Filomat 37:6 (2023), 1751–1765
https://doi.org/10.2298/FIL2306751Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Drazin invertibility for sum and product of two elements in a ring

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Abstract. In a ring, the expressions for the Drazin inverses of the sum $a + b$ and the product ab have been studied in some literature under the assumption that the two Drazin invertible elements a, b are commutative. In this paper, we will extend the known research results under the weaker conditions. Meanwhile, we characterize the relations of $a + b$, $(a + b)bb^D$, $\mathcal{I} + a^D b$, $aa^D(a + b)$ and $aa^D(a + b)bb^D$ and find the expressions of $(a + b)^D$, $[(a + b)bb^D]^D$, $(\mathcal{I} + a^D b)^D$, etc.

1. Introduction

In this paper, $\mathcal{R}$ will denote an associative ring whose unity is $\mathcal{I}$. The commutant of an element $a \in \mathcal{R}$ is defined as $\text{comm}(a) = \{x \in \mathcal{R} : ax = xa\}$. Let us recall that an element $a \in \mathcal{R}$ has a Drazin inverse [1] if there exists $b \in \mathcal{R}$

$$bab = b, \qquad ab = ba, \qquad a^k = a^{k+1}b \tag{1.1}$$

for some positive integer k. The element b satisfying (1.1) is unique if it exists and is denoted by a^D. The smallest integer k satisfying (1.1) is called the Drazin index of a, denoted by $\text{ind}(a)$. If $\text{ind}(a) = 1$, then b is called the group inverse of a and is denoted by $a^\#$. The subset of $\mathcal{R}$ composed of Drazin invertible elements will be denote by $\mathcal{R}^D$.

The conditions in (1.1) are equivalent to

$$bab = b, \qquad ab = ba, \qquad a - a^2 b \text{ is nilpotent.}$$

The notation a^π means $\mathcal{I} - aa^D$ for any Drazin invertible element $a \in \mathcal{R}$. Observe that by the definition of the Drazin inverse, $aa^\pi = a^\pi a$ is nilpotent.

The research for Drazin invertibility of the sum of two elements a, b in a ring is attractive. Many authors have studied such problems from different views, see, e.g. [1, 2, 6, 8, 9, 11–13]. In the articles of Wei and Deng [9], Zhuang et al. [12] and Liu and Qin [2], the commutativity $ab = ba$ was assumed. In [9], they characterized the relationships of the Drazin inverse between $A + B$ and $\mathcal{I} + A^D B$ by Jordan canonical decomposition for complex matrices A and B. Zhuang et al. [12] extended the result in [9] to a ring $\mathcal{R}$, and

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 32A65, 16E50.
Keywords. Drazin inverse; Ring; Nilpotent; Weakly commutative condition.
Received: 7 February 2022; Accepted: 8 April 2022
Communicated by Yimin Wei
Research supported by National Natural Science Foundation of China (12161020).
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it was proved that if $a, b \in \mathcal{R}^D$ and $ab = ba$, then $a + b \in \mathcal{R}^D$ if and only if $\mathcal{I} + a^D b \in \mathcal{R}^D$. In [2], Liu and Qin deduced that $a + b \in \mathcal{R}^D$ if and only if $aa^D(a + b) \in \mathcal{R}^D$ under the condition $ab = ba$ for $a, b \in \mathcal{R}^D$. In resent years, several authors focused on the problem under some weaker conditions. Liu et al. [3] considered the relations between the Drazin inverses of $P + Q$ and $\mathcal{I} + P^D Q$, under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$ for complex matrices P and Q by using the method of splitting complex matrices into blocks. In [11], Zhu and Chen generalized the results in [3] to a ring case. More results on the Drazin inverse can also be found in [4, 5, 7, 10]. In this paper, we will further consider the results of [11] and [3] for the Drazin inverse, which extend [9, Theorem 2], [12, Theorem 3] and [2, Theorem 2.1].

In section 2, we present some lemmas which are used in the proof of the main results.

In section 3, we characterize the relations of $a + b$, $(a + b)bb^D$, $\mathcal{I} + a^D b$, $aa^D(a + b)$ and $aa^D(a + b)bb^D$. Also we obtain some expressions for $(a + b)^D$, $(a + b)^D bb^D$, $(\mathcal{I} + a^D b)^D$, etc.

Finally, in the last section, we investigate Drazin invertibility of the product of $a, b \in \mathcal{R}^D$ which will be used in the sequel. Then we introduce some new conditions and give the Drazin inverse of the sum $a + b$, where a, b are Drazin invertible in $\mathcal{R}$.

2. Preliminaries

We give some previous results which will be useful in proving our results.

Lemma 2.1. [11, Lemma 2.4] *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab$. Then*

$$(1)\ \{ab, a^D b, ab^D, a^D b^D\} \subseteq comm(a). \tag{2.1}$$

$$(2)\ \{ba, b^D a, ba^D, b^D a^D\} \subseteq comm(b). \tag{2.2}$$

Lemma 2.2. [11, Lemma 2.6] *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab$. Then for any positive integer i, the following hold:*

$$(1)\ (a^D b)^{i+1} = a^D b(ba^D)^i = (a^D)^{i+1} b^{i+1}. \tag{2.3}$$

$$(2)\ (ba^D)^{i+1} = ba^D(a^D b)^i = b^{i+1}(a^D)^{i+1}. \tag{2.4}$$

Lemma 2.3. [11, Theorem 3.1] *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab$. Then $ab \in \mathcal{R}^D$ and $(ab)^D = a^D b^D$.*

Lemma 2.4. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab$. If $c_1 = aa^\pi b^\pi$ and $c_2 = aa^D bb^\pi$, then $c_1 - c_2$ is nilpotent.*

Proof. Firstly, we prove that $c_1 = aa^\pi b^\pi$ is nilpotent. According to Lemma 2.1, we have the following equalities:

$$aa^\pi b^\pi a = a^2 a^\pi b^\pi \tag{2.5}$$

and

$$ab^\pi a^\pi = aa^\pi b^\pi. \tag{2.6}$$

Hence, we get

$$\begin{aligned}(aa^\pi b^\pi)^2 &= (aa^\pi b^\pi a)a^\pi b^\pi \overset{(2.5)}{=} (a^2 a^\pi b^\pi)a^\pi b^\pi = aa^\pi(ab^\pi a^\pi)b^\pi \\ &\overset{(2.6)}{=} aa^\pi(aa^\pi b^\pi)b^\pi = (aa^\pi)^2(b^\pi)^2 = (aa^\pi)^2 b^\pi.\end{aligned}$$

By induction, $(aa^\pi b^\pi)^n = (aa^\pi)^n b^\pi$ for every integer $n \geq 1$. Since aa^π is nilpotent, $aa^\pi b^\pi = c_1$ is nilpotent.

