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New characterizations of g-Drazin inverse in a Banach algebra

Huanyin Chen[a], Marjan Sheibani Abdolyousefi[b,*]

[a]*School of Mathematics, Hangzhou Normal University, Hangzhou, China*
[b]*Farzanegan Campus, Semnan University, Semnan, Iran*

Abstract. In this paper, we present a new characterization of g-Drazin inverse in a Banach algebra. We prove that an element a in a Banach algebra has g-Drazin inverse if and only if there exists $x \in \mathcal{A}$ such that $ax = xa, a - a^2x \in \mathcal{A}^{qnil}$. As an application, we obtain the sufficient and necessary conditions for the existence of the g-Drazin inverse for certain 2×2 anti-triangular matrices over a Banach algebra. These extend the results of Koliha (Glasgow Math. J., **38**(1996), 367–381), Nicholson (Comm. Algebra, **27**(1999), 3583–3592 and Zou et al. (Studia Scient. Math. Hungar., **54**(2017), 489–508).

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. We define $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $b \in \mathcal{A}$ such that

$$ab = ba, b = bab, a - a^2b \in \mathcal{A} \text{ is quasinilpotent.}$$

Such b is unique, if exists, and denote it by a^d. If we replace quasinilpotent in the above definition with nilpotent, then b is called the Drazin inverse of a. Following Mosić ,see[15], an element $a \in \mathcal{A}$ has gs-Drazin inverse if there exists $b \in \mathcal{A}$ such that $b = bab, b \in comm(a)$ *and* $a - ab \in \mathcal{A}^{qnil}$. The g-Drazin inverse plays an important role in matrix and operator theory. Many authours have been studying this subject from different views (see [12, 14] and [17]). In this paper we provide some new characterizations for the g-Drazin inverse of an element in a Banach algebra. In Section 2, we drop the regular condition for the g-Drazin invertibility of the definition. We then thereby prove that an element a in a Banach algebra $\mathcal{A}$ has g-Drazin inverse if and only if there exist an idempotent e, a unit u and a quasinilpotent w which commute each other such that $a = eu + w$. This helps us to generalize [16, Theorem 3] and prove that an element $a \in \mathcal{A}$ has g-Drazin inverse if and only if there exists an idempotent $e \in comm(a)$ such that $eae \in [e\mathcal{A}e]^{-1}$ and $(1-e)a(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$. It was firstly posed by Campbell that the solutions to singular systems of differential equations are determined by the g-Drazin invertibility of the 2×2 anti-triangular block matrix (see [2]). The g-Drazin inverse of such special matrices attracts many authors (see [3, 7, 10, 13] and [18]). In Section 3, we apply the results in section 2 for certain anti-triangular block matrices over a Banach algebra

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* Corresponding author: Marjan Sheibani Abdolyousefi
Email addresses: huanyinchen@aliyun.com (Huanyin Chen), m.sheibani@semnan.ac.ir (Marjan Sheibani Abdolyousefi)

and provide some necessary and sufficient conditions for such matrices to be g-Drazin invertible. These also extend [3, Theorem 4.1] and [19, Theorem 2.6] for the g-Drazin inverse.

Throughout the paper, we use $\mathcal{A}^{-1}$ to denote the set of all units in $\mathcal{A}$. $\mathcal{A}^d$ indicates the set of all g-Drazin invertible elements in $\mathcal{A}$. Let $a \in \mathcal{A}$. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. $\mathbb{N}$ stands for the set of all natural numbers.

2. g-Drazin inverse

The aim of this section is to provide a new characterization of g-Drazin inverse in a Banach algebra. We shall prove that regular condition "$x = xax$" can be dropped from the definition of g-Drazin inverse. An element $a \in \mathcal{A}$ has strongly g-Drazin inverse if it is the sum of an idempotent and a quasinilpotent that commute (see [6]). We begin with a characterization of strongly Drazin inverse.

Lemma 2.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}$ has strongly g-Drazin inverse.*
(2) *$a - a^2 \in \mathcal{A}^{qnil}$.*

Proof. See [6, Lemma 2.2]. □

We come now to the demonstration for which this paper has been developed.

Theorem 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*
(2) *There exists some $x \in comm(a)$ such that $a - a^2x \in \mathcal{A}^{qnil}$.*

Proof. (1) ⇒ (2) This is obvious by choosing $x = a^d$.

