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Generalized fractional integrals in the vanishing generalized weighted local and global Morrey spaces

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Abstract. In this paper, we prove the boundedness of generalized fractional integral operators I_ρ in the vanishing generalized weighted Morrey-type spaces, such as vanishing generalized weighted local Morrey spaces and vanishing generalized weighted global Morrey spaces by using weighted L_p estimates over balls.

In more detail, we obtain the Spanne-type boundedness of the generalized fractional integral operators I_ρ in the vanishing generalized weighted local Morrey spaces with $w^q \in A_{1+\frac{q}{p'}}$ for $1 < p < q < \infty$, and from the vanishing generalized weighted local Morrey spaces to the vanishing generalized weighted weak local Morrey spaces with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. We also prove the Adams-type boundedness of the generalized fractional integral operators I_ρ in the vanishing generalized weighted global Morrey spaces with $w \in A_{p,q}$ for $1 < p < q < \infty$ and from the vanishing generalized weighted global Morrey spaces to the vanishing generalized weighted weak global Morrey spaces with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. The our all weight functions belong to Muckenhoupt-Weeden classes $A_{p,q}$.

1. Introduction

The classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ defined by Morrey in [25] to study the local behavior of solutions to second order elliptic PDEs. Morrey spaces have important applications to potential theory, function spaces and applied mathematics, for instance see the papers [1, 23, 34].

The boundedness of some important classical operators on the weighted Lebesgue spaces $L_p(\mathbb{R}^n, w)$ were obtained by Muckenhoupt [27], Mukenhoupt and Wheeden [26], and Coifman and Fefferman [5].

Weighted Morrey spaces $L_{p,\kappa}(\mathbb{R}^n, w)$ were defined by Komori and Shirai in [17]. They studied the boundedness of the classical operators of harmonic analysis such as Hardy-Littlewood maximal operator, Calderon-Zygmund operator, fractional integral operator in these spaces. These results were extended to several other spaces (see [13, 20] for examples). Weighted inequalities for fractional operators have good applications to potential theory and quantum mechanics.

Firstly, Vitanza in [37] defined the vanishing Morrey space $\mathcal{VM}_{p,\lambda}(\mathbb{R}^n)$ of the classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ and applied in this study to get a regularity result for elliptic PDEs. Later in [38], Vitanza proved an existence theorem for a Dirichlet problem, under weaker conditions than those introduced by Miranda

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in [24], and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa [31] obtained a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $\mathcal{VM}_{p,\lambda}(\mathbb{R}^n)$. A deep research on commutator operators in vanishing Morrey spaces can be seen in [30].

The vanishing generalized global Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n)$ and vanishing generalized local Morrey space $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ were introduced by Samko in [35, 36]. The boundedness of the multi-dimensional Hardy type operators, maximal, potential and singular operators in these spaces were proved in [35, 36]. Guliyev et al. proved the commutators of Riesz potential in the vanishing generalized weighted Morrey spaces with variable exponent in [15].

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The generalized fractional integral operator I_ρ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy,$$

where $\rho : (0,\infty) \to (0,\infty)$ is a positive and measurable function. If $\rho(t) \equiv t^\alpha$, then $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential operator.

The generalized fractional integral operator I_ρ was initilally investigated in [7, 16, 28]. Nakai [28] introduced the the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ and proved the boundedness of the generalized fractional integral operator I_ρ in these spaces. Recently, many authors have been culminating important observations about the operator I_ρ especially in connection with Morrey-type spaces (see [6, 9, 14, 19–21, 32, 33]). But, the boundedness of generalized fractional integral operators I_ρ in the vanishing generalized weighted Morrey-type spaces, such as vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$ and vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ have not been studied, yet.

Guliyev [12] proved the Spanne and Adams types boundedness of Riesz potential operator I_α from the spaces $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ without any assumption on monotonicity of φ_1, φ_2.