Secondly, we will show that $c_2 = aa^D bb^\pi$ is nilpotent. As

$$\begin{aligned}(aa^D bb^\pi)^2 &= (aa^D bb^\pi)(aa^D bb^\pi) = aa^D b(\mathcal{I} - bb^D)aa^D b(\mathcal{I} - bb^D)\\ &= aa^D(\mathcal{I} - bb^D)baa^D b(\mathcal{I} - bb^D)\\ &\overset{(2.2)}{=} aa^D ba(\mathcal{I} - bb^D)a^D b(\mathcal{I} - bb^D)\\ &\overset{(2.1)}{=} aa^D ab(\mathcal{I} - bb^D)a^D b(\mathcal{I} - bb^D)\\ &= aa^D a(\mathcal{I} - bb^D)ba^D b(\mathcal{I} - bb^D)\\ &\overset{(2.2)}{=} aa^D aba^D(\mathcal{I} - bb^D)b(\mathcal{I} - bb^D)\\ &\overset{(2.1)}{=} aa^D aa^D b(\mathcal{I} - bb^D)b(\mathcal{I} - bb^D)\\ &= aa^D(bb^\pi)^2.\end{aligned}$$

By induction, $(aa^D bb^\pi)^n = aa^D(bb^\pi)^n$ for every integer $n \geq 1$. Since bb^π is nilpotent, $aa^D bb^\pi = c_2$ is nilpotent.

Finally, we shall prove that $c_1 - c_2$ is nilpotent. Since $a^\pi a^D = a^D a^\pi = 0$, combining Lemma 2.1, we derive

$$c_1^2 c_2 = aa^\pi b^\pi aa^\pi b^\pi aa^D bb^\pi \overset{(2.1)}{=} aa^\pi b^\pi aa^\pi aa^D b^\pi bb^\pi = 0$$

and

$$\begin{aligned}c_2 c_1 &= aa^D bb^\pi aa^\pi b^\pi = aa^D b(\mathcal{I} - bb^D)aa^\pi b^\pi\\ &= \left[a^D(ab) - a^D(ab)bb^D\right]aa^\pi b^\pi\\ &\overset{(2.1)}{=} \left[aba^D - a(ba^D)bb^D\right]aa^\pi b^\pi\\ &\overset{(2.2)}{=} (aba^D - abb^D ba^D)aa^\pi b^\pi\\ &= abb^\pi a^D aa^\pi b^\pi = 0.\end{aligned}$$

Therefore, we can prove that $c_1^2 c_2 = c_1 c_2 c_1 = 0$ and $c_2^2 c_1 = c_2 c_1 c_2 = 0$.

As c_1 and c_2 are nilpotent, $aa^\pi b^\pi - aa^D bb^\pi = c_1 - c_2$ is nilpotent by [11, Lemma 2.2 (2)]. □

Lemma 2.5. *Let $a, b \in \mathcal{R}^D$ with $a^2 b = aba$ and $b^2 a = bab$ and $c = (a+b)bb^D \in \mathcal{R}^D$. Suppose $d_1 = bb^\pi + cc^\pi$ and $d_2 = aa^\pi b^\pi - aa^D bb^\pi$. Then $d_1 + d_2$ is nilpotent.*

Proof. First, we will give some useful equalities. From $b^\pi b^D = 0$ and $a^\pi a^D = 0$, we get

$$bb^\pi c = bb^\pi(a+b)bb^D = (bb^\pi a)b^D b + bb^\pi bbb^D \overset{(2.2)}{=} b^D bb^\pi ab = 0 \tag{2.7}$$

and

$$\begin{aligned}aa^\pi bb^\pi aa^D &= a^\pi ab(\mathcal{I} - bb^D)a^D a = a^\pi aba^D a - a^\pi abb(b^D a^D)a\\ &\overset{(2.2)}{=} a^\pi(ab)a^D a - a^\pi(ab^D)a^D bba\\ &\overset{(2.1)}{=} a^\pi a^D(ab)a - a^\pi a^D(ab^D)bba = 0.\end{aligned}$$

Similarly

$$caa^\pi b^\pi = caa^D bb^\pi = ab^\pi c = bb^\pi c = 0 \tag{2.8}$$

and

$$aa^\pi b^\pi aa^D = aa^D bb^\pi aa^\pi = aa^D b^2 b^\pi aa^\pi = aa^\pi bb^\pi aa^D = 0. \tag{2.9}$$

Next, we will show that d_1 is nilpotent. Let $d_1 = x + y$, where $x = bb^\pi$, $y = cc^\pi$. It is not difficult to see that $x^2y = bb^\pi(bb^\pi c)c^\pi \overset{(2.7)}{=} 0$. The equality $cb^\pi = (a+b)bb^Db^\pi = 0$ implies $yx = cc^\pi bb^\pi = c^\pi(cb^\pi)b = 0$. Consequently, $x^2y = xyx = 0$ and $y^2x = yxy = 0$.

Since bb^π, cc^π are nilpotent, it follows from [11, Lemma 2.2 (2)] that $d_1 = bb^\pi + cc^\pi$ is nilpotent. By virtue of Lemma 2.4, $d_2 = aa^\pi b^\pi - aa^Dbb^\pi$ is nilpotent.