(2) ⇒ (1) By hypothesis, there exists some $x \in comm(a)$ such that $a - a^2x \in \mathcal{A}^{qnil}$. Set $z = xax$. Then $z \in comm(a)$. As $(a - a^2x) \in \mathcal{A}^{qnil}$ and $x \in comm(a)$,we see that,

$$\begin{array}{rcl} a - a^2z &=& a - axaxa \\ &=& (1 + ax)(a - a^2x) \\ &\in& \mathcal{A}^{qnil}, \\ z - z^2a &=& xax - xaxaxax \\ &=& x(a - a^2x)x + xax(a - a^2x)x \\ &\in& \mathcal{A}^{qnil}. \end{array}$$

$$az - (az)^2 = (a - a^2z)z \in \mathcal{A}^{qnil}.$$

By Lemma 2.1, az is strongly g-Drazin invertible and so by [9, Theorem 3.2], we have an idempotent $e \in comm^2(az)$ such that $az - e \in \mathcal{A}^{qnil}$. We easily check that

$$(a + 1 - az)(z + 1 - az) = 1 + (a - a^2z)(1 - z) + (z - z^2a).$$

Hence,

$$\begin{array}{rcl} a + 1 - e &=& (a + 1 - az) + (az - e) \in \mathcal{A}^{-1} \textit{ and }, \\ a(1 - e) &=& (a - a^2z) + a(az - e) \in \mathcal{A}^{qnil}. \end{array}$$

Since $a \in comm(az)$, we have $ea = ae$. That is, $a \in \mathcal{A}$ is quasipolar. As every quasipolar element is g-Drazin invertible so, $a \in \mathcal{A}^d$, by [11, Theorem 4.2]. □

Corollary 2.3. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*
(2) *There exists an invertible $u \in comm(a)$ such that $a - a^2u \in \mathcal{A}^{qnil}$.*

(3) *au has strongly g-Drazin inverse for some invertible $u \in comm(a)$.*

Proof. (1) ⇒ (3) In view of [11, Theorem 4.2], there exists an idempotent $p \in comm(a)$ such that $u := a+p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$. Hence, $ap = a(u-a) \in \mathcal{A}^{qnil}$. Then $a - a^2u^{-1} \in \mathcal{A}^{qnil}$. Thus $au^{-1} - [au^{-1}]^2 \in \mathcal{A}^{qnil}$. Therefore au has strongly g-Drazin inverse by Lemma 2.1.

(3) ⇒ (2) In light of Lemma 2.1, $au - (au)^2 \in \mathcal{A}^{qnil}$ for some invertible $u \in comm(a)$. Hence $a - a^2u \in \mathcal{A}^{qnil}$, as required.

(2) ⇒ (1) This is obvious by Theorem 2.2. □

We are now ready to extend [11, Theorem 4.2] as follows.

Corollary 2.4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*
(2) *There exists some $p \in comm(a)$ such that $a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.*

Proof. (1) ⇒ (2) This is clear by [11, Theorem 4.2].

(2) ⇒ (1) Set $b = (a+p)^{-1}(1-p)$. Then $b \in comm(a)$ and

$$\begin{array}{rcl} ab & = & a(a+p)^{-1}(1-p) \\ & = & (a+p)(a+p)^{-1}(1-p) - p(a+p)^{-1}(1-p) \\ & = & 1 - p - p(a+p)^{-1}(1-p). \end{array}$$

In view of [19, Lemma 2.11], we have

$$\begin{array}{rcl} a - a^2b & = & a(1-ab) \\ & = & ap[1 + (a+p)^{-1}(1-p)] \ , \\ & \in & \mathcal{A}^{qnil} \end{array}$$

as $1 - ab = p + p(a+p)^{-1}(1-p)$. This completes the proof by Theorem 2.2. □

The next result generalizes [4, Proposition 13.1.18].

Theorem 2.5. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*
(2) *There exist an idempotent e, a unit u and a quasinilpotent w which commute each other such that $a = eu + w$.*

Proof. (1) ⇒ (2) By hypothesis, there exists a invertible $u \in comm(a)$ such that $a - a^2u^{-1} \in \mathcal{A}^{qnil}$. Then $(u^{-1}a)^2 - u^{-1}a \in \mathcal{A}^{qnil}$. In light of Lemma 2.1, $u^{-1}a$ has strongly g-Drazin inverse and so by [9, Theorem 3.2], there exists $e^2 = e \in comm^2(u^{-1}a)$ such that $w := u^{-1}a - e \in \mathcal{A}^{qnil}$. Hence, $a = ue + uw$. Clearly, $eu = ue$ and $ea = ae$; hence, $uw = wu$, $(ue)(uw) = (uw)(ue)$ and $uw \in \mathcal{A}^{qnil}$, as required.

