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Some results of Fredholm perturbations of multivalued linear operator in normed spaces

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Abstract. In the present paper, we establish first the relation between the perturbation of upper Fredholm and strictly singular, and then the relation between lower semi-Fredholm and strictly cosingular linear relations. Most importantly in Theorem 3.4, we show that $\mathcal{P}(\mathcal{F}_-(X,Y))$ coincides with $SC(X,Y)$. We bring to light, the relationship between the essential spectra of a multivalued linear operator and its selection

1. Introduction

The theory of multivalued linear operators has proved to be distinct from the theory of single valued linear operators (especially see [7]). In fact, there arose some problems when applying the theory of single valued to multivalued linear operators. This is why there emerged a need to formulate fundamental concepts related exclusively to multivalued linear operators or simply linear relations. In general, linear relations appeared in Functional Analysis with J. Von Neumann [10] who shed light on the adjoints of non-densely defined linear operators and the inverses of certain operators.

In this paper, let X and Y be two normed linear spaces. A linear relation T from X to Y is a mapping from a subspace $\mathcal{D}(T) = \{x \in X : Tx \neq \emptyset\} \subseteq X$, called the domain of T, into $P(Y)\backslash\{\emptyset\}$ (the collection of nonempty subsets of Y) such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all nonzero scalars α_1, α_2 and x_1, $x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be single valued or simply an operator, that is equivalent to $T(0) = \{0\}$. The collection of linear relations is denoted by $L\mathcal{R}(X,Y)$ and we write $L\mathcal{R}(X) = L\mathcal{R}(X,X)$. A linear relation $T \in L\mathcal{R}(X,Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \Big\{(x,y) \in X \times Y : x \in \mathcal{D}(T) \text{ and } y \in Tx\Big\}.$$

The inverse of $T \in L\mathcal{R}(X,Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) = \Big\{(y,x) \in Y \times X : \ (x,y) \in G(T)\Big\}.$$

Let $T,\ S \in L\mathcal{R}(X,Y)$, then the linear relation $T+S$ is defined by

$$G(T+S) = \Big\{(x,u+v) \in X \times Y : \ (x,u) \in G(T) \text{ and } (x,v) \in G(S)\Big\}.$$

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If $T \in L\mathcal{R}(X, Y)$ and $S \in L\mathcal{R}(Y, Z)$, then the composition or product $ST \in L\mathcal{R}(X, Z)$ is defined by

$$G(ST) = \Big\{(x, z) \in X \times Z : \ (x, y) \in G(T) \text{ and } (y, z) \in G(S) \text{ for some } y \in Y\Big\}.$$

If M is a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$, then $T_{|M\cap\mathcal{D}(T)} = T_{|M}$ is defined by

$$G(T_{|M}) = \Big\{(x, y) \in G(T) : x \in M\Big\}.$$

We write $S \subset T$ if $G(S) \subset G(T)$ and we say that T is an extension of S if $T/_{\mathcal{D}(S)} = T$. The notations $R(T)$ and $N(T)$ for a linear relation T denote respectively the range and the null space of T, defined by $R(T) = \Big\{y : \ (x, y) \in G(T)\Big\}$ and $N(T) = \Big\{x \in \mathcal{D}(T) : \ (x, 0) \in G(T)\Big\}$ T is said to be surjective if $R(T) = Y$. Similarly, T is said to be injective if the null spaces $N(T) = T^{-1}(0) = \{0\}$. When T is both injective and surjective, we say that T is bijective.

The quotient map from Y into $Y/\overline{T(0)}$ is denoted by Q_T. Clearly $Q_T T$ is single valued and the norm of T is defined by $\|T\| = \|Q_T T\|$ and $\|Tx\| := \|Q_T Tx\|$ for all $x \in \ \mathcal{D}(T)$. We say that $T \in L\mathcal{R}(X, Y)$ is continuous if $\|T\| < \infty$; bounded if it is continuous and $\mathcal{D}(T) = X$; open if T^{-1} is continuous; closed if its graph is a closed subspace. We denote the set of all closed and bounded linear relations from X to Y by $\mathcal{CR}(X, Y)$ and $\mathcal{BR}(X, Y)$ respectively. If $X = Y$, we have $\mathcal{CR}(X, X) = \mathcal{CR}(X)$ and $\mathcal{BR}(X, X) = \mathcal{BR}(X)$.
Let $T \in \mathcal{LR}(X, Y)$. We say that T is compact if $\overline{Q_T T(B_X)}$ is compact and B_X is the unit ball of X. T is precompact if $Q_T T(B_X)$ is totally bounded. We denote the class of all compact linear relations by $\mathcal{KR}(X, Y)$.

If M and N are subspaces of X and of the dual space X' respectively, then

$$\begin{array}{rcl} M^{\perp} & = & \Big\{x' \in X' : \ x'(x) = 0 \ \text{ for all } x \in M\Big\}, \text{ and} \\ N^{\top} & = & \Big\{x \in X : \ x^{'}(x) = 0 \ \text{ for all } x^{'} \in N\Big\}. \end{array}$$

Let $T \in L\mathcal{R}(X, Y)$. The adjoint of T, which is T', is defined by

$$G(T') = G(-T^{-1})^{\perp} \subset Y' \times X',$$

where $\langle (y, x), (y', x')\rangle = \langle x, x'\rangle + \langle y, y'\rangle$. This means that

$$(y', x') \in G(T') \text{ if, and only if, } y'y - x'x = 0 \text{ for all } (x, y) \in G(T).$$

The adjoint or conjugate T' of a linear relation $T \in L\mathcal{R}(X, Y)$ is defined by

$$G(T') = G(-T^{-1})^{\perp} \subset Y' \times X',$$

where $\langle (y, x), (y', x')\rangle := \langle x, x'\rangle + \langle y, y'\rangle = xx' + y'y$. For $(y', x') \in G(T')$ we have $y'y = x'x$ whenever $x \in \mathcal{D}(T)$.