In this present paper, by using the method given by Guliyev in [12], we obtain the Spanne-type boundedness of the generalized fractional integral operators I_ρ from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n, w^p)$ to another one $\mathcal{VM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w^q \in A_{1+\frac{q}{p'}}$ for $1 < p < q < \infty$, and from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{1,\varphi_1}$ to the vanishing generalized weighted weak local Morrey spaces $\mathcal{VWM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. We also prove the Adams-type boundedness of the generalized fractional integral operators I_ρ from the vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to $\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{p,q}$ for $1 < p < q < \infty$ and from the vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{1,\varphi}(\mathbb{R}^n, w)$ to the vanishing generalized weighted weak global Morrey spaces $\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. The all weight functions belong to Muckenhoupt-Weeden class $A_{p,q}$.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C.

2. Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r) \subset \mathbb{R}^n$ the open ball centered at x of radius r. Let $|B(x,r)|$ be the Lebesgue measure of ball $B(x,r)$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, in the special case of $w \equiv 1$ we get $w(E) = |E|$. The characteristic function of E by χ_E. If w is a weight function, for all $f \in L_1^{loc}(\mathbb{R}^n)$ and $1 \le p < \infty$ we denote by

$L_p^{loc}(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n,w)} = \left(\int_{B(x,r)} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

We recall that a weight function w belongs to the Muckenhoupt-Wheeden classes $A_{p,q}$ (see [26]) for $1 < p < q < \infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B w(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq C$$

and, if $p = 1$, w is in the $A_{1,q}$ with $1 < q < \infty$ then

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}} \left(ess\sup_{x\in B} \frac{1}{w(x)}\right) \leq C,$$

where $C > 0$ and the supremum is taken with respect to all balls B.

Lemma 2.1. *[8, 10] If $w \in A_{p,q}$ with $1 < p < q < \infty$, then the following statements are true.*
(i) $w^q \in A_r$ with $r = 1 + \frac{q}{p'}$.
(ii) $w^{-p'} \in A_{r'}$ with $r' = 1 + \frac{p}{q'}$.
(iii) $w^p \in A_s$ with $s = 1 + \frac{p}{q'}$.
(iv) $w^{-q'} \in A_{s'}$ with $s' = 1 + \frac{q'}{p}$.

For convenience, we use the following definition of generalized weighted global Morrey spaces.

Definition 2.2. *([4]). Let $1 \leq p < \infty$, w be a weight function on $\mathbb{R}^n$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We denote by $M_{p,\varphi}(\mathbb{R}^n, w)$ the generalized weighted global Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n, w)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)}.$$

Also by $WM_{p,\varphi}(\mathbb{R}^n, w)$ we denote the generalized weighted weak global Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n, w)$ for which

$$\|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)},$$

where $WL_p(B(x,r), w)$ denotes the weighted weak L_p space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r),w)} = \sup_{t>0} \left(\int_{\{y\in B(x,r):|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Definition 2.3. *([4]). Let $1 \leq p < \infty$, w be a weight function on $\mathbb{R}^n$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ the generalized weighted local Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n, w)$ with finite norm*

$$\|f\|_{M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n,w)} = \|f(x_0+\cdot)\|_{M_{p,\varphi}(\mathbb{R}^n,w)}.$$

Also by $WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ we denote the weak generalized weighted local Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n, w)$ for which

$$\|f\|_{WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n,w)} = \|f(x_0+\cdot)\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} < \infty.$$

Since the generalized weighted local Morrey space $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ is an expansion of the generalized weighted global Morrey space $M_{p,\varphi}(\mathbb{R}^n, w)$ then we have the following embeddings between in these spaces:

$$M_{p,\varphi}(\mathbb{R}^n) \subset M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n), \ \ \|f\|_{M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)} \le \|f\|_{M_{p,\varphi}(\mathbb{R}^n)},$$

$$WM_{p,\varphi}(\mathbb{R}^n) \subset WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n), \ \ \|f\|_{WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)} \le \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)}.$$

Definition 2.4. *([35]). Let $1 \le p < \infty$, w be a weight function on $\mathbb{R}^n$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. The vanishing generalized weighted global Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ is defined as the space of functions $f \in M_{p,\varphi}(\mathbb{R}^n, w)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0.$$

The vanishing generalized weighted weak global Morrey space $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n, w)$ is defined as the space of functions $f \in WM_{p,\varphi}(\mathbb{R}^n, w)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{WL_p(B(x,r),w)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \quad \text{and} \quad \sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{2.1}$$

which makes the spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ and $VWM_{p,\varphi}(\mathbb{R}^n, w)$ non-trivial, because bounded functions with compact support belong to this space. If the function φ satisfies the assumptions in (2.1) then we say that φ belongs to the class $\mathfrak{M}_{\text{glob}}$.

The spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ and $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n, w)$ are Banach spaces with respect to the norm

$$\|f\|_{\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)},$$

$$\begin{aligned} \|f\|_{\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)} &\equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)}, \end{aligned}$$

respectively.

Extending the definition of vanishing generalized weighted global Morrey spaces to the case of weighted local Morrey spaces, we introduce the following definition.

Definition 2.5. *Let $1 \le p < \infty$, w be a weight function on $\mathbb{R}^n$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. For any fixed $x_0 \in \mathbb{R}^n$, the vanishing generalized weighted local Morrey space $\mathcal{VM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ and its weak version $\mathcal{VWM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ are defined as the spaces of functions $f \in M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ and $f \in WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ such that*

$$\lim_{r\to 0} \frac{w(B(x_0,r))^{-\frac{1}{p}}}{\varphi(x_0,r)} \|f\|_{L_p(B(x_0,r),w)} = 0,$$

$$\lim_{r\to 0} \frac{w(B(x_0,r))^{-\frac{1}{p}}}{\varphi(x_0,r)} \|f\|_{WL_p(B(x_0,r),w)} = 0,$$

respectively.

Theorem 2.6. *(Spanne, but published by Peetre, [29]). Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the Riesz potential operator I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$.*

In particular, the following statement containing Theorem 2.6.

Theorem 2.7. *([2, 3]) Let $1 \leq p < q < \infty$, $0 < \lambda, \mu < n$ and $0 < \alpha = \frac{n-\lambda}{p} - \frac{n-\mu}{q} < \frac{n}{p}$. Then, for $p > 1$, the operator I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, and, for $p = 1$, I_α is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$.*

The following theorem which is the Spanne-type results for the boundedness of the operator I_ρ on the generalized local Morrey spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$.

Theorem 2.8. *(Spanne-type result, [14]). Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$, the function ρ satisfy the conditions* (3.1)-(3.2) *and* (3.3)*. Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \leq C\, \varphi_2\Big(x_0, \frac{t}{2}\Big)\, t^{\frac{n}{q}},$$

$$\int_r^\infty \Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}\Big) \frac{\rho(t)}{t^{\frac{n}{p}}} \frac{dt}{t} \leq C\, \varphi_2(x_0, r),$$

where C does not depend on x_0 and r. Then the operator I_ρ is bounded from $M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)$ to $M^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M^{\{x_0\}}_{1,\varphi_1}(\mathbb{R}^n)$ to $WM^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)$ for $p = 1$. Moreover, for $p > 1$

$$\|I_\rho f\|_{M^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)} \leq C \|f\|_{M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)},$$

and for $p = 1$

$$\|I_\rho f\|_{WM^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)} \leq C \|f\|_{M^{\{x_0\}}_{1,\varphi_1}(\mathbb{R}^n)}.$$

The followings are sufficient conditions for the non-triviality of the spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $\mathcal{VWM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$:

$$\lim_{r\to 0} \frac{1}{\varphi(x_0, r)} = 0 \quad \text{and} \quad \sup_{r>0} \frac{1}{\varphi(x_0, r)} < \infty, \tag{2.2}$$

since bounded functions with compact support belong to these spaces, (see [36]).

Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. If the function φ satisfies the assumptions in (2.2) then we say that φ belongs to the class $\mathfrak{M}_{\text{loc}}$.

Under the suitable conditions, the spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $VWM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ are closed subspaces of the Banach spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $WM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$, respectively, which may be shown by standard means.