(2) ⇒ (1) Write $a = ue + w$ for an idempotent e, an invertible u and a quasinilpotent w which commute each other. Then $(u^{-1}a)^2 - u^{-1}a \in \mathcal{A}^{qnil}$. Then $a - a^2u^{-1} \in \mathcal{A}^{qnil}$, since $-u^{-1}(a - a^2u^{-1}) \in \mathcal{A}^{qnil}$. □

Corollary 2.6. *Let $a \in \mathcal{A}^d$. Then a is the sum of two units in $\mathcal{A}$.*

Proof. Since $a \in \mathcal{A}^d$, it follows by [19, Theorem 3.11] that $\frac{a}{2} \in \mathcal{A}^d$. In view of Theorem 2.5, there exist an idempotent e, a unit u and a quasinilpotent w which commute each other such that $\frac{a}{2} = eu + w$. Hence, $a = 2eu + 2w = (2e-1)u + u + 2w = (2e-1)u + u(1 + 2u^{-1}w)$. Since $(2e-1)^2 = 1$ and $1 + 2u^{-1}w \in \mathcal{A}^{-1}$, a is the sum of two units, as asserted. □

Theorem 2.7. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}^d$.*
(2) *There exist an idempotent $e \in comm(a)$ such that $eae \in [e\mathcal{A}e]^{-1}$, $(1-e)a(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$.*

Proof. (1) ⇒ (2) By virtue of Theorem 2.5, there exist an idempotent e, a unit u and a quasinilpotent w which commute each other such that $a = eu + w$. Then $eae = eu(1 + u^{-1}w) \in [e\mathcal{A}e]^{-1}$. Moreover, we have $(1-e)a(1-e) = (1-e)w \in [(1-e)\mathcal{A}(1-e)]^{qnil}$, as desired.

(2) ⇒ (1) Suppose there exists an idempotent $e \in comm(a)$ such that $eae \in [e\mathcal{A}e]^{-1}, (1-e)a(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$. Then $a = ea + (1-e)a = e[eae + 1 - e] + (1-e)a$. In view of [19, Lemma 2.11], $(1-e)a \in \mathcal{A}^{qnil}$. Obviously, $eae + 1 - e \in \mathcal{A}^{-1}$. According to Theorem 2.5, a has g-Drazin inverse, as asserted. □

Let $\alpha \in \mathcal{A} = End(M)$. The submodule P of M is α-invariant provided that $\alpha(P) \subseteq P$ (see [16]). We now derive

Corollary 2.8. *Let $\alpha \in \mathcal{A} = End(M)$. Then the following are equivalent:*

(1) $\alpha \in \mathcal{A}^d$.
(2) $M = P \oplus Q$, *where P and Q are α-invariant,* $\alpha|_P \in [End(P)]^{-1}$, $\alpha|_Q \in End(Q)^{qnil}$. *The corresponding PQPQ-decomposition looks like*

$$\begin{array}{ccccc} M & = & P & \bigoplus & Q \\ & \alpha\,|_P = unit & \downarrow & & \downarrow \quad \alpha\,|_Q = quasinilpotent \\ M & = & P & \bigoplus & Q \end{array}.$$

Proof. (1) ⇒ (2) In view of Theorem 2.7, there exist an idempotent $e \in comm(\alpha)$ such that $e\alpha e \in [e\mathcal{A}e]^{-1}, (1-e)\alpha(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$. Set $P = Me$ and $Q = M(1-e)$. Then $M = P \oplus Q$. As $e \in comm(\alpha)$, we see that P and Q are α-invariant.