Let E be a subspace of a normed linear space X. We denote the natural injection map from E into X by J_E i.e., for $x \in E$, $J_E x = x \in E$. The families of infinite dimensional and the class of all closed infinite codimensional subspaces of X are denoted by $\mathcal{I}(X)$ and $\varepsilon(X)$ respectively. Perturbation theorems recalled below are the following operational quantities, see Cross [7, Definition IV.1.1].

$$\Gamma(T) = \inf\Big\{\|T|_M\| : M \in \mathcal{I}(\mathcal{D}(T))\Big\},$$

$$\Delta(T) = \sup\Big\{\Gamma(T|_M) : M \in \mathcal{I}(\mathcal{D}(T))\Big\},$$

$$\Gamma'(T) = \inf\Big\{\|Q_M J_Y T\| : M \in \varepsilon(Y)\Big\},$$

$$\Delta'(T) = \sup\Big\{\Gamma'(Q_M T) : M \in \varepsilon(Y)\Big\}.$$

We say that T is strictly cosingular if $\Delta'(T) = 0$. It is proved in [7, Theorem V.2.6] that T is strictly singular if, and only if, $\Delta(T) = 0$.

For $T \in L\mathcal{R}(X, Y)$, we write $\alpha(T) = \dim N(T)$, $\beta(T = \dim Y/R(T)$ and the index of T is the quantity $i(T) = \alpha(T) - \beta(T)$.
A linear relation $T \in L\mathcal{R}(X, Y)$ is said to be upper semi-Fredholm and denoted by $T \in \mathcal{F}_+(X, Y)$, if there exists a finite codimensional subspace M of X for which the restriction $T|_M$ is injective and open.
T is said to be lower semi-Fredholm and denoted by $T \in \mathcal{F}_-(X, Y)$, if its conjugate T' is upper semi-Fredholm. The class of Fredholm linear relation is defined by $\mathcal{F}(X, Y) = \mathcal{F}_+(X, Y) \cap \mathcal{F}_-(X, Y)$.

In the case where X and Y are two Banach spaces, we extend the classes of closed single valued Fredholm type operators given earlier to include closed multivalued operators. Note that the definitions of the classes $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ are consistent

$$\Phi_+(X, Y) = \Big\{T \in C\mathcal{R}(X, Y)\ \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\Big\},$$

$$\Phi_-(X, Y) = \Big\{T \in C\mathcal{R}(X, Y) :\ \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\Big\}.$$

T is said to be semi-Fredholm (resp. Fredholm) relation if $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y) = \Phi_\pm(X, Y)$, (resp. $T \in \Phi_+(X, Y) \cap \Phi_-(X, Y) = \Phi(X, Y)$).
If $X = Y$, we have by $\mathcal{F}_+(X, X) = \mathcal{F}_+(X)$, $\mathcal{F}_-(X, X) = \mathcal{F}_-(X)$, $SS(X, X) = SS(X)$, $SC(X, X) = SC(X)$, $\mathcal{F}(X, X) = \mathcal{F}(X)$, $\Phi_+(X, X) = \Phi_+(X)$, $\Phi_+(X, X) = \Phi_+(X)$, $\Phi_\pm(X, X) = \Phi_\pm(X)$ and $\Phi(X, X) = \Phi(X)$.

It is worth noting that the study of multivalued Fredholm linear operators was tackled by D.Wilcox in his PhD thesis. In fact, he investigated some properties of multivalued Fredholm linear operators in normed linear spaces. For more information, we may refer to [11].

We denote by $\mathcal{L}(X, Y)$ the classes of all bounded operators. An operator $T \in \mathcal{L}(X, Y)$ is called Riesz operator if $\lambda - T \in \Phi(X, Y)$ for all scalars $\lambda \neq 0$.
Let $T \in L\mathcal{R}(X, Y)$ and let G_T denote the graph operator of T, i.e., G_T is the identity injection of X_T into X ($G_T x = x$) and X_T is the vector space $\mathcal{D}(T)$ endowed with the norm $\|x\|_T = \|x\| + \|Tx\|$ for $x \in \mathcal{D}(T)$.

Let $T \in L\mathcal{R}(X)$. The resolvent set T is the set defined by

$$\rho(T = \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is bijective, open a with dense range}\Big\}.$$

So, by virtue of Closed Theorem for linear relations (see [7, Theorem III.4.2]), when A is closed and X is a Banach space, this coincides with the set

$$\rho(T) = \Big\{\lambda \in \mathbb{C} :\ (\lambda - T)^{-1} \text{ is everywhere defined and single valued}\Big\}.$$

The spectrum of T is the set $\sigma(T) = \mathbb{C}\backslash\rho(T)$.

In recent years, several authors have extended the notion of the essential spectra to linear relations. We can cite as example [2, 3, 6, 11]. Let $T \in L\mathcal{R}(X)$. We define the essential spectra of T by

$$\begin{aligned}
\sigma_{e1}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{F}_+(X)\},\\
\sigma_{e2}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{F}_-(X)\},\\
\sigma_{e3}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{F}_\pm(X)\},\\
\sigma_{e4}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{F}(X)\},\\
\sigma_{e5}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{F}(X) \text{ and } i(\lambda - T) = 0\}.
\end{aligned}$$

Remark 1.1. (*i*) *In* [11, Proposition 8.2.9], *the other proved that*

$$\sigma_{e3}(T) \subset \sigma_{e1}(T) \subset \sigma_{e4}(T) \subset \sigma_{e5}(T) \subset \sigma(T).$$

(*ii*) $\sigma_{e1}(T') = \sigma_{e2}(T)$. *In fact, let* $\lambda \notin \sigma_{e1}(T')$ *if, and only if,* $(\lambda - T') \in \mathcal{F}_+(X)$ *if, and only if,* $(\lambda - T)' \in \mathcal{F}_+(X)$ *if, and only if,* $(\lambda - T) \in \mathcal{F}_-(X)$ *if, and only if,* $\lambda \notin \sigma_{e2}(T)$.