We will also use the following notation

$$\mathfrak{A}_{p,\varphi,w}(f; x_0, r) := \varphi(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r),w)}$$

and

$$\mathfrak{A}^W_{p,\varphi,w}(f; x_0, r) := \varphi(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r),w)}$$

for brevity, so that

$$\mathcal{VM}^{\{x_0\}}_{p,\varphi,w}(\mathbb{R}^n) = \Big\{ f \in M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w) : \lim_{r\to 0} \mathfrak{A}_{p,\varphi,w}(f; x_0, r) = 0 \Big\}$$

and similarly we will use for the space $\mathcal{VWM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$.

3. Spanne-type result for the operators I_ρ on the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$

In this section, we show the Spanne-type boundedness of the generalized fractional integral operators I_ρ in the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$.

In the following theorem Spanne studied boundedness of the Riesz potential operator I_α in the Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$.

In order to achieve our purpose, we assume that

$$\int_1^\infty \frac{\rho(t)}{t^n}\frac{dt}{t} < \infty, \tag{3.1}$$

so that the generalized fractional integrals $I_\rho f$ are well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n\setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we will assume that ρ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r<s\le 2r} \frac{\rho(s)}{s^n} \le C \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t^n}\frac{dt}{t}, \quad r > 0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C}\frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C\frac{\rho(t)}{t^n},$$

whenever r and t satisfy $r, t > 0$ and $\frac{1}{2} \le \frac{r}{t} \le 2$.

The following two lemmas are our basic tools to prove our main results.

Lemma 3.1. *([21]). Let $1 \le p < q < \infty, w^q \in A_{1+\frac{q}{p'}}$, the function ρ satisfies the conditions* (3.1)- (3.2)*, and $f \in L_1^{loc}(\mathbb{R}^n, w)$.*

(i) If $1 < p < q < \infty$ then there exist $C > 0$ for all $r > 0$ such that the inequality

$$\rho(r) \le C r^{\frac{n}{p}-\frac{n}{q}} \tag{3.3}$$

is sufficient condition for the boundedness of generalized fractional integral operator I_ρ from $L_p(\mathbb{R}^n, w^p)$ to $L_q(\mathbb{R}^n, w^q)$.

(ii) If $p = 1, 1 < q < \infty$ then there exist $C > 0$ for all $r > 0$ such that the inequality

$$\rho(r) \le C r^{n-\frac{n}{q}} \tag{3.4}$$

is sufficient condition for the boundedness of generalized fractional integral operator I_ρ from $L_1(\mathbb{R}^n, w)$ to $WL_q(\mathbb{R}^n, w^q)$, where the constant C does not depend on f.

The following lemma is strong and weak weighted local L_p-estimates for the operator I_ρ.

Lemma 3.2. *([22]). Let fixed $x_0 \in \mathbb{R}^n$, and $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$ and $\rho(t)$ satisfy the conditions* (3.1) *and* (3.2)*.*

(i) If $1 < p < q < \infty$ and the condition (3.3) *is fulfill, then the inequality*

$$\|I_\rho f\chi_{B(x_0,r)}\|_{L_q(\mathbb{R}^n,w^q)} \lesssim \|f\chi_{B(x_0,2r)}\|_{L_p(\mathbb{R}^n,w^p)} + (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^\infty \|f\chi_{B(x_0,t)}\|_{L_p(\mathbb{R}^n,w^p)} (w^q(B(x_0,t)))^{-\frac{1}{q}} \frac{\rho(t)}{t^n}\frac{dt}{t} \tag{3.5}$$

holds for the ball $B(x_0, r)$ *and for all* $f \in L_p^{loc}(\mathbb{R}^n, w^p)$ *and,*

(*ii*) *if* $p = 1, 1 < q < \infty$ *and the condition* (3.4) *is fulfill, then the inequality*

$$\|I_\rho f \chi_{B(x_0,r)}\|_{WL_q(\mathbb{R}^n,w^q)} \lesssim \|f\chi_{B(x_0,2r)}\|_{L_1(\mathbb{R}^n,w)} + (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\chi_{B(x_0,t)}\|_{L_1(\mathbb{R}^n,w)} (w^q(B(x_0,t)))^{-\frac{1}{q}} \frac{\rho(t)}{t^n}\frac{dt}{t} \tag{3.6}$$

hold for the ball $B(x_0, r)$ *and for all* $f \in L_1^{loc}(\mathbb{R}^n, w)$.