Finally, we will prove that $d_1 + d_2$ is nilpotent. Using the previous equations and combining $cb^\pi = 0$, we obtain that

$$\begin{aligned}
d_1^2d_2 &= (bb^\pi + cc^\pi)^2(aa^\pi b^\pi - aa^Dbb^\pi)\\
&= (b^2b^\pi + bb^\pi cc^\pi + cc^\pi bb^\pi + c^2c^\pi)(aa^\pi b^\pi - aa^Dbb^\pi)\\
&= b^2b^\pi aa^\pi b^\pi + bb^\pi cc^\pi aa^\pi b^\pi + c^2c^\pi aa^\pi b^\pi\\
&\quad - b^2b^\pi aa^Dbb^\pi - bb^\pi cc^\pi aa^Dbb^\pi - c^2c^\pi aa^Dbb^\pi\\
&\overset{(2.8)}{=} b^2b^\pi aa^\pi b^\pi - b^2b^\pi aa^Dbb^\pi
\end{aligned}$$

and

$$\begin{aligned}
d_1d_2d_1 &= (bb^\pi + cc^\pi)(aa^\pi b^\pi - aa^Dbb^\pi)(bb^\pi + cc^\pi)\\
&= bb^\pi aa^\pi bb^\pi + bb^\pi aa^\pi b^\pi cc^\pi + cc^\pi aa^\pi bb^\pi + cc^\pi aa^\pi b^\pi cc^\pi\\
&\quad - bb^\pi aa^Db^2b^\pi - bb^\pi aa^Dbb^\pi cc^\pi - cc^\pi aa^Db^2b^\pi - cc^\pi aa^Dbb^\pi cc^\pi\\
&\overset{(2.8)}{=} b^\pi(baa^\pi)bb^\pi - b^\pi(baa^D)b^2b^\pi \overset{(2.2)}{=} b^\pi b(baa^\pi)b^\pi - b^\pi b(baa^D)bb^\pi\\
&= b^2b^\pi aa^\pi b^\pi - b^2b^\pi aa^Dbb^\pi.
\end{aligned}$$

Hence, $d_1^2d_2 = d_1d_2d_1$. And, similarly $d_2^2d_1 = d_2d_1d_2$.

By [11, Lemma 2.2 (2)], it follows that $d_1 + d_2$ is nilpotent. □

3. Main result 1

Now we will characterize the relations of $a+b$, $(a+b)bb^D$, $\mathcal{I}+a^Db$, $aa^D(a+b)$ and $aa^D(a+b)bb^D$ for $a, b \in \mathcal{R}^D$. Furthermore we deduce the expressions of $(a+b)^D$, $\left[aa^D(a+b)\right]^D$, $(\mathcal{I}+a^Db)^D$, etc. The results extend those given in [9, Theorem 2], [12, Theorem 3] and [2, Theorem 2.1].

Theorem 3.1. *Let $a, b \in \mathcal{R}^D$ be such that $a^2b = aba$, $b^2a = bab$ and $ind(a) = s$, $ind(b) = t$. Then the following conditions are equivalent:*

(1) $a + b \in \mathcal{R}^D$;
(2) $c = (a+b)bb^D \in \mathcal{R}^D$;
(3) $\xi = \mathcal{I} + a^Db \in \mathcal{R}^D$;
(4) $e = aa^D(a+b) \in \mathcal{R}^D$;
(5) $w = aa^D(a+b)bb^D \in \mathcal{R}^D$.

In this case,

$$(a+b)^D = c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi + a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi, \tag{3.1}$$

$$\begin{aligned}
(a+b)^D &= e^D + a^\pi b(e^D)^2 + \sum_{i=0}^{s-1}(b^D)^{i+1}(-a)^ia^\pi + b^\pi a\sum_{i=0}^{s-2}(i+1)(b^D)^{i+2}(-a)^ia^\pi\\
&= a^D\xi^D + a^\pi b(a^D\xi^D)^2 + \sum_{i=0}^{s-1}(b^D)^{i+1}(-a)^ia^\pi + b^\pi a\sum_{i=0}^{s-2}(i+1)(b^D)^{i+2}(-a)^ia^\pi,
\end{aligned} \tag{3.2}$$

where

$$c^D = (a+b)^D bb^D, \xi^D = a^\pi + a^2a^D(a+b)^D = a^\pi + ae^D \tag{3.3}$$

and $e^D = aa^D(a+b)^D = a^D\xi^D = \xi^D a^D, w^D = aa^D(a+b)^D bb^D$.

Proof. (1) ⇒ (2) To show that $c \in \mathcal{R}^D$, we write $c = f_1 f_2$, where $f_1 = a+b$, $f_2 = bb^D$. By Lemma 2.1, we have

$$\begin{aligned} f_1^2 f_2 &= (a+b)^2 bb^D = a(ab)b^D + abbb^D + babb^D + b^3b^D \\ &\overset{(2.1)}{=} a(ba)b^D + abbb^D + (ba)bb^D + b^3b^D \\ &\overset{(2.2)}{=} ab^Dba + abbb^D + bb^Dba + b^3b^D \\ &= (a+b)bb^D(a+b) = f_1f_2f_1, \end{aligned}$$

and

$$\begin{aligned} f_2^2 f_1 &= bb^Dbb^D(a+b) = bb^Db(b^Da) + b^Dbb^Dbb \\ &\overset{(2.2)}{=} bb^Dab^Db + b^Dbb^Dbb \\ &= bb^D(a+b)bb^D = f_2f_1f_2. \end{aligned}$$

Applying Lemma 2.3, we deduce that $c \in \mathcal{R}^D$ and $c^D = \left[(a+b)bb^D\right]^D = (a+b)^D bb^D$.

(2) ⇒ (1) Let

$$x = c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + a^\pi b \sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi = x_1 + x_2,$$

where $x_1 = c^D$, $x_2 = \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + a^\pi b \sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi$.

Assume that c is Drazin invertible. We will prove that x is the Drazin inverse of $a+b$, i.e., we will prove that $x(a+b) = (a+b)x$, $x(a+b)x = x$ and $(a+b)-(a+b)^2x$ is nilpotent.

Step 1 First we prove that $x(a+b) = (a+b)x$. In view of Lemma 2.1, we have

$$\begin{aligned} (a+b)a^\pi b(a^D)^2 &= a^\pi ab(a^D)^2 + b^2(a^D)^2 - ba(a^Db)(a^D)^2 \\ &\overset{(2.1)}{=} a^\pi a^D aba^D + b^2(a^D)^2 - ba^Dba^D \overset{(2.2)}{=} 0. \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} (a+b)x &= (a+b)\left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + a^\pi b \sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi\right] \\ &\overset{(3.4)}{=} (a+b)\left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi\right] = y_1 + y_2, \end{aligned} \tag{3.5}$$

where $y_1 = (a+b)c^D$, $y_2 = (a+b)\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi$.