Write $(e\alpha e)^{-1} = e\beta e$. Then one easily checks that $[\alpha|_P]^{-1} = \beta|_P$. Let $\gamma \in End(Q) \cap comm(\alpha|_Q)$. We will suffice to prove $1_Q - \alpha|_Q\gamma \in [End(P)]^{-1}$.

$$\begin{array}{cccc} 1_Q - \alpha|_Q\gamma : & Q & \to & Q \\ & p & \mapsto & q - (q)\alpha\gamma. \end{array}$$

Define $\overline{\gamma} : M \to M$ given by $(p+q)\overline{\gamma} = (q)\gamma$ for any $p \in P, q \in Q$. Set $f = 1 - e$. If $(q)\big(1_Q - \alpha|_Q\gamma\big) = 0$, then $(qf)\big(f - (f\alpha)f\overline{\gamma}f\big) = 0$. As $\alpha f \in \big(f\mathcal{A}f\big)^{qnil}$, we get $qf = 0$. This implies that $1_Q - \alpha|_Q \in End(Q)$ is an R-monomorphism. For any $q \in Q$. Choose $z = (qf)\big(f - (f\alpha)f\overline{\gamma}f\big)^{-1} \in Q$. Then $(z)\big(1_Q - \alpha|_Q\gamma\big) = q$; hence, $1_Q - \alpha|_Q\gamma \in End(Q)$ is an $\mathcal{A}$-epimorphism. Thus $1_Q - \alpha|_Q\gamma \in [End(Q)]^{-1}$, and so $\alpha|_Q \in End(Q)^{qnil}$.

(2) ⇒ (1) Let $e : M = P \oplus Q \to P$ be the projection on P. In view of [16, Lemma 2], $e^2 = e \in comm(\alpha)$. Moreover, $P = Me$ and $Q = M(1-e)$. Since $\alpha|_P \in [End(P)]^{-1}$, we see that $e\alpha e \in (e\mathcal{A}e)^{-1}$. It follows from $(1-e)\alpha(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$ that $(1-e)\alpha(1-e) \in [(1-e)\mathcal{A}(1-e)]^{qnil}$. This completes the proof by Theorem 2.7. □

3. Anti-triangular matrices

In this section we apply Theorem 2.2 to block matrices over a Banach algebra and present necessary and sufficient conditions for the existence of the g-Drazin inverse for a class of 2×2 anti-triangular block matrices. We now derive

Lemma 3.1. *Let* $M = \begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *Then*

(1) *For any* $n \in \mathbb{N}$, $M^n = \begin{pmatrix} U(n) & U(n-1) \\ U(n-1)a & U(n-2)a \end{pmatrix}$, *where* $U(m) = \sum\limits_{i=0}^{[\frac{m}{2}]} \begin{pmatrix} m-i \\ i \end{pmatrix} a^i, m \geq 0; U(-1) = 0$.
(2) $U(n) - U(n-1) = U(n-2)a$ *for any* $n \in \mathbb{N}$.

Proof. See [3, Proposition 3.1]. □

Lemma 3.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) $\begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A})^d$.

Proof. (1) ⇒ (2) As 1 and a are g-Drazin invertible then we obtain the result by [8, Lemma 2.2] and [5, Corollary 2.4].

(2) ⇒ (1) Write $M^d = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then $MM^d = M^dM$, and so

$$\begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_{11} + x_{21} & x_{12} + x_{22} \\ ax_{11} & ax_{12} \end{pmatrix} = \begin{pmatrix} x_{11} + x_{12}a & x_{11} \\ x_{21} + x_{22}a & x_{21} \end{pmatrix}.$$

Hence, we have

$$x_{11} + x_{21} = x_{11} + x_{12}a,$$
$$ax_{12} = x_{21}.$$

Therefore $ax_{12} = x_{21} = x_{12}a$.

Write $(M^2M^d - M)^n = W_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} (n \in \mathbb{N})$. Since $M^{n+1}M^d - M^n = W_n$, we see that

$$\lim_{n\to\infty} \| W_n \|^{\frac{1}{n}} = 0,$$

and then

$$\lim_{n\to\infty} \| \begin{pmatrix} 0 & \beta_n \\ 0 & 0 \end{pmatrix} \|^{\frac{1}{n}} = \lim_{n\to\infty} \| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \|^{\frac{1}{n}} = 0.$$

This implies that

$$\lim_{n\to\infty} \| \beta_n \|^{\frac{1}{n}} = 0.$$

Likewise,

$$\lim_{n\to\infty} \| \delta_n \|^{\frac{1}{n}} = 0.$$

Clearly, we have

$$\begin{aligned} M^{n+1}M^d &= \begin{pmatrix} U(n+1) & U(n) \\ U(n)a & U(n-1)a \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ &= M^n + W_n \\ &= \begin{pmatrix} U(n) & U(n-1) \\ U(n-1)a & U(n-2)a \end{pmatrix} + \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}. \end{aligned}$$