The following theorem which is an extension theorem of Theorem 2.8 containing Theorem 2.6 and Theorem 2.7, is one of our main results in which we generalize the Spanne-type boundedness of the operator I_ρ in vanishing generalized weighted local Morrey spaces $\mathcal{VM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w^p)$.

Theorem 3.3. *Let* $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$, $\varphi_1, \varphi_2, \in \mathfrak{M}_{\rm loc}$ *and the function* ρ *satisfy the conditions* (3.1), (3.2), (3.3) *and* (3.4). *Let also* φ_1, φ_2 *satisfy the conditions*

$$\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(x_0, s)\,(w^p(B(x_0,s)))^{\frac{1}{p}} \le C\,\varphi_2\Big(x_0, \frac{r}{2}\Big)\,(w^q(B(x_0,r)))^{\frac{1}{q}}, \tag{3.7}$$

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)(w^p(B(x_0,s)))^{\frac{1}{p}}\rho(t)}{(w^q(B(x_0,t)))^{\frac{1}{q}}\, t^{\frac{n}{p}}}\frac{dt}{t} \le C\,\varphi_2(x_0,r), \tag{3.8}$$

where C *does not depend on* x_0 *and* r. *Then the operator* I_ρ *is bounded from vanishing generalized weighted local Morrey spaces* $\mathcal{VM}_{p,\varphi_1}^{\{x_0\}}(\mathbb{R}^n, w^p)$ *to* $\mathcal{VM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n, w^q)$ *for* $p > 1$ *and from the space* $\mathcal{VM}_{1,\varphi_1}^{\{x_0\}}(\mathbb{R}^n, w)$ *to the weak space* $\mathcal{VWM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n, w^q)$ *for* $p = 1$. *Additionally the following norm inequalities, for* $p > 1$

$$\|I_\rho f\|_{\mathcal{VM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n,w^q)} \lesssim \|f\|_{\mathcal{VM}_{p,\varphi_1}^{\{x_0\}}(\mathbb{R}^n,w^p)},$$

and for $p = 1$

$$\|I_\rho f\|_{\mathcal{VWM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n,w^q)} \lesssim \|f\|_{\mathcal{VM}_{1,\varphi_1}^{\{x_0\}}(\mathbb{R}^n,w)}$$

hold.

Proof. Since the norm inequalities are provided in the Theorem 2.8, then we only have to prove the undermentioneds:

$$\lim_{r\to 0} \mathfrak{A}_{p,\varphi_1,w^p}(f; x_0, r) = 0 \implies \lim_{r\to 0} \mathfrak{A}_{q,\varphi_2,w^q}(M_\rho f; x_0, r) = 0, \tag{3.9}$$

and

$$\lim_{r\to 0} \mathfrak{A}_{1,\varphi_1,w}^{W}(f; x_0, r) = 0 \implies \lim_{r\to 0} \mathfrak{A}_{q,\varphi_2,w^q}^{W}(M_\rho f; x_0, r) = 0, \tag{3.10}$$

To control (3.9), i.e., to prove that

$$\frac{(w^q(B(x_0,r)))^{-\frac{1}{q}}\, \|I_\rho f\|_{L^q(B(x_0,r),w^q)}}{\varphi_2(x_0,r)} < \varepsilon \quad \text{for infinitesimal } r,$$

we use the inequality (4.1) where we split the right-hand side:

$$\frac{(w^q(B(x_0,r)))^{-\frac{1}{q}}\, \|I_\rho f\|_{L^q(B(x_0,r),w^q)}}{\varphi_2(x_0,r)} \lesssim I(x_0,r) + J_{\delta_0}(x_0,r) + K_{\delta_0}(x_0,r), \tag{3.11}$$

with $\delta_0 > 0$ and $r < \delta_0$, where

$$I(x_0,r) := \frac{(w^q(B(x_0,r)))^{-\frac{1}{q}}\,\|f\|_{L^p(B(x_0,2r),w^p)}}{\varphi_2(x_0,r)},$$