This paper is organized as follows: In the next section, we recall some definitions and results from the theory of linear relation which will be used extensively in the sequel. In section 3, we establish some perturbation results. Most importantly in Theorem 3.4, we show hat $\mathcal{P}(\mathcal{F}_-(X,Y))$ coincides with $SC(X,Y)$. We bring to light, the relationship between the essential spectra of a multivalued linear operator and its selection.

2. Preliminaries

In this section we recall some results of the theory of linear relations which will be needed in the following sections. That's why, we start by giving some auxiliary results from the theory of linear relations.

Proposition 2.1. [7, Definition II.5.1 and Proposition II.5.3] *Let $T \in L\mathcal{R}(X,Y)$. Then*

(i) *T is closed if, and only if, T^{-1} is closed if, and only if, $T(0)$ is closed and $Q_T T$ is closed.*

(ii) *If T is continuous, $\mathcal{D}(T)$ and $T(0)$ are closed, then T is closed.*

Proposition 2.2. [7, Proposition II.5.13] *Let T be closed and $F \subset Y$ be finite dimensional, then $Q_F T$ is closed.*

Definition 2.3. [7, Definition I.5.1] *A single-valued linear operator, S, is called a linear selection of a linear relation T if*

$$T = \underbrace{S}_{\text{single valued part}} + T - T \ \ \text{with } \mathcal{D}(S) = \mathcal{D}(T).$$

Then for $x \in \mathcal{D}(T)$, we have $Tx = Sx + T(0)$.

Remark 2.4. *Let $T \in L\mathcal{R}(X,Y)$.*

(i) *If P is a linear projection with $\mathcal{D}(P) = R(T)$ and $N(P) = T(0)$, then PT is a linear selection of T.*

(ii) *If T has a continuous linear selection S, then T is continuous with $\|T\| \leq \|S\|$.*

Proposition 2.5. [7, Proposition VII.2.2] *Let $T \in L\mathcal{R}(X,Y)$ and suppose $S \in L\mathcal{R}(X,Y)$ satisfies $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$ and $S(0) \subset T(0)$, and is T-bounded with $a, b > 0, b < 1$ such that for $x \in \mathcal{D}(T)$, $\|Sx\| \leq a\|x\| + b\|Tx\|$. Then the norm $\|.\|_T$ and $\|.\|_{T+S}$ are equivalent.*

Proposition 2.6. [7, Definition V.1.1] *The following equivalences hold:*
(i) *$T \in \mathcal{F}_+(X,Y)$ if, and only if, $Q_T T \in \mathcal{F}_+(X, Y/\overline{T(0)})$.*

(ii) *$T \in \mathcal{F}_-(X,Y)$ if, and only if, $Q_T T \in \mathcal{F}_-(X, Y/\overline{T(0)})$.*

Lemma 2.7. [7, Corollary V.7.7 and Proposition V.5.11] *Let $T \in L\mathcal{R}(X,Y)$ and let $M \subset Y$ such that $dim(M) < \infty$. Then*

(i) *$T \in \mathcal{F}_+(X,Y)$ if, and only if, $Q_M T \in \mathcal{F}_+(X, Y/M)$.*

(ii) *$T \in \mathcal{F}_-(X,Y)$ if, and only if, $Q_M T \in \mathcal{F}_-(X, Y/M)$.*

Proposition 2.8. (i) [7, Corollary V.2.5] *Let $T \in L\mathcal{R}(X,Y)$.*

$$T \in \mathcal{F}_+(X,Y) \text{if, and only if, } TG_T \in \mathcal{F}_+(X_T, Y).$$

(ii) [7, Proposition V.5.24] *If $TG_T \in \mathcal{F}_-(X_T, Y)$, then $T \in \mathcal{F}_-(X,Y)$.*

(iii) [7, Corollary V.5.27] *Let $T \in L\mathcal{R}(X,Y)$ such that T is closable.*

$$T \in \mathcal{F}_-(X,Y) \text{ if, and only if, } TG_T \in \mathcal{F}_-(X_T, Y).$$

Lemma 2.9. (*i*) [7, Proposition IV.5.11] *Let* $T, S \in L\mathcal{R}(X, Y)$. *Then*

$$\Gamma'(T + S) \leq \Delta'(J_Y T) + \Gamma'(S).$$

(*ii*) [11, Corollary 4.2.9] *Let* $T \in L\mathcal{R}(X, Y)$. *Then*

$$\Gamma'(J_Y T) = \Gamma'(T).$$

(*iii*) [11, Proposition 4.2.5] *Let* $M \subset Y$ *such that* $dim(M) < \infty$ *and* $\dim(Y) = \infty$. *Then*

$$\Gamma'(Q_M T) = \Gamma'(T) \text{ and } \Delta'(Q_M T) = \Delta'(T).$$

In his book [7], R.W.Cross has introduced some results for the perturbations of $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$.