Second we show $x_1(a+b) = y_1$ and $x_2(a+b) = y_2$. In light of Lemma 2.1, we get

$$\begin{aligned} c(a+b) &= (a+b)bb^D(a+b) = ab(b^Da) + abb^Db + bb^D(ba) + b^2b^Db \\ &\overset{(2.2)}{=} (ab^D)ab + abb^Db + babb^D + b^2b^Db \\ &\overset{(2.1)}{=} a^2bb^D + abb^Db + babb^D + b^2b^Db \\ &= (a^2 + ab + ba + b^2)bb^D \\ &= (a+b)c. \end{aligned}$$

Then, by [1, Theorem 1], we get

$$c^D(a+b) = (a+b)c^D. \tag{3.6}$$

Thus, $x_1(a+b) = y_1$.

By mathematical induction, for every integer $i \geq 1$, a calculation yields

$$aa^D(ba^D)^i = (aa^Dba^D)^i \overset{(2.1)}{=} (a^Db)^i. \tag{3.7}$$

From the equality $b^tb^\pi = 0$ and

$$\begin{aligned} a^Db^\pi a &= a^D(\mathcal{I} - bb^D)a = aa^D - a^Db(b^Da) \overset{(2.2)}{=} aa^D - (a^Db^D)ab \\ &\overset{(2.1)}{=} aa^D - aa^Db^Db = aa^Db^\pi. \end{aligned} \tag{3.8}$$

So we have

$$\begin{aligned} x_2(a+b) - y_2 &= -\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^{i+1}b^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi a \\ &\quad - a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^{i+1}b^\pi + a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi a \\ &\quad - (a+b)\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi \\ &\overset{(2.3)}{=} -\sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi + \sum_{i=0}^{t-1}a^D(-a^Db)^ib^\pi a \\ &\quad - a^\pi ba^D\sum_{i=0}^{t-2}(i+1)(-a^Db)^{i+1}b^\pi + a^\pi b(a^D)^2\sum_{i=0}^{t-2}(i+1)(-a^Db)^ib^\pi a \\ &\quad - b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi - a\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi \\ &= -\sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi + \sum_{i=0}^{t-1}(-ba^D)^{i+1}b^\pi \\ &\quad + a^\pi\left[\sum_{i=0}^{t-2}(i+1)(-ba^D)^{i+2}b^\pi - \sum_{i=0}^{t-2}(i+1)(-ba^D)^{i+1}b^\pi\right] \\ &= -\sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi + \sum_{i=0}^{t-1}(-ba^D)^{i+1}b^\pi - a^\pi\sum_{i=1}^{t-1}(-ba^D)^ib^\pi \\ &= -\sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi + aa^D\sum_{i=1}^{t-1}(-ba^D)^ib^\pi \\ &\overset{(3.7)}{=} -\sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi + \sum_{i=1}^{t-1}(-aa^Dba^D)^ib^\pi \\ &\overset{(2.3)}{=} -\sum_{i=1}^{t-1}(-a^Db)^ib^\pi - (-a^D)^tb^tb^\pi + \sum_{i=1}^{t-1}(-a^Db)^ib^\pi \\ &= 0. \end{aligned}$$

Hence, $x_2(a+b) = y_2$. It follows that $x(a+b) = (a+b)x$.

Step 2 We give the proof of $x(a+b)x = x$. From the equality (3.5), we obtain

$$\begin{aligned} x(a+b)x &= x(a+b)\left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi\right] \\ &= (a+b)\left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi\right] \times \left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi\right] \\ &= m_1 + m_2 + m_3, \end{aligned}$$

where

$$m_1 = (a+b)(c^D)^2, m_2 = (a+b)c^D \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi, m_3 = (a+b)\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi.$$

Now we prove $m_1 + m_2 + m_3 = x$. Also, the following equalities will be useful:

$$a + b = c + (a+b)b^\pi, \tag{3.9}$$

and

$$(a+b)b^\pi c = ab^\pi c + bb^\pi c \overset{(2.8)}{=} 0. \tag{3.10}$$

Firstly, we have

$$\begin{aligned} m_1 &= (a+b)(c^D)^2 = [c + (a+b)b^\pi]\,(c^D)^2 \\ &= c(c^D)^2 + (a+b)b^\pi(c^D)^2 \\ &= c(c^D)^2 + (a+b)b^\pi c(c^D)^3 \\ &\overset{(3.10)}{=} c^D, \end{aligned}$$

and

$$\begin{aligned} m_2 &= (a+b)c^D \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi = (a+b)(c^D)^2 c \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\ &= (a+b)(c^D)^2(a+b)bb^D \sum_{i=0}^{t-1} b^D ba^D(a^D)^i(-b)^i b^\pi \\ &\overset{(2.4)}{=} -(a+b)c^D \sum_{i=0}^{t-1} b^D(-ba^D)^{i+1} b^\pi \\ &\overset{(2.2)}{=} -(a+b)c^D \sum_{i=0}^{t-1}(-ba^D)^{i+1} b^D b^\pi \\ &= 0. \end{aligned}$$

Secondly, we prove that

$$m_3 = \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + a^\pi b \sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi. \tag{3.11}$$

Then simple computations show that

$$
\begin{aligned}
m_3 &= (a+b)\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= a\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= \left[aa^D b^\pi + \sum_{i=1}^{t-1}(-a^D b)^i b^\pi\right]\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= aa^D\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi - aa^D bb^D\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&\quad + \sum_{i=1}^{t-1}(-a^D b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi - aa^D bb^D\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + \sum_{i=1}^{t-1}(-a^D b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&\quad + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&\overset{(2.4)}{=} \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + aa^D\sum_{i=0}^{t-1}(-ba^D)^{i+1} b^D b^\pi + \sum_{i=1}^{t-1}(-a^D b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&\quad + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + \sum_{i=1}^{t-1}(-a^D b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \\
&= \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + z_1 + z_2,
\end{aligned}
$$

where

$$z_1 = \sum_{i=1}^{t-1}(-a^D b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi,\ z_2 = b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi.$$

In view of the equality (3.11), it is enough to prove

$$z_1 + z_2 = a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi.$$

Since

$$
\begin{aligned}
a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi &= (\mathcal{I} - aa^D)b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi \\
&= b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi - aa^D b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^i b^\pi,
\end{aligned}
$$

we only need to show

$$z_1 = -aa^Db\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi,\ z_2 = b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi.$$

