the metric space X is equivalent to the existence of fixed point. On the other hand, the next theorem establishes that completeness of the space is equivalent to fixed point property for the larger class of k-continuous mappings satisfying contractive conditions (i) and (ii) of Theorems 2.1 and 2.5. In the next theorem, given two real numbers r and s, we shall use the notation $r << s$ to mean that r is much less than s.

Theorem 2.9. *Let (X, d) be a metric space or a b-metric space. If every k-continuous self-mapping of X satisfying the conditions (i) and (ii) has a fixed point, then X is complete.*

Proof. Suppose that every k-continuous self-mapping of X satisfying conditions (i) and (ii) of Theorem 2.1 possesses a fixed point. We assert that X is complete. If possible, suppose X is not complete. Then there exists a Cauchy sequence in X, say $S = \{u_1, u_2, u_3, \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the sequence $S, d(x, S - \{x\}) > 0$ and there exists a least positive integer $N(x)$ such that $x \neq u_{N(x)}$ and for each $m \geq N(x)$ we have

$$d(u_{N(x)}, u_m) << d(x, u_{N(x)}). \tag{4}$$

Let us define a mapping $f : X \to X$ by $f(x) = u_{N(x)}$. Then, $f(x) \neq x$ for each x and, using (4), for any x, y in X we get

$$\begin{aligned} d(fx, fy) &= d(u_{N(x)}, u_{N(y)}) << d(x, u_{N(x)}) = d(x, fx) \text{ if } N(x) \leq N(y) \\ \text{or} \quad d(fx, fy) &= d(u_{N(x)}, u_{N(y)}) << d(y, u_{N(y)}) = d(y, fy) \text{ if } N(x) > N(y). \end{aligned}$$

This implies that

$$d(fx, fy) << max\{d(x, fx), d(y, fy)\}. \tag{5}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon < max\{d(x, fx), d(y, fy)\} \leq \epsilon + \delta \implies d(fx, fy) \leq \epsilon. \tag{6}$$

It is clear from (5) and (6) that the mapping f satisfies conditions (i) and (ii) of Theorem 2.1. Moreover, f is a fixed point free mapping whose range is contained in the non-convergent Cauchy sequence $S = \{u_n\}$. Hence, there exists no sequence $\{x_n\}$ in X for which $\{fx_n\}$ converges, that is, there exists no sequence $\{x_n\}$ in X for which the condition $fx_n \to t \implies f^2x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping. Thus, we have a 2-continuous self-mapping f of X satisfying (i) and (ii) which does not possess a fixed point. This contradicts our hypothesis. Hence X is complete. □

Example 2.2 shows that a k-continuous self-mapping that satisfies conditions (i) and (ii) possesses a unique fixed point if X is complete. The next example shows that a k-continuous self-mapping satisfying (i) and (ii) may not possess a fixed point if X is not complete.

Example 2.10. *Let $X = [0, 1) \cup (1, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = \frac{(1 + x)}{2} \text{ if } 0 \leq x < 1, \quad fx = 0 \text{ if } 1 < x \leq 2.$$

Then f satisfies the contractive conditions (i) and (ii) with $\delta(\epsilon) = 1 - \frac{\epsilon}{2}$ for $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ but f does not have a fixed point. The mapping f is continuous and, hence, k- continuous for each $k \geq 1$.

Remark 2.11. *In Theorem 2.9 if the mappings satisfy Kannan contractive condition then the contractive conditions (i) and (ii) are obviously satisfied and Theorem 2.9 not only contains Subrahmanyam's theorem as a particular case but also extends it to b-metric spaces.*

Remark 2.12. *In a family of mappings containing both continuous and discontinuous mappings and satisfying the conditions of Theorem 2.1 we may define a measure of discontinuity of a mapping f_r in the following manner:*
Measure of discontinuity of f_r at $z = [\lim_n sup\{d(z, f_r x_n)\} + lim_n inf\{d(z, f_r x_n)\}]/[2sup\{d(z, fx) : x \in X\}]$, where $\{x_n\}$ is any sequence such that $\lim_{n\to\infty} x_n = z$. Thus
Measure of discontinuity of f_0 at $1 = [1 + 0]/2 = 1/2$,
Measure of discontinuity of f_r, $0 < r < 1$, at $1 = [(1 - r) + 0]/2 = (1 - r)/2$,
Measure of discontinuity of f_r, $0 < r < 1$, at $2 = [2 + (2 - r)]/4 = (4 - r)/4$,
Measure of discontinuity of f_1 at $1 = [0 + 0]/2 = 0$,
Measure of discontinuity of f_1 at $2 = [2 + 1]/4 = 3/4$.
The mapping f_1 is obviously continuous at $x = 1$ and the measure of discontinuity at $x = 1$ is found to be 0.

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