Comparing two-sides of the preceding equality, we have

$$U(n+1)x_{12} + U(n)x_{22} = U(n-1) + v_0, v_0 := \beta_n \qquad (i)$$
$$U(n)ax_{12} + U(n-1)ax_{22} = U(n-2)a + v_1, v_1 := \delta_n \qquad (ii)$$

Multiplying a from the left side of (i), we get

$$U(n+1)ax_{12} + U(n)ax_{22} = U(n-1)a + a\beta_n \qquad (iii)$$

In view of Lemma 3.1, $U(n+1)-U(n) = U(n-1)a, U(n)-U(n-1) = U(n-2)a, U(n-1)-U(n-2) = U(n-3)a$. By (iii) subtracted (ii), we derive

$$U(n-1)a^2x_{12} + U(n-2)a^2x_{22} = U(n-3)a^2 + v_2, v_2 := av_0 - v_1 \qquad (iv)$$

Moreover, by (iv) subtracted (ii), we have

$$U(n-2)a^3x_{12} + U(n-3)a^3x_{22} = U(n-4)a^3 + v_3, v_3 := av_1 - v_2 \qquad (v)$$

By iteration of this process, we have

$$\begin{array}{l} U(n-(n-2))a^{n-1}x_{12} + U(n-(n-1))a^{n-1}x_{22} \\ = U(n-n)a^{n-1} + v_{n-1}; \\ v_{n-1} := av_{n-3} - v_{n-2}, \\ U(n-(n-1))a^n x_{12} + U(n-n)a^n x_{22} = U(n-(n+1))a^n + v_n, \\ v_n := av_{n-2} - v_{n-1}. \end{array}$$

That is,

$$\begin{array}{c} (1+a)a^{n-1}x_{12} + a^{n-1}x_{22} = a^{n-1} + v_{n-1}, v_{n-1} := av_{n-3} - v_{n-2}; \\ a^n x_{12} + a^n x_{22} = v_n, v_n := av_{n-2} - v_{n-1}. \end{array}$$

Therefore

$$\begin{array}{rcl} a^n & = & a^n a^{n-1} \\ & = & a[(1+a)a^{n-1}x_{12} + a^{n-1}x_{22} - v_{n-1}] \\ & = & (1+a)a^n x_{12} + a^n x_{22} - av_{n-1} \\ & = & (1+a)a^n x_{12} + (v_n - a^n x_{12}) - av_{n-1} \\ & = & a^{n+1}x_{12} + v_n - av_{n-1}. \end{array}$$

Hence,

$$a^n - a^{n+1}x_{12} = v_n - av_{n-1}.$$

By the preceding construction, we have a recurrence relations

$$v_0 = \beta_n, v_1 = \delta_n, v_n = -v_{n-1} + av_{n-2}.$$

Obviously,

$$\| v_2 \| \leq \| v_1 \| + \| a \| \| v_0 \| \leq (1+ \| a \|)^2 (\| v_0 \| + \| v_1 \|).$$

By induction, we show that

$$\begin{array}{rl} & \| v_n \| \\ \leq & \| v_{n-1} \| + \| a \| \| v_{n-2} \| \\ \leq & (1+ \| a \|)^{n-1}(\| v_0 \| + \| v_1 \|) + \| a \| \| (1+ \| a \|)^{n-2}(\| v_0 \| + \| v_1 \|) \\ = & [(1+ \| a \|)^{n-1} + \| a \| \| (1+ \| a \|)^{n-2}](\| v_0 \| + \| v_1 \|) \\ = & (1+ \| a \|)^{n-2}(1 + 2 \| a \| \|)(\| v_0 \| + \| v_1 \|) \\ \leq & (1+ \| a \|)^n (\| v_0 \| + \| v_1 \|). \end{array}$$

Likewise, we have

$$\| v_{n-1} \| \leq (1+ \| a \|)^{n-1}(\| v_0 \| + \| v_1 \|).$$

Therefore we have

$$\begin{array}{rcl} \| v_n - av_{n-1} \| & \leq & \| v_n \| + \| a \| \| v_{n-1} \| \\ & \leq & [(1+ \| a \|)^n + \| a \| (1+ \| a \|)^{n-1}](\| v_0 \| + \| v_1 \|) \\ & \leq & (1+ \| a \|)^{n+1}(\| v_0 \| + \| v_1 \|). \end{array}$$