From $b^tb^\pi = 0$ and aa^D commutes with a^Db, we obtain

$$\begin{aligned}
z_1 &= \sum_{i=1}^{t-1}(-a^Db)^ib^\pi\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi = \sum_{i=1}^{t}(-a^Db)^ib^\pi\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi\\
&\overset{(2.1)}{=} \sum_{i=1}^{t}(-aa^Da^Db)^ib^\pi\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi = aa^D\sum_{i=1}^{t}(-a^Db)^ib^\pi\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi\\
&= -aa^D\sum_{i=1}^{t}(-a^Db)^{i-1}a^Dbb^\pi\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi \overset{(2.1)}{=} -aa^D\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^i(a^Dbb^\pi)a^Db^\pi\\
&= -aa^D\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ia^D(a^Dbb^\pi)b^\pi \overset{(2.1)}{=} -aa^D(a^D)^2b\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ib^\pi\\
&\overset{(2.1)}{=} -aa^Db(a^D)^2\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ib^\pi = -aa^Db(a^D)^2\sum_{i=0}^{t-2}(i+1)(-a^Db)^ib^\pi\\
&\overset{(2.3)}{=} -aa^Db\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi,
\end{aligned}$$

and similarly,

$$\begin{aligned}
z_2 &= b\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi\sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi \overset{(2.1)}{=} b\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi\\
&\overset{(2.1)}{=} b\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi a^Db^\pi \overset{(2.1)}{=} b\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ia^Da^Db^\pi b^\pi\\
&= b\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^i(a^D)^2b^\pi \overset{(2.1)}{=} b(a^D)^2\sum_{i=0}^{t-1}(-a^Db)^i\sum_{i=0}^{t-1}(-a^Db)^ib^\pi\\
&= b(a^D)^2\sum_{i=0}^{t-2}(i+1)(-a^Db)^ib^\pi \overset{(2.3)}{=} b(a^D)^2\sum_{i=0}^{t-2}(i+1)(a^D)^i(-b)^ib^\pi\\
&= b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi.
\end{aligned}$$

Therefore

$$m_3 = \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi + a^\pi b\sum_{i=0}^{t-2}(i+1)(a^D)^{i+2}(-b)^ib^\pi.$$

So, we get $x(a+b)x = x$.

Step 3 Now we will prove that $a+b-(a+b)^2x$ is nilpotent.

According to the equality (3.5), we have

$$(a+b)^2x = \left[c^D + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi\right](a+b)^2 = c^D(a+b)^2 + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi(a+b)^2. \tag{3.12}$$

By using (3.6), (3.9) and (3.10), we get

$$\begin{aligned} c^D(a+b)^2 &= (a+b)^2c^D = (a+b)^2c^2(c^D)^3 \\ &= [(c+(a+b)b^\pi)c]^2(c^D)^3 \\ &= c^4(c^D)^3 = c - cc^\pi. \end{aligned} \tag{3.13}$$

By elementary computations, we obtain

$$\begin{aligned} \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^ib^\pi(a+b)^2 &\overset{(2.3)}{=} -\sum_{i=0}^{t-1}(-a^Db)^{i+1}bb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a \\ &\quad + \sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi ab + \sum_{i=0}^{t-1}a^D(-a^Db)^ib^\pi a^2 \\ &\overset{(3.8)}{=} -\sum_{i=0}^{t-1}(-a^Db)^{i+1}bb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a \\ &\quad + \sum_{i=0}^{t-1}(-a^Db)^iaa^Dbb^\pi + \sum_{i=0}^{t-1}a^D(-a^Db)^ib^\pi a^2 \\ &= aa^Dbb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a + \sum_{i=0}^{t-1}a^D(-a^Db)^ib^\pi a^2 \\ &\overset{(2.1)}{=} aa^Dbb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a + \sum_{i=0}^{t-1}(-a^Db)^ia^Db^\pi a^2 \\ &\overset{(3.8)}{=} aa^Dbb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a + \sum_{i=0}^{t-1}(-a^Db)^iaa^Db^\pi a \\ &\overset{(2.1)}{=} aa^Dbb^\pi - \sum_{i=0}^{t-1}(-a^Db)^{i+1}b^\pi a + \sum_{i=0}^{t-1}aa^D(-a^Db)^ib^\pi a \\ &= aa^Dbb^\pi + a(a^Db^\pi)a \overset{(2.1)}{=} aa^Dbb^\pi + aa^Dab^\pi. \end{aligned} \tag{3.14}$$

Combining (3.9), (3.12), (3.13) and (3.14) gives

$$\begin{aligned} &(a+b) - (a+b)^2x \\ &= [c+(a+b)b^\pi] - (c - cc^\pi) - (aa^Dbb^\pi + aa^Dab^\pi) \\ &= bb^\pi + cc^\pi + aa^\pi b^\pi - aa^Dbb^\pi \\ &= d_1 + d_2. \end{aligned}$$

It follows from Lemma 2.5, $(a+b) - (a+b)^2x = d_1 + d_2$ is nilpotent.

(1) ⇔ (4) This is similar to (1) ⇔ (2).

(3) ⇒ (4) In order to prove that $e \in \mathcal{R}^D$, let $e = aa^D(a+b) = a^2a^D + aa^Db = a^2a^D + aa^Daa^Db = a^2a^D(\mathcal{I} + a^Db) = g_1g_2$, where $g_1 = a^2a^D$, $g_2 = \mathcal{I} + a^Db$. Obviously $(a^2a^D)^D = a^D$ and

$$g_1g_2 = a^2a^D(\mathcal{I} + a^Db) = a^2a^D + aa^Da(a^Db) \overset{(2.1)}{=} a^2a^D + (a^Db)aa^Da = (\mathcal{I} + a^Db)a^2a^D = g_2g_1,$$

by [12, Lemma 2], we have $e \in \mathcal{R}^D$ and

$$e^D = (a^2a^D)^D(\mathcal{I} + a^Db)^D = (\mathcal{I} + a^Db)^D(a^2a^D)^D = a^D\xi^D = \xi^Da^D.$$

(4) ⇒ (3) We can write $\mathcal{I} + a^D b = h_1 + h_2$, where $h_1 = a^\pi$, $h_2 = a^D(a+b) = a^D aa^D(a+b) = a^D e$. It follows from Lemma 2.1 that

$$a^D e = a^D aa^D(a+b) = aa^D(a+b)a^D = ea^D,$$

utilizing [12, Lemma 2] gets $a^D(a+b) = a^D aa^D(a+b) \in \mathcal{R}^D$ and

$$\left[a^D(a+b)\right]^D = \left[a^D aa^D(a+b)\right]^D = (a^D)^D\left[aa^D(a+b)\right]^D = a^2a^D(a+b)^D = ae^D.$$

Applying again Lemma 2.1, we obtain that $a^D b$ commutes with aa^D. Then $a^D(a+b) \in comm(a^\pi)$ and $h_1h_2 = h_2h_1 = 0$. It follows from [1, corollary 1] that $\xi^D = a^\pi + a^2a^D(a+b)^D = a^\pi + ae^D$.