$$J_{\delta_0}(x_0,r) := \frac{1}{\varphi_2(x_0,r)}\left(\sup_{r<t<\delta_0}\|f\|_{L_p(B(x_0,t),w^p)}\,\frac{\rho(t)}{(w^p(B(x_0,t)))^{\frac{1}{p}}}\right)$$

and

$$K_{\delta_0}(x_0,r) := \frac{1}{\varphi_2(x,r)}\left(\sup_{t>\delta_0}\|f\|_{L_p(B(x_0,t),w^p)}\,\frac{\rho(t)}{(w^p(B(x_0,t)))^{\frac{1}{p}}}\right).$$

For the first expression from (3.15) we have

$$I(x_0,r) \lesssim \frac{r^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,r))}}{\varphi_1(x_0,r)}.$$

By conjecture we get $H(x_0,r) < \frac{\varepsilon}{3}$ for infinitesimal r.

We use the fact that $f \in \mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)$ and choose any fixed $\delta_0 > 0$, in order to guarantee its finite in the limiting case, such that

$$\frac{t^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,t))}}{\varphi_1(x_0,t)} < \frac{\varepsilon}{3C}, \quad t \le \delta_0,$$

where C is constant from (3.11) and (3.16), which satisfies the calculation of the second expression uniform in $r \in (0,\delta_0)$:

$$J_{\delta_0}(x_0,r) < \frac{\varepsilon}{3C}, \;\; 0 < r < \delta_0.$$

For the third expression, we have

$$K_{\delta_0}(x_0,r) \le C_{\delta_0}\frac{\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)}}{\varphi_2(x_0,r)},$$

where

$$C_{\delta_0} = \sup_{t>\delta_0}\varphi_1(x_0,t)\rho(t).$$

Let's point out $C_{\delta_0} < \infty$ follows from (3.16). Then, by (2.2) we choose infinitesimal r such that

$$\frac{1}{\varphi_2(x_0,r)} \le \frac{\varepsilon}{3C_{\delta_0}\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)}},$$

which completes the estimation of the third expression and the proof. The proof of (3.10) is, step by step, the same as in the proof of (3.9) by using (4.2). □

In the Theorem 3.3, in the special case of the weight function for $w \equiv 1$ we get the following which was proved in ([18], Theorem 3.4, p. 284).

Corollary 3.4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $\varphi_1,\varphi_2 \in \mathfrak{M}_{\rm loc}$ and the function ρ satisfy the conditions* (3.1)-(3.4). *Let also φ_1,φ_2 satisfy the conditions*

$$\varphi_1(x_0,r)r^{\frac{n}{p}} \le C\,\varphi_2\Big(x_0,\frac{r}{2}\Big)\,r^{\frac{n}{q}}, \tag{3.12}$$

$$\int_r^\infty \varphi_1(x_0,t)\rho(t)\frac{dt}{t} \le C\,\varphi_2(x_0,r), \tag{3.13}$$

where C does not depend on x_0 and r. Then the operator I_ρ is bounded from vanishing generalized local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{VM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from the vanishing space $\mathcal{VM}^{\{x_0\}}_{1,\varphi_1}(\mathbb{R}^n)$ to the vanishing weak space $\mathcal{VWM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n)$ for $p = 1$.

Also, from the Theorem 3.3 for $w \equiv 1$, if the constant c_δ exists as follows then we get the following.

Corollary 3.5. *Let $1 \le p < q < \infty$, $\varphi \in \mathfrak{M}_{\mathrm{glob}}$ and the function ρ satisfy the conditions* (3.1)-(3.4)*. Let also φ_1, φ_2 satisfy the conditions for every $\delta > 0$*

$$c_\delta = \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\rho(t)\frac{dt}{t} < \infty, \tag{3.14}$$

and

$$\varphi_1(x,r)r^{\frac{n}{p}} \le C\,\varphi_2\Big(x,\frac{r}{2}\Big)\,r^{\frac{n}{q}}, \tag{3.15}$$

$$\int_r^\infty \varphi_1(x,t)\rho(t)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{3.16}$$

where C does not depend on x and r. Then the operator I_ρ is bounded from vanishing generalized global Morrey spaces $\mathcal{VM}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{VM}_{q,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from the vanishing space $\mathcal{VM}_{1,\varphi_1}(\mathbb{R}^n)$ to the vanishing weak space $\mathcal{VWM}_{q,\varphi_2}(\mathbb{R}^n)$ for $p = 1$.