Theorem 2.10. (*i*) [7, Theorem V.2.4] *Let* $T \in L\mathcal{R}(X, Y)$ *and* $dim(\mathcal{D}(T)) = \infty$.

$$T \in \mathcal{F}_+(X, Y) \text{ if, and only if, } \Gamma(T) > 0.$$

(*ii*) [11, Corollary 4.2.9] *Let* $\dim(Y) = \infty$ *and* $\dim(T(0)) < \infty$.

$$T \in \mathcal{F}_-(X, Y) \text{ if, and only if, } \Gamma'(T) > 0.$$

Theorem 2.11. [7, Theorem V.3.2] *Let* $S, T \in L\mathcal{R}(X, Y)$ *and let* $S(0) \subset \overline{T(0)}$.

$$\text{If } \Delta(S) < \Gamma(T), \text{ then } T + S \in \mathcal{F}_+(X, Y).$$

The following Corollary is a direct consequence of Theorem 2.11.

Corollary 2.12. *Let* $S, T \in L\mathcal{R}(X, Y)$ *and let* $S(0) \subset \overline{T(0)}$.
If $T \in \mathcal{F}_+(X, Y)$ *and* $S \in SS(X, Y)$, *then* $T + S \in \mathcal{F}_+(X, Y)$ *and* $T - S \in \mathcal{F}_+(X, Y)$.

Remark 2.13. *The inclusion* $S(0) \subset \overline{T(0)}$ *is believed to be necessary* (*see* [7, Example V.3.1]).

Theorem 2.14. [7, Theorem IV.2.9] *Let* $T \in L\mathcal{R}(X, Y)$ *and* $S \in L\mathcal{R}(Y, Z)$.
If T *is an operator, then* $\Delta(ST) \leq \Delta(S)\Delta(T)$.

Proposition 2.15. [7, Proposition IV.5.8] *Let* $T \in L\mathcal{R}(X, Y)$ *and* $S \in L\mathcal{R}(Y, Z)$ *such that* $T(0) \subset \mathcal{D}(S)$. *Then*

$$\Delta'(ST) \leq \|T\|\Delta'(S).$$

3. Main results

In the last years there have been many studies of the peturbation of upper, lower semi Fredholm and Fredohlm perturbation for single valued operators (see, for example [4, 5, 8, 9]). These studies of Fredholm theory and perturbation results are of a great importance in the description of the essential spectrum. For this, it seems interesting to study some perturbation results of multivalued linear in normed linear spaces.

Lemma 3.1. *Let* $S \in L\mathcal{R}(X, Y)$ *such that* $\dim(S(0)) < \infty$.

(*i*) $T + S - S \in \mathcal{F}_+(X, Y)$ *if, and only if,* $T \in \mathcal{F}_+(X, Y)$.

(*ii*) $T + S - S \in \mathcal{F}_-(X, Y)$ *if, and only if,* $T \in \mathcal{F}_-(X, Y)$.

(*iii*) $T + S - S \in \mathcal{F}(X, Y)$ *if, and only if,* $T \in \mathcal{F}(X, Y)$.

Proof. (*i*) [7, Lemma V.7.8].
(*ii*) We have

$$Q_S(T+S-S) = Q_S(T)+Q_S(S-S) = Q_S(T). \tag{1}$$

Since $T+S-S \in \mathcal{F}_-(X,Y)$ and $\dim(S(0)) < \infty$, then by using Lemma 2.7 (*iii*), $Q_S(T+S-S) \in \mathcal{F}_-(X,Y)$. It follows from Eq. (1) that $Q_S T \in \mathcal{F}_-(X,Y)$. Hence, $T \in \mathcal{F}_-(X,Y)$. Conversely, we suppose that $T \in \mathcal{F}_-(X,Y)$, then by Proposition 2.6 (*ii*), we have $Q_S T \in \mathcal{F}_-(X,Y)$. From Eq.(1), it follows that $Q_S(T+S-S) \in \mathcal{F}_-(X,Y)$. Thus using Lemma 2.7 (*ii*), we have $T+S-S \in \mathcal{F}_-(X,Y)$.
(*iii*) The proof may be checked in the same way as the proof of (*ii*). □

Definition 3.2. *Let $S \in L\mathcal{R}(X,Y)$ such that* $\dim S(0) < \infty$.

(*i*) *S is called an upper semi-Fredholm perturbation if $T+S \in \mathcal{F}_+(X,Y)$ whenever $T \in \mathcal{F}_+(X,Y)$.*

(*ii*) *S is called lower semi-Fredholm perturbation if $T+S \in \mathcal{F}_-(X,Y)$ whenever $T \in \mathcal{F}_-(X,Y)$.*

(*iii*) *S is called a Fredholm perturbation if $T+S \in \mathcal{F}(X,Y)$ whenever $T \in \mathcal{F}(X,Y)$.*

The sets of upper, lower semi-Fredholm and Fredholm perturbations are denoted by $\mathcal{P}(\mathcal{F}_+(X,Y))$, $\mathcal{P}(\mathcal{F}_-(X,Y))$ and $\mathcal{P}(\mathcal{F}(X,Y)) = \mathcal{P}(\mathcal{F}_+(X,Y)) \cap \mathcal{P}(\mathcal{F}_-(X,Y))$ respectively. If $X = Y$, we get $\mathcal{P}(\mathcal{F}_+(X)) := \mathcal{P}(\mathcal{F}_+(X,X))$, $\mathcal{P}(\mathcal{F}_-(X)) := \mathcal{P}(\mathcal{F}_-(X,X))(X,X)$ and $\mathcal{P}(\mathcal{F}(X)) := \mathcal{P}(\mathcal{F}(X,X))$.

$$\widetilde{\mathcal{K}}(X,Y) := \{K \in \mathcal{KR}(X,Y) : \dim K(0) < \infty\}$$

and

$$\widetilde{\mathcal{K}_p}(X,Y) := \{K \text{ is precompact} : \dim K(0) < \infty\}.$$

In general we have :

$$\widetilde{\mathcal{K}}(X,Y) \subset \mathcal{P}(\mathcal{F}_+(X,X))$$

and

$$\widetilde{\mathcal{K}}(X,Y) \subset \widetilde{\mathcal{K}_p}(X,Y) \subset \mathcal{P}(\mathcal{F}_-(X,X)).$$

Remark 3.3. *If X is a Banach space and $S \in \mathcal{L}(X)$, the pre-mentioned sets coincide with the sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations respectively in* [9, Definition 2.1.13].