(4) ⇒ (5) In order to verify that $w \in \mathcal{R}^D$, we write $aa^D(a+b)bb^D = l_1l_2$, where $l_1 = aa^D(a+b)$, $l_2 = bb^D$. In view of Lemma 2.1, we deduce that

$$aa^D(a+b) = (aa^D)^2(a+b) = (aa^D)^2a + aa^Da(a^Db) = aa^Daaa^D + aa^Dbaa^D = aa^D(a+b)aa^D$$

and $abb^Da \overset{(2.2)}{=} (ab^D)ab \overset{(2.1)}{=} aabb^D$, it follows by [1, Theorem 1] that $abb^Da^D = a^Dabb^D$. So, we get

$$\begin{aligned}
l_1l_2l_1 &= aa^D(a+b)a^D(abb^Da)a^D(a+b)\\
&= aa^D(a+b)a^Daa^Dabb^D(a+b)\\
&= aa^D(a+b)a^Daa^D\left(abb^Da + abb^Db\right)\\
&= aa^D(a+b)a^Daa^D(aabb^D + abb^Db)\\
&= aa^D(a+b)aa^D(a+b)bb^D = l_1^2l_2.
\end{aligned}$$

In a similar way, $l_2l_1l_2 = l_2^2l_1$. Thus, applying Lemma 2.3, we have $w \in \mathcal{R}^D$ and

$$w^D = \left[aa^D(a+b)bb^D\right]^D = \left[aa^D(a+b)\right]^D(bb^D)^D = aa^D(a+b)^Dbb^D.$$

(2) ⇒ (5) This is similar to (4) ⇒ (5).

(5) ⇒ (4) To check that $e \in \mathcal{R}^D$, let $p_1 = a^2a^D$, $p_2 = aa^Db$. Further, we can write $aa^Db = q_1q_2$, where $q_1 = aa^D$, $q_2 = b$. In view of Lemma 2.1, $q_1q_2q_1 = q_1^2q_2, q_2q_1q_2 = q_2^2q_1$. Then $aa^Db \in \mathcal{R}^D$ and $(aa^Db)^D = (aa^D)^Db^D = aa^Db^D$ by Lemma 2.3.

It is easy to verify that $p_1p_2p_1 = p_1^2p_2, p_2p_1p_2 = p_2^2p_1$ and $(p_1+p_2)p_2p_2^D = aa^D(a+b)bb^D \in \mathcal{R}^D$. Applying (1) ⇔ (2) to p_1 and p_2, we conclude that $aa^D(a+b) = p_1 + p_2 \in \mathcal{R}^D$, as required. □

Remark 3.2. *As mentioned in the introduction, in the papers of Zhuang et al. [12] and Liu and Qin [2], the commutativity $ab = ba$ was assumed. In [12, Theorem 3], they proved that if $a, b \in \mathcal{R}^D$ and $ab = ba$, then $a + b \in \mathcal{R}^D$ if and only if $\mathcal{I} + a^Db \in \mathcal{R}^D$. Moreover, the expressions of $(a+b)^D$ and $(\mathcal{I} + a^Db)^D$ are presented. In [2, Theorem 2.1], Liu and Qin assumed that $aa^D(a+b)$ instead of $\mathcal{I} + a^Db$, they deduced another expression for $(a+b)^D$. In Theorem 3.1, we relax this hypothesis $ab = ba$ by assuming two conditions $a^2b = aba$ and $b^2a = bab$. It also can be seen from Theorem 3.1 that the condition $\mathcal{I} + a^Db \in \mathcal{R}^D$ of [12, Theorem 3] and $aa^D(a+b) \in \mathcal{R}^D$ of [2, Theorem 2.1] are equivalent. Moreover, the expressions for $(a+b)^D$ in [12, Theorem 3] will be exactly the same as in [2, Theorem 2.1], we will prove them in Corollary 3.4.*

First we show that $ab = ba$ implies the conditions of Theorem 3.1. From $ab = ba$, we get $a^2b = a(ab) = aba$. Symmetrically, $b^2a = bab$. To prove that our conditions are strictly weaker than $ab = ba$, we construct matrices a, b satisfying the conditions of Theorem 3.1, but not $ab = ba$.

Example 3.3. *Let $R = M_3(C)$, and take*

$$a = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{bmatrix} \in \mathcal{R}^D.$$

It is easy to check $a^2b = aba$ and $b^2a = bab$. But $ab \neq ba$. Then, applying Theorem 3.1 and after simple computations, we obtain

$$(a+b)^D = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

The following corollary follows from Theorem 3.1. For the sake of clarity of presentation, the short proof is given.

Corollary 3.4. *Let $a, b \in \mathcal{R}^D$ be such that $ab = ba$. Then the following conditions are equivalent:*

(1) $a + b \in \mathcal{R}^D$;

(2) $\xi = \mathcal{I} + a^D b \in \mathcal{R}^D$;

(3) $e = aa^D(a + b) \in \mathcal{R}^D$.

In this case,

$$\begin{aligned}(a+b)^D &= \xi^D a^D + b^D(\mathcal{I} + aa^\pi b^D)^{-1}a^\pi \\ &= e^D + a^\pi(\mathcal{I} + b^D aa^\pi)^{-1}b^D = e^D + a^\pi\left(\sum_{i=0}^{\mathrm{ind}(a)-1}(-b^D a)^i\right)b^D \\ &= a^D\xi^D bb^D + b^\pi(\mathcal{I} + bb^\pi a^D)^{-1}a^D + b^D(\mathcal{I} + aa^\pi b^D)^{-1}a^\pi,\end{aligned} \tag{3.15}$$

where $\xi^D = a^\pi + a^2a^D(a+b)^D$, $e^D = aa^D(a+b)^D$.