4. Adams-type result for the operators I_ρ on the vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$

It is well-known that for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function Mf of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n.$$

The following lemma is weighted local strong and weak L_p-estimates for the operator I_ρ which is our main tool to prove our main results.

Lemma 4.1. *[22] Let $1 \le p < q < \infty$, $w \in A_{p,q}$ and $\rho(t)$ satisfy the conditions* (3.1)-(3.2)*.*
(i) If the condition (3.3) *is fulfill, then the inequality*

$$\begin{aligned}\|I_\rho f\chi_{B(x,r)}\|_{L_q(\mathbb{R}^n,w)} &\lesssim \|f\chi_{B(x,2r)}\|_{L_p(\mathbb{R}^n,w)} \\ &+ (w(B(x,r)))^{\frac{1}{q}} \int_{2r}^\infty \|f\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,w)}\,(w(B(x,r)))^{-\frac{1}{q}}\,\frac{\rho(t)}{t^{n+1}}dt\end{aligned} \tag{4.1}$$

holds for the ball $B(x,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n, w)$.
(ii) If the condition (3.3) *is fulfill, then for $p = 1$ the inequality*

$$\begin{aligned}\|I_\rho f\chi_{B(x,r)}\|_{WL_q(\mathbb{R}^n,w)} &\lesssim \|f\chi_{B(x,2r)}\|_{L_1(\mathbb{R}^n,w)} \\ &+ (w(B(x,r)))^{\frac{1}{q}} \int_{2r}^\infty \|f\chi_{B(x,t)}\|_{L_1(\mathbb{R}^n,w)}\,(w(B(x,t)))^{-\frac{1}{q}}\,\frac{\rho(t)}{t^{n+1}}dt\end{aligned} \tag{4.2}$$

holds for the ball $B(x,r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n, w)$.

The following is an Adams-type result for generalized fractional integral operator I_ρ in generalized Morrey spaces.

Theorem 4.2. *(Adams-type result, [14]). Let $1 \le p < \infty$, $q > p$, $\rho(t)$ satisfy the conditions* (3.1)-(3.4). *Let also $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} \varphi(x,t) \le C\,\varphi(x,r), \tag{4.3}$$

and

$$\int_r^\infty \varphi(x,t)^{\frac{1}{p}} \frac{\rho(t)}{t} dt \le C\rho(r)^{-\frac{p}{q-p}}, \tag{4.4}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator I_ρ is bounded from generalized Morrey spaces $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n)$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n)$ for $p > 1$ and from the space $M_{1,\varphi}(\mathbb{R}^n)$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n)$ for $p = 1$.

In Theorem 4.2, if we take $\rho(t) = t^\alpha$, then we get Adams type result on generalized Morrey spaces proved in [11] (Theorem 5.7, p. 182) and if we take $\rho(t) = t^\alpha$ and $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$, then we get Adams's result in [1].

The following theorem is the second main result of our paper in which we prove the Adams-type boundedness of the operator I_ρ in vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$.

Let $1 \le p < q < \infty$, $\varphi \in \mathfrak{M}_{\text{glob}}$, $\rho(t)$ satisfy the conditions (3.1)-(3.4). Let also $\varphi(x,t)$ satisfy the conditions

$$\sup_{r<t<\infty} \varphi(x,t) \le C\,\varphi(x,r), \tag{4.5}$$

$$m_\delta = \sup_{\delta<t<\infty} \sup_{x\in\mathbb{R}^n} \varphi(x,t) < \infty, \tag{4.6}$$