Then we get

$$\begin{array}{rcl} \| v_n - av_{n-1} \|^{\frac{1}{n}} & \leq & (1+ \| a \|)^{\frac{n+1}{n}} (\| v_0 \| + \| v_1 \|)^{\frac{1}{n}} \\ & \leq & (1+ \| a \|)^{\frac{n+1}{n}} (\| \beta_n \| + \| \delta_n \|)^{\frac{1}{n}} \\ & \leq & (1+ \| a \|)^{1+\frac{1}{n}} (\| \beta_n \|^{\frac{1}{n}} + \| \delta_n \|^{\frac{1}{n}}). \end{array}$$

Thus,

$$\lim_{n\to\infty} \| a^n - a^{n+1}x_{12} \|^{\frac{1}{n}} = 0.$$

Since $\| (a - a^2x_{12})^n \| \leq \| a^n - a^{n+1}x_{12} \| \| 1 - ax_{12} \|^{n-1}$, we deduce that

$$\lim_{n\to\infty} \| (a - a^2x_{12})^n \|^{\frac{1}{n}} = 0.$$

Therefore $a - a^2x_{12} \in \mathcal{A}^{qnil}$. In light of Theorem 2.2, $a \in \mathcal{A}^d$, as asserted. □

We are ready to extend [18, Theorem 2.6] for the g-Drazin inverse.

Theorem 3.3. *Let* $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$*. If* $a^2 = a \in \mathcal{A}$ *and* $ab = b$*, then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.
(2) $bc \in \mathcal{A}^d$.

Proof. (1) ⇒ (2) One easily checks that

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

By using Cline's formula, $\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. Moreover, we have

$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix},$$
$$\begin{pmatrix} a & a \\ bc & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix}\begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}.$$

By using Cline's formula again, $\begin{pmatrix} a & a \\ bc & 0 \end{pmatrix}$ has g-Drazin inverse. Since

$$\begin{pmatrix} 1 & a \\ bc & 0 \end{pmatrix} = \begin{pmatrix} 1-a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & a \\ bc & 0 \end{pmatrix},$$

it follows by [8, Theorem 2.2] that $\begin{pmatrix} 1 & a \\ bc & 0 \end{pmatrix}$ has g-Drazin inverse. Let $S = \begin{pmatrix} 1 & 1 \\ bc & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. Then

$$ST = \begin{pmatrix} 1 & a \\ bc & 0 \end{pmatrix}, TS = \begin{pmatrix} 1 & 1 \\ bc & 0 \end{pmatrix}.$$

In view of Cline's formula, $\begin{pmatrix} 1 & 1 \\ bc & 0 \end{pmatrix}$ has g-Drazin inverse. In light of Lemma 3.2, $bc \in \mathcal{A}^d$, as asserted.

(2) ⇒ (1) Since $bc = abc \in \mathcal{A}^d$, it follows by Cline's formula that bca has g-Drazin inverse. In light of Lemma 3.2, $\begin{pmatrix} 1 & 1 \\ bca & 0 \end{pmatrix}$ has g-Drazin inverse. As

$$\begin{pmatrix} 1 & 1 \\ bca & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ bca & 0 \end{pmatrix},$$

it follows by [11, Theorem 5.5] that $\begin{pmatrix} a & a \\ bca & 0 \end{pmatrix}$ has g-Drazin inverse. Since

$$\begin{pmatrix} a & a \\ bc & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ bc(1-a) & 0 \end{pmatrix} + \begin{pmatrix} a & a \\ bca & 0 \end{pmatrix},$$

it follows by [8, Theorem 2.2] that $\begin{pmatrix} a & a \\ bc & 0 \end{pmatrix}$ has g-Drazin inverse. We easily check that

$$\begin{pmatrix} a & a \\ bc & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix}\begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix}.$$

In view of Cline's formula, $\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. Furthermore, we have

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

By using Cline's formula again, we conclude that M has g-Drazin inverse. □

Corollary 3.4. *Let* $M = \begin{pmatrix} a & a \\ b & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a^2 = a \in \mathcal{A}$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.
(2) $ab \in \mathcal{A}^d$.

Proof. This is obvious by Theorem 3.3. □

Lemma 3.5. *Let* $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a \in \mathcal{A}^d$, $caa^d = c$ *and* $a^d bc = bca^d$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.
(2) $bc \in \mathcal{A}^d$.