Proof. Since $ab = ba$, we get $a^2b = aba$ and $b^2a = bab$. Using Theorem 3.1, the following are equivalent:

(1) $a + b \in \mathcal{R}^D$;

(2) $\xi = \mathcal{I} + a^D b \in \mathcal{R}^D$;

(3) $e = aa^D(a + b) \in \mathcal{R}^D$.

Recall that aa^π is nilpotent and its index of nilpotency is the Drazin index of a. Let s=index(a). From the assumption $ab = ba$, we have a, b, a^D and b^D commute with each other by [1, Theorem 1]. From this, we conclude that $a^\pi b = ba^\pi$ and $b^\pi a = ab^\pi$. Applying again [1, Theorem 1], we get $a^\pi b^D = b^D a^\pi$. Hence $a^\pi b(e^D)^2 = a^\pi b(a^D\xi^D)^2 = 0$ and $b^\pi a\sum_{i=0}^{s-2}(i+1)(b^D)^{i+2}(-a)^i a^\pi = 0$.

Since $b^D aa^\pi$ is nilpotent, $\mathcal{I} + b^D aa^\pi$ is invertible and $a^\pi b^D = b^D a^\pi$, we get

$$\begin{aligned}(\mathcal{I} + b^D aa^\pi)^{-1} &= \mathcal{I} + (-b^D aa^\pi) + (-b^D aa^\pi)^2 + \cdots + (-b^D aa^\pi)^{s-1} \\ &= \sum_{i=0}^{s-1}(-b^D aa^\pi)^i = \sum_{i=0}^{s-1}(-a^\pi b^D a)^i = a^\pi\sum_{i=0}^{s-1}(-b^D a)^i.\end{aligned}$$

From $(\mathcal{I} + b^D aa^\pi)b^D = b^D(\mathcal{I} + aa^\pi b^D)$, we obtain

$$\begin{aligned}b^D(\mathcal{I} + aa^\pi b^D)^{-1}a^\pi = a^\pi(\mathcal{I} + b^D aa^\pi)^{-1}b^D &= a^\pi\left(a^\pi\sum_{i=0}^{s-1}(-b^D a)^i\right)b^D \\ &= a^\pi\left(\sum_{i=0}^{\mathrm{ind}(a)-1}(-b^D a)^i\right)b^D.\end{aligned}$$

Note that $e^D = \xi^D a^D$ by Theorem 3.1, then we have

$$\begin{aligned}(a+b)^D &= \xi^D a^D + b^D(\mathcal{I} + aa^\pi b^D)^{-1}a^\pi = e^D + a^\pi(\mathcal{I} + b^D aa^\pi)^{-1}b^D \\ &= e^D + a^\pi\left(\sum_{i=0}^{\mathrm{ind}(a)-1}(-b^D a)^i\right)b^D.\end{aligned}$$

The last equality $(a+b)^D = a^D\xi^D bb^D + b^\pi(\mathcal{I} + bb^\pi a^D)^{-1}a^D + b^D(\mathcal{I} + aa^\pi b^D)^{-1}a^\pi$ appearing in (3.15) follows from the one in [12, Theorem 3]. □

4. Main result 2

In this section, we consider some results on the expressions of $(ab)^D$ and $(a+b)^D$, by using a, b, a^D and b^D, where $a, b \in \mathcal{R}^D$. We begin with

Lemma 4.1. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba = ba^2$, then $aa^Db = baa^D$.*

Proof. Since $a^2b = aba$, by [1, Theorem 1], $aba^D = a^Dab$. Then $baa^D = ba^2(a^D)^2 = aba(a^D)^2 = aba^D = a^Dab$. □

We come now to the demonstration of the main result of this section which extends [12, Lemma 2].

Theorem 4.2. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba = ba^2$ and $b^2a = bab$, then $ab \in \mathcal{R}^D$ and $(ab)^D = b^Da^D = a^Db^D$.*

Proof. Let $x = b^Da^D$. Since $aa^Db = baa^D$, by [1, Theorem 1], $aa^Db^D = b^Daa^D$.

Step 1 We can verify that

$$xab = b^Da^Dab = a^D(ab)b^D \overset{(2.1)}{=} a(ba^D)b^D \overset{(2.2)}{=} abb^Da^D = abx.$$

Step 2 It is easy to check that

$$xabx = b^D(a^Dab)b^Da^D = bb^D(a^Dab^D)a^D = b^Dbb^Da^Daa^D = b^Da^D = x.$$

Step 3 Take $k = max\{ind(a), ind(b)\}$. Since $a^2b = aba$, by [11, Lemma 2.1(2)], $(ab)^k = a^kb^k$. From the definition of the Drazin inverse and $(ab)^k = a^kb^k$, we have

$$\begin{aligned}(ab)^{k+1}x &= (ab)^{k+1}b^Da^D = a^{k+1}(b^{k+1}b^D)a^D = a^{k+1}b^ka^D \\ &= a(a^kb^k)a^D = a(ab)^ka^D \overset{(2.1)}{=} a^Da(ab)^k \\ &= (a^Da^{k+1})b^k = a^kb^k = (ab)^k.\end{aligned}$$

Hence, $(ab)^D = b^Da^D$. Similarly, we can check that $(ab)^D = a^Db^D$. □

Corollary 4.3. [12, Lemma 2] *Let $a, b \in \mathcal{R}^D$ with $ab = ba$, then $ab \in \mathcal{R}^D$ and $(ab)^D = b^Da^D = a^Db^D$.*

Proof. From $ab = ba$, we have $a^2b = a(ab) = (ab)a = ba^2$ and $b^2a = b(ba) = bab$. This completes the proof by Theorem 4.2. □

Remark 4.4. *In Theorem 4.2, the conditions $a^2b = aba = ba^2$ and $b^2a = bab$ are weaker than $ab = ba$. Since $ab = ba$, by the proof of Corollary 4.3 we get $a^2b = aba = ba^2$ and $b^2a = bab$. However, in general, the converse is false. The following example can illustrate this fact.*

Example 4.5. *Let $R = M_3(C)$, and take*

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{R}^D.$$

It is clear that $a^2b = aba = ba^2$ and $b^2a = bab$. However, $ab \neq ba$. Therefore we can apply Theorem 4.2 and we obtain

$$(ab)^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the rest of the paper, we look for simplifying equation (3.2) for $(a+b)^D$ under some stronger hypotheses than those of Theorem 3.1. First, we give a result which recovers a known result in [9, Theorem 3(2)] for matrices and [12, Corollary 5(2)] for elements of a ring.