Theorem 3.4. *Let X and Y be two normed linear spaces, we have*

(*i*) $\mathcal{P}(\mathcal{F}_+(X,Y)) = SS(X,Y)$.

(*ii*) $\mathcal{P}(\mathcal{F}_-(X,Y)) = SC(X,Y)$.

(*iii*) $\mathcal{P}(\mathcal{F}(X,Y)) = SS(X,Y) \cap SC(X,Y)$.

Proof. (*i*) Let $T \in \mathcal{F}_+(X,Y)$ and $S \in SS(X,Y)$ such that $dim(S(0)) < \infty$. We have

$$Q_S(T+S) = Q_S T + Q_S S.$$

Since $T \in \mathcal{F}_+(X,Y)$ and $dim(S(0)) < \infty$, then by Lemma 2.7 (*i*), $Q_S T \in \mathcal{F}_+(X, Y/S(0))$. Moreover, $Q_S S$ is strictly singular and single valued. Therefore, applying Theorem 2.11 we obtain $Q_S S + Q_S T = Q_S(S+T)$ is upper semi-fredholm, which implies that $T+S \in F_+(X,Y)$ (see Proposition 2.6 (*i*)). Thus $SS(X,Y) \subset \mathcal{P}(\mathcal{F}_+(X,Y))$. For the reverse inclusion, let $S \notin SS(X,Y)$ then by Theorem 2.10 (*iii*) $\Delta(S) > 0$, there exists $M \in \mathcal{I}(\mathcal{D}(T))$ such that $\Gamma(S|_M) > 0$. Let $T \in L\mathcal{R}(X,Y)$ such that $\mathcal{D}(S) = M$ and $T = -S|_M$.Then

$$\begin{aligned} \Gamma(T) &= \inf_{N \in \mathcal{I}(\mathcal{D}(T))} \|T|_N\| \\ &= \inf_{N \in \mathcal{I}(M)} \|S|_{M \cap N}\| \\ &= \Gamma(S|_M) > 0. \end{aligned}$$

Hence by Theorem 3.1 $T \in \mathcal{F}_+(X,Y)$. Moreover, we have $T+S = S - S|_M \notin \mathcal{F}_+(X,Y)$, then $S \notin P(\mathcal{F}_+(X,Y))$. Therefore $\mathcal{P}(\mathcal{F}_+(X,Y)) \subset SS(X,Y)$.
(*ii*) Let $T \in \mathcal{F}_-(X,Y)$ and $S \in SC(X,Y)$ with $\dim(S(0)) < \infty$. We have

$$\begin{aligned} Q_S T &= Q_S(T) + Q_S S - Q_S S \\ &= Q_S(T+S) - Q_S S, \end{aligned}$$

and so from Lemma 2.9, we obtain

$$\Gamma'(Q_S T) \leq \Gamma'(Q_S(T+S)) + \Delta'(Q_S S). \tag{2}$$

Since $\dim(S(0)) < \infty$, then by Lemma 2.9 (*iii*) $\Delta'(Q_S S) = \Delta'(S) = 0$. Moreover, since $T \in \mathcal{F}_-(X,Y)$ and $\dim(S(0)) < \infty$, thus $Q_S T \in \mathcal{F}_-(X,Y)$. Hence $\Gamma'(Q_S T)) > 0$. So that by Eq.(2), we obtain $\Gamma'(Q_S(T+S)) > 0$, then $T+S \in \mathcal{F}_-(X,Y)$. Therefore $SC(X,Y) \subset \mathcal{P}(\mathcal{F}_-(X,Y))$
For the reverse inclusion, let $S \notin SC(X,Y)$. Then $\Delta'(S) > 0$ and hence there exist $N \in \varepsilon(Y)$ such that $\Gamma'(Q_N S) > 0$.
Let $T \in L\mathcal{R}(X,Y)$ such that $\dim(T(0)) < \infty$, $\mathcal{D}(T) = M \in \varepsilon(Y)$, $N \subset M$ and $J_Y Tx = -J_Y Sx$ $(x \in M)$. Then

$$\begin{aligned} \Gamma'(T) &= \inf\{\|Q_M J_Y T\| | M \in \varepsilon(Y)\} \\ &= \inf\{\|Q_M J_Y S\| | M \in \varepsilon(Y) \textit{ and } N \subset M\} \\ &= \Gamma'(Q_N S) > 0. \end{aligned}$$

Hence by Theorem 3.1(ii) $T \in \mathcal{F}_-(X,Y)$. While

$$\begin{aligned} \Gamma'(T+S) &= \Gamma'(J_Y T + J_Y S) \ \ (\textit{By Lemma } 2.9) \\ &= \Gamma'(J_Y T - J_Y T) \\ &= 0, \end{aligned}$$

then $T+S \notin \mathcal{F}_-(X,Y)$. Thus $S \notin \mathcal{P}(\mathcal{F}_-(X,Y)$. Therefore $\mathcal{P}(\mathcal{F}_-(X,Y)) \subset SC(X,Y)$.
(*iii*) we have

$$\begin{aligned} \mathcal{P}(\mathcal{F}(X,Y)) &= \mathcal{P}(\mathcal{F}_+(X,Y)) \cap \mathcal{P}(\mathcal{F}_-(X,Y)) \\ &= \mathcal{P}(\mathcal{F}(X,Y)) = SS(X,Y) \cap SC(X,Y). \end{aligned}$$