and

$$\int_r^\infty \varphi(x,t)^{\frac{1}{p}} \frac{\rho(t)}{t} dt \le C\rho(r)^{-\frac{p}{q-p}}, \tag{4.7}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator I_ρ is bounded from vanishing generalized weighted global Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to $\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ for $p > 1$ and from the vanishing space $\mathcal{VM}_{1,\varphi}(\mathbb{R}^n, w)$ to the vanishing weak space $\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ for $p = 1$. Additionally the following norm inequalities, for $p > 1$

$$\|I_\rho f\|_{\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)} \lesssim \|f\|_{\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n,w)},$$

and for $p = 1$

$$\|I_\rho f\|_{\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)} \lesssim \|f\|_{\mathcal{VM}_{1,\varphi}(\mathbb{R}^n,w)}$$

hold.

Proof. Since the norm inequalities are provided in the Theorem 2.8, then we only have to prove the undermentioneds:

$$\text{If } \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi^{1/p},w}(f;x,r) = 0, \text{ then } \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{q,\varphi^{1/q},w}(I_\rho f;x,r) = 0, \tag{4.8}$$

and

$$\text{if } \lim_{r\to 0}\mathfrak{A}^{W}_{1,\varphi,w}(f;x,r)=0, \text{ then } \lim_{r\to 0}\mathfrak{A}^{W}_{q,\varphi^{1/q},w}(I_\rho f;x,r)=0. \tag{4.9}$$

Under the conditions (3.2), (4.5) and (4.7) we know that (see [14]) for all $x\in\mathbb{R}^n$

$$|I_\rho f(x)| \le C(Mf(x))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}}. \tag{4.10}$$

To test (4.8), i.e. to prove that

$$\sup_{x\in\mathbb{R}^n}\frac{w(B(x,r))^{-\frac{1}{q}}\|I_\rho f\|_{L^q(B(x,r))}}{\varphi(x,r)^{1/q}} < \varepsilon \quad \text{for infinitesimal } r,$$

we use the expressions (4.1) and (4.10) where we split the right-hand side:

$$\frac{w(B(x,r))^{-\frac{1}{q}}\|I_\rho f\|_{L^q(B(x,r),w)}}{\varphi(x,r)^{1/q}} \le C\left(J_{\delta_0}(x,r)+K_{\delta_0}(x,r)\right), \tag{4.11}$$

with $\delta_0>0$ and $r<\delta_0$, where

$$J_{\delta_0}(x,r) := \frac{1}{\varphi(x,r)^{1/q}}\sup_{r<t<\delta_0} t^{-\frac{n}{q}}\|f\|^{p/q}_{L_p(B(x,t),w)}$$

and

$$K_{\delta_0}(x,r) := \frac{1}{\varphi(x,r)^{1/q}}\sup_{t>\delta_0} w(B(x,t))^{-\frac{1}{q}}\|f\|^{p/q}_{L_p(B(x,t),w)}.$$

We use the fact that $f\in VM_{p,\varphi^{1/p}}(\mathbb{R}^n,w)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{w(B(x,t))^{-\frac{1}{q}}\|f\|_{L^p(B(x,t),w)}}{\varphi(x,t)^{1/p}} < \left(\frac{\varepsilon}{2C^{p/q^2}}\right)^{q/p}, \quad t\le\delta_0,$$

where C is constants from (4.5) and (4.11), which satisfies the estimate of the second expression uniform in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CJ_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0<r<\delta_0.$$

For the second term, we have

$$K_{\delta_0}(x,r) \le \frac{m_{\delta_0}^{1/q}\|f\|^{p/q}_{M_{p,\varphi^{1/p}}(\mathbb{R}^n,w)}}{\varphi(x,r)^{1/q}},$$

where m_{δ_0} is the constant from (4.5) with $\delta=\delta_0$. Then, by (2.1) we choose small r such that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi(x,r)} \le \left(\frac{\varepsilon}{2m_{\delta_0}^{1/q}\|f\|^{p/q}_{M_{p,\varphi^{1/p}}(\mathbb{R}^n,w)}}\right)^q,$$

which completes the estimation of the second expression and the proof. The proof of (4.9) is, step by step, the same as in the proof of (4.8). □

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