Proof. (2) ⇒ (1) Since $a^d bc = bca^d$, it follows by [11, Theorem 5.5] that $(a^d)^2 bc \in \mathcal{A}^d$. In view of Lemma 3.2,

$$\begin{pmatrix} 1 & 1 \\ (a^d)^2 bc & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$

We easily check that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ (a^d)^2 bc & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ (a^d)^2 bc & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

we see that
$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ (a^d)^2bc & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
This shows that
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ ca^d & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
By using Cline's formula,
$$M = \begin{pmatrix} 1 & 1 \\ ca^d & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})^d.$$

(1) ⇒ (2) Since M has g-Drazin inverse, we prove that
$$\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})^d.$$
By Cline's formula,
$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
That is,
$$\begin{pmatrix} a & 1 \\ bc & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
Since $a^d(bc) = (bc)a^d$, by virtue of [19, Theorem 3.1], we have
$$\begin{pmatrix} a^da & a^d \\ a^dbc & 0 \end{pmatrix} = \begin{pmatrix} a^d & 0 \\ 0 & a^d \end{pmatrix}\begin{pmatrix} a & 1 \\ bc & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
By using Cline's formula,
$$\begin{pmatrix} a^da & aa^d \\ (a^d)^2bc & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^d \end{pmatrix}\begin{pmatrix} a^da & a^d \\ a^dbc & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in M_2(\mathcal{A})^d.$$
One easily checks that
$$\begin{pmatrix} 1 & 1 \\ (a^d)^2bc & 0 \end{pmatrix} = \begin{pmatrix} a^\pi & a^\pi \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} aa^d & aa^d \\ (a^d)^2bc & 0 \end{pmatrix}.$$
Hence,
$$\begin{pmatrix} 1 & 1 \\ (a^d)^2bc & 0 \end{pmatrix} \in M_2(\mathcal{A})^d.$$
In light of Lemma 3.2, $(a^d)^2bc \in \mathcal{A}^d$. Since $a(a^d)^2bc = (a^d)^2bca$, we see that $a^2(a^d)^2bc = (a^d)^2bca^2$. In view of [19, Theorem 3.1],
$$bc = bc(a^d)^2a^2 = (a^d)^2bca^2 \in \mathcal{A}^d,$$
as asserted. □

The following result is a generalization of [3, Theorem 4.1] for the g-Drazin inverse.

Theorem 3.6. *Let* $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a \in \mathcal{A}^d$, $bca^\pi = 0$ *and* $a^dbc = bca^d$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.
(2) $bc \in \mathcal{A}^d$.

Proof. (2) ⇒ (1) Let $c' = caa^d$. Since $bca^\pi = 0$, we have $bc = bcaa^d$. We see that

$$M = P + Q, P = \begin{pmatrix} a & b \\ c' & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ ca^\pi & 0 \end{pmatrix}.$$

Clearly, $PQ = 0$ and $Q^2 = 0$. Since $c'a^\pi = 0, a^d bc' = bc'a^d$ and $bc' = bc \in \mathcal{A}^d$, it follows by Lemma 3.5 that P has g-Drazin inverse. In light of [8, Theorem 2.2], M has g-Drazin inverse, as required.

(1) ⇒ (2) One easily checks that

$$\begin{pmatrix} a & b \\ c' & 0 \end{pmatrix} = M + N, N = \begin{pmatrix} 0 & 0 \\ ca^\pi & 0 \end{pmatrix}.$$

Clearly, $MN = 0$ and $N^2 = 0$. In view of [8, Theorem 2.2], $\begin{pmatrix} a & b \\ c' & 0 \end{pmatrix}$ has g-Drazin inverse. Moreover, $c'a^\pi = 0, a^d bc' = bc'a^d$ and $bc' = bc \in \mathcal{A}^d$. According to Lemma 3.5, $bc = bc'$ has g-Drazin inverse, as asserted. □

Corollary 3.7. *Let* $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M_2(\mathcal{A})$. *If* $a \in \mathcal{A}^d, a^\pi bc = 0$ *and* $abc = bca$, *then the following are equivalent:*

(1) $M \in M_2(\mathcal{A})^d$.
(2) $bc \in \mathcal{A}^d$.

Proof. Since $a(bc) = (bc)a$ and a has g-Drazin inverse, by [11, Theorem 4.4], $a^d(bc) = (bc)a^d$, and so $0 = a^\pi bc = (1 - aa^d)bc = bc(1 - aa^d) = bca^\pi$. The corollary is therefore established by Theorem 3.6. □

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