□

Proposition 3.5. *Let $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$.*

i) If T is an operator continuous and $S \in \mathcal{P}(\mathcal{F}_+(Y,Z))$, then $ST \in \mathcal{P}(\mathcal{F}_+(X,Z))$.

ii) If $T \in L\mathcal{R}(X,Y)$ and $S \in \mathcal{P}(\mathcal{F}_-(Y,Z))$ such that $T(0) \subset \mathcal{D}(S)$ and S is continuous,

then $ST \in \mathcal{P}(\mathcal{F}_-(X,Z))$.

Proof. *i*) Since $S \in \mathcal{P}(\mathcal{F}_+(Y,Z))$, then it follows from Theorem 3.4 that $S \in SS(Y,Z)$ which implies that $\Delta(S) = 0$. The use of Theorem2.14 leads to $\Delta(ST) \leq \Delta(S)\Delta(T) = 0$. Thus $\Delta(ST) = 0$ which implies that $ST \in SS(X,Z)$. Therefore $ST \in \mathcal{P}(\mathcal{F}_+(X,Z))$.

ii) Since $S \in \mathcal{P}(\mathcal{F}_-(Y,Z))$, then it follows from Theorem 3.4 that $S \in SC(Y,Z)$ which implies that $\Delta'(S) = 0$. Since $S(0) \subset \mathcal{D}(T)$ and S is continuous, then by Proposition 2.15, we get $\Delta'(ST) \leq \|T\|\Delta'(S) = 0$. Hence $\Delta'(ST) = 0$ and therefore $ST \in SC(X,Z)$. This yields $ST \in \mathcal{P}(\mathcal{F}_-(X,Z))$. □

Definition 3.6. *Let X and Y be two normed spaces, $A \in L\mathcal{R}(X,Y)$ and let $T \in L\mathcal{R}(X,Y)$ be an arbitrary linear relation. We say that T is A-Fredholm perturbation if $TG_A \in \mathcal{P}(\mathcal{F}(X_A,Y)$.*
T is called upper (resp., lower) A-Fredholm perturbation if $TG_A \in \mathcal{P}(\mathcal{F}_+(X_A,Y)$ (resp., $TG_A \in \mathcal{P}(\mathcal{F}_-(X_A,Y))$.

Let $A\mathcal{P}(\mathcal{F}(X,Y)), A\mathcal{P}(\mathcal{F}_+(X,Y))$ and $A\mathcal{P}(\mathcal{F}_-(X,Y))$ designate the set of A-Fredholm, upper A-Fredholm and lower A-Fredholm perturbation respectively.

If $X = Y$, we get $A\mathcal{P}(\mathcal{F}(X)) = A\mathcal{P}(\mathcal{F}(X,X))$, $A\mathcal{P}(\mathcal{F}_+(X)) = A\mathcal{P}(\mathcal{F}_+(X,X))$ and $A\mathcal{P}(\mathcal{F}_-(X)) = A\mathcal{P}(\mathcal{F}_-(X,X))$.

Proposition 3.7. *Let X and Y be two normed spaces, $A \in L\mathcal{R}(X, Y)$ such that A is continuous.*

(*i*) $\mathcal{P}(\mathcal{F}_+(X)) \subset A\mathcal{P}(\mathcal{F}_+(X))$.

(*ii*) $\mathcal{P}(\mathcal{F}_-(X)) \subset A\mathcal{P}(\mathcal{F}_-(X))$.

Proof. (*i*) Let $T \in \mathcal{P}(\mathcal{F}_+(X))$. Since $A \in L\mathcal{R}(X, Y)$ such that A is continuous, then G_A is the graph operator of A. Let $x \in X_A$, then

$$\begin{aligned} \|G_A x\|_A &= \|x\| + \|Ax\| \\ &\leq \|x\| + M\|x\| = (1 + M)\|x\| \text{ (as A is continuous),} \end{aligned}$$

this implies that $G_A \in \mathcal{L}(X_A, X)$. From Proposition3.5, it follows $TG_A \in \mathcal{P}(\mathcal{F}_+(X_A, Y))$. Hence $T \in A\mathcal{P}(\mathcal{F}_+(X, Y))$.

(*ii*) Let $T \in \mathcal{P}(\mathcal{F}_-(X, Y))$ and we have $G_A \in \mathcal{L}(X_A, X)$. Using Proposition3.5, we get $TG_A \in \mathcal{P}(\mathcal{F}_-(X_A, Y))$. Hence $T \in A\mathcal{P}(\mathcal{F}_+(X, Y))$. □

Theorem 3.8. *Let $T \in L\mathcal{R}(X, Y)$ and $A \in L\mathcal{R}(X, Y)$ such that $T(0) \subset A(0)$, $\mathcal{D}(A) \subset \mathcal{D}(T)$ and T is A-bounded.*

(*i*) *If $A \in \mathcal{F}_+(X, Y)$ and $T \in A\mathcal{P}(\mathcal{F}_+(X, Y))$, then $A + T \in \mathcal{F}_+(X, Y)$.*

(*ii*) *If $A \in \mathcal{F}_-(X, Y)$ such that A is closable and $T \in A\mathcal{P}(\mathcal{F}_-(X, Y))$, then $A + T \in \mathcal{F}_-(X, Y)$.*

Proof. (*i*) Since $A \in \mathcal{F}_+(X, Y)$, then it follows from Proposition2.8 that $AG_A \in \mathcal{F}_+(X_A, Y)$. Assume that $T \in A\mathcal{P}(\mathcal{F}_+(X, Y))$, hence $TG_A \in \mathcal{F}_+(X_A, Y)$, allows us to deduce that $TG_A + AG_A \in \mathcal{F}_+(X_A, Y)$. This shows that $(T + A)G_A \in \mathcal{F}_+(X_A, Y)$. Hence by the equivalence $\|.\|_A$ and $\|.\|_{T+A}$, we get $(T + A)G_{A+T} \in \mathcal{F}_+(X_{A+T}, Y)$. by Proposition2.8, we get $T + A \in \mathcal{F}_+(X, Y)$.

(*ii*) Since $A \in \mathcal{F}_-(X, Y)$ such that A is closable, then it follows from Proposition2.8 that $AG_A \in \mathcal{F}_-(X_A, Y)$. Assume that $T \in A\mathcal{P}(\mathcal{F}_-(X, Y))$, hence $TG_A \in \mathcal{F}_-(X_A, Y)$, allows us to deduce that $TG_A + AG_A \in \mathcal{F}_-(X_A, Y)$. Which implies that $(T + A)G_{A+T} \in \mathcal{F}_-(X_{A+T}, Y)$. by Proposition2.8, we get $T + A \in \mathcal{F}_-(X, Y)$. □

Proposition 3.9. *Let $T \in L\mathcal{R}(X)$.*

(*i*) *If $S \in \mathcal{P}(\mathcal{F}_+(X))$. then $\sigma_{e1}(T + S) = \sigma_{e1}(T)$.*

(*ii*) *If $S \in \mathcal{P}(\mathcal{F}_-(X))$, then $\sigma_{e2}(T + S) = \sigma_{e2}(T)$.*

(*iii*) *If $S \in \mathcal{P}(\mathcal{F}_\pm(X))$, then $\sigma_{e3}(T + S) = \sigma_{e3}(T)$.*

(*iv*) *If $S \in \mathcal{P}(\mathcal{F}(X))$, then $\sigma_{e4}(T + S) = \sigma_{e4}(T)$.*

Proof. (*i*) Let $\lambda \notin \sigma_{e1}(T)$. Then $\lambda - T \in \mathcal{F}_+(X)$. Also $S \in \mathcal{P}(\mathcal{F}_+(X))$, which implies that $\lambda - T - S \in \mathcal{F}_+(X)$. Hence $\lambda - (T + S) \in \mathcal{F}_+(X)$, which yields that $\lambda \notin \sigma_{e1}(T + S)$. Therefore $\sigma_{e1}(T + S) \subset \sigma_{e1}(T)$.
Conversely, let $\lambda \notin \sigma_{e1}(T + S)$. Then $(\lambda - (T + S)) \in \mathcal{F}_+(X)$. Moreover, since $S \in \mathcal{P}(\mathcal{F}_+(X))$, hence $\lambda - T - S + S \in \mathcal{F}_+(X)$. By Lemma 3.1 (*i*), we deduce that $\lambda - T \in \mathcal{F}_+(X)$. Thus $\lambda \notin \sigma_{e1}(T)$. Therefore $\sigma_{e1}(T) \subset \sigma_{e1}(T + S)$.
Statements (*ii*), (*iii*) and (*iv*) can be checked in the same way as (*i*). □

Theorem 3.10. *Let $T \in L\mathcal{R}(X, Y)$ and $A \in L\mathcal{R}(X, Y)$ such that $T(0) \subset A(0)$, $\mathcal{D}(A) \subset \mathcal{D}(T)$ and T is A-bounded.*

(*i*) *If $T \in A\mathcal{P}(\mathcal{F}_+(X))$, then*

$$\sigma_{e1}(T + A) = \sigma_{e1}(A).$$

(*ii*) *If $T \in A\mathcal{P}(\mathcal{F}_-(X))$ and A is closable, then*

$$\sigma_{e2}(T + A) = \sigma_{e1}(A).$$

Proof. (*i*) Let $\lambda \notin \sigma_{e1}(T + A)$. Then, $\lambda - T - A \in \mathcal{F}_+(X)$ and we have $T \in A\mathcal{P}(\mathcal{F}_+(X))$, using Theorem3.8, we get $\lambda - T - A + A \in \mathcal{F}_+(X)$. This show that $\lambda - A \in \mathcal{F}_+(X)$. Thus $\lambda \notin \sigma_{e1}(A)$. $\sigma_{e1}(A) \subset \sigma_{e1}(T + A)$
Conversely, Let $\lambda \notin \sigma_{e1}(A)$. Then, $\lambda - A \in \mathcal{F}_+(X)$ and we have $T \in A\mathcal{P}(\mathcal{F}_+(X))$. Using Theorem3.8, we get $\lambda - T - A \in \mathcal{F}_+(X)$. $\lambda \notin \sigma_{e1}(T + A)$. Therefore $\sigma_{e1}(T + A) \subset \sigma_{e1}(A)$.
The proofs of (*ii*) may be achieved by following the same reasoning in (*i*). □

In what follows, we will introduce some properties of linear selections and we will show that the essential spectra of a linear relation is stable with the essential spectra of its selection.

Proposition 3.11. *Let $T \in L\mathcal{R}(X,Y)$ such that* $\dim(T(0)) < \infty$ *and let S be a selection of T.*

(i) If $S \in C\mathcal{R}(X,Y)$, then $T \in C\mathcal{R}(X,Y)$.

(ii) If $S \in \mathcal{F}_+(X,Y)$, then $T \in \mathcal{F}_+(X,Y)$.

(iii) If $S \in \mathcal{F}_-(X,Y)$, then $T \in \mathcal{F}_-(X,Y)$.

(iv) If $S \in \mathcal{F}(X,Y)$, then $T \in \mathcal{F}(X,Y)$.

Proof. (*i*) Let S be a closed selection of T, then $T = S + T - T$ and

$$Q_T T = Q_T(S + T - T) = Q_T S. \tag{3}$$

Since S is closed and $\dim(T(0)) < \infty$, then by Proposition 2.2 (*i*) $Q_T S$ is closed.This implies that $Q_T T$ is closed by referring back to Eq.(3). Moreover $T(0)$ is closed. Then we have by Proposition 2.1 that $T \in C\mathcal{R}(X,Y)$.
(*ii*) Since $S \in \mathcal{F}_+(X,Y)$ and $\dim(T(0)) < \infty$, then using Lemma 2.7 (*i*), we infer that $Q_T S \in \mathcal{F}_+(X,Y)$. Hence it follows from Eq.(3), that $Q_T T \in \mathcal{F}_+(X,Y)$. Therefore, $T \in \mathcal{F}_+(X,Y)$.
Statements (*iii*) and (*iv*) can be checked in the same way as (*ii*). □

Theorem 3.12. *Let $T \in L\mathcal{R}(X)$ such that* $\dim(T(0)) < \infty$ *and let S be a selection of T. Then*

$$\sigma_{ei}(T) = \sigma_{ei}(S), i = 1,2,3,4.$$

Proof. Let $\lambda \notin \sigma_{e1}(T)$. Then $\lambda - T \in \mathcal{F}_+(X)$. Also S is a selection of T, thus $T = S + T - T$, which implies that $\lambda - S + T - T \in \mathcal{F}_+(X)$. Moreover, we have $\dim(T(0)) < \infty$. From Lemma 3.1 (*i*), it follows that $\lambda - S \in \mathcal{F}_+(X)$. Then $\lambda \notin \sigma_{e1}(S)$. Therefore $\sigma_{e1}(S) \subset \sigma_{e1}(T)$.
Conversely, let $\lambda \notin \sigma_{e1}(S)$. Then $\lambda - S \in \mathcal{F}_+(X)$. Thus using Lemma 3.1 (*i*), we have $\lambda - S + T - T \in \mathcal{F}_+(X)$ which implies that $\lambda - T \in \mathcal{F}_+(X)$ (as S is a selection of T). Therefore $\sigma_{e1}(T) \subset \sigma_{e1}(S)$.
For $i = 2,3,4$, their proof may be checked in the same way. □

Remark 3.13. *Let S be a selection of T such that* $\dim(T(0)) < \infty$.
If S is a Riesz operator, then $\sigma_{ei}(T) = \{0\}$, $i = 1,2,3,4$.
In fact, using the preceding Theorem, we have $\sigma_{e1}(T) = \sigma_{e1}(S)$. Since S is a Riesz operator, then $\sigma_{e1}(S) = \{0\}$, which implies that $\sigma_{e1}(T) = \{0\}$.
For $i = 2,3,4$, their proof may be checked in the same way.

Theorem 3.14. *Let X Banach spaces. Let S_1 and S_2 be selections of T_1 and T_2 respectively such that $S_1, S_2 \in \mathcal{L}(X)$,* $\dim(T_1(0)) < \infty$ *and* $\dim(T_2(0)) < \infty$.

(i) If $S_1S_2 \in \mathcal{P}(\mathcal{F}_+)(X)$ and $S_2S_1 \in \mathcal{P}(\mathcal{F}_+)(X)$, then

$$\sigma_{e1}(T_1 + T_2)\backslash\{0\} = [\sigma_{e1}(T_1) \cup \sigma_{e1}(T_2)]\backslash\{0\}.$$

(ii) If $S_1S_2 \in \mathcal{P}(\mathcal{F}_-(X))$ and $S_2S_1 \in \mathcal{P}(\mathcal{F}_-(X))$, then

$$\sigma_{e2}(T_1 + T_2)\backslash\{0\} = [\sigma_{e2}(T_1) \cup \sigma_{e2}(T_2)]\backslash\{0\}.$$

(iii) If $S_1S_2 \in \mathcal{P}(\mathcal{F}(X))$ and $S_2S_1 \in \mathcal{P}(\mathcal{F})(X)$, then

$$\sigma_{e4}(T_1 + T_2)\backslash\{0\} = [\sigma_{e4}(T_1) \cup \sigma_{e4}(T_2)]\backslash\{0\}.$$

Proof. (*i*) Let $\lambda \in \sigma_{e1}(T_1 + T_2)\backslash\{0\}$. Since $\dim((T_1 + T_2)(0)) < \infty$, then by using Theorem 3.12, we have $\lambda \in \sigma_{e1}(S_1 + S_2)\backslash\{0\}$. Hence, it follows from [1, Theorem 2.2] that $\lambda \in [\sigma_{e1}(S_1) \cup \sigma_{e1}(S_2)]\backslash\{0\}$. Moreover, $\dim(T_1(0)) < \infty)$ and $\dim(T_2(0)) < \infty$. Then, by Theorem 3.12, we have $\lambda \in [\sigma_{e1}(T_1) \cup \sigma_{e1}(T_2)]\backslash\{0\}$. Conversely, by using the same reasoning, we find the result.
Statements (*ii*) and (*iii*) can be checked in the same way as (*i*). □

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