Theorem 4.6. *Let $a, b \in \mathcal{R}^D$ be such that $a^2b = aba = ba^2$, $b^2a = bab$ and $ind(a) = s$. Then the following conditions are equivalent:*

(1) $a + b \in \mathcal{R}^D$;

(2) $\varsigma = a(\mathcal{I} + a^D b) \in \mathcal{R}^D$.

In this case,

$$(a+b)^D = \varsigma^D + \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i a^\pi + b^\pi a (b^D)^2, \tag{4.1}$$

where $\varsigma^D = aa^D(a+b)^D$.

Proof. (1) ⇒ (2) Let ς have the following representation

$$\varsigma = a(\mathcal{I} + a^D b) = aa^D(a+b) + aa^\pi = r_1 + r_2,$$

where $r_1 = aa^D(a+b)$, $r_2 = aa^\pi$.

By Lemma 4.1, we have $aa^D(a+b) = (a+b)aa^D$. Then, in view of Corollary 4.3, it follows that $aa^D(a+b) \in \mathcal{R}^D$ and $\left[aa^D(a+b)\right]^D = aa^D(a+b)^D$.

From $aa^D(a+b) = (a+b)aa^D$ and $a^D a^\pi = 0$, we have $r_1 r_2 = r_2 r_1 = 0$. Observe that aa^π is nilpotent. Hence, we can apply [1, Corollary 1] to get an expression of ς^D obtaining

$$\varsigma^D = \left[aa^D(a+b)\right]^D + (aa^\pi)^D = \left[aa^D(a+b)\right]^D = aa^D(a+b)^D.$$

(2) ⇒ (1) Obviously, $aa^D(a+b) = a^2a^D(\mathcal{I} + a^D b) = aa^D a(\mathcal{I} + a^D b)$. By virtue of Lemma 4.1, $aa^D b = baa^D$, and so $aa^D a(\mathcal{I} + a^D b) = a(\mathcal{I} + a^D b)aa^D$. It follows from Corollary 4.3 that $aa^D a(\mathcal{I} + a^D b) \in \mathcal{R}^D$. Hence $aa^D(a+b) \in \mathcal{R}^D$. This completes the proof by Theorem 3.1. In this case, $(a+b)^D$ is represented as in (3.2), where $\varsigma^D = e^D = aa^D(a+b)^D$.

Now, let us calculate $a^\pi b(e^D)^2$ appearing in (3.2). The hypothesis $a^2b = aba = ba^2$ implies that $a^\pi b = ba^\pi$, by Lemma 4.1. From this and $a^\pi a^D = 0$, we get $a^\pi b(e^D)^2 = a^\pi baa^D(a+b)^D e^D = 0$.

Finally, let us observe that the expression $b^\pi a \sum_{i=0}^{s-2}(i+1)(b^D)^{i+2}(-a)^i a^\pi$ given in (3.2) can be simplified. By using the condition $a^2b = ba^2$, [1, Theorem 1] leads to $a^2b^D = b^D a^2$ and

$$b^D a^2 = a(ab^D) \overset{(2.1)}{=} ab^D a. \tag{4.2}$$

Using the equation $b^\pi b^D = 0$, we have

$$\begin{aligned}
b^\pi a \sum_{i=0}^{s-2}(i+1)(b^D)^{i+2}(-a)^i a^\pi &= b^\pi a(b^D)^2 a^\pi + b^\pi a \sum_{i=1}^{s-2}(i+1)(b^D)^{i+2}(-a)^i a^\pi \\
&= b^\pi a(b^D)^2 - b^\pi ab^D(b^D a)a^D - b^\pi a \sum_{i=1}^{s-2}(i+1)(b^D)^{i+1}(b^D a)(-a)^{i-1}a^\pi \\
&\overset{(2.2)}{=} b^\pi a(b^D)^2 - b^\pi(ab^D a)b^D a^D - b^\pi \sum_{i=1}^{s-2}(i+1)(ab^D a)(b^D)^{i+1}(-a)^{i-1}a^\pi \\
&\overset{(4.2)}{=} b^\pi a(b^D)^2 - b^\pi b^D a^2 b^D a^D - b^\pi \sum_{i=1}^{s-2}(i+1)b^D a^2(b^D)^{i+1}(-a)^{i-1}a^\pi = b^\pi a(b^D)^2,
\end{aligned}$$

then (3.2) becomes (4.1). □

Remark 4.7. *In Theorem 4.6, the conditions $a^2b = aba = ba^2$, $b^2a = bab$ and $a(\mathcal{I} + a^D b) \in \mathcal{R}^D$ are weaker than $ab = ba$ and $a^D b = 0$ which were used in the paper* [12, Corollary 5(2)]*(or* [9, Theorem 3(2)]*). In fact, Example 4.5 can also illustrate this fact.*

Adding a condition $a^D b = 0$ in Theorem 4.6, we obtain the next result.

Corollary 4.8. *Let $a, b \in \mathcal{R}^D$ be such that $a^2b = aba = ba^2$, $b^2a = bab$, $a^Db = 0$ and $ind(a) = s$. Then $a + b \in \mathcal{R}^D$ and*

$$(a+b)^D = a^D + \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i + b^\pi a(b^D)^2. \tag{4.3}$$

Proof. From $a^Db = 0$, we get $a(\mathcal{I} + a^Db) = a \in \mathcal{R}^D$. Hence Theorem 4.6 is applicable. Since $a^2b = aba = ba^2$ and $b^2a = bab$, we have $aa^Db^D = b^Daa^D$ by Lemma 4.1 and [1, Theorem 1], combining $a^Db = 0$, we derive

$$\begin{aligned}
\sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i a^\pi &= \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i(\mathcal{I} - aa^D) \\
&= \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i - \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i aa^D \\
&= \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i - \sum_{i=0}^{s-1} aa^D(b^D)^{i+1}(-a)^i \\
&= \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i - \sum_{i=0}^{s-1} a(a^Db)(b^D)^{i+2}(-a)^i \\
&= \sum_{i=0}^{s-1} (b^D)^{i+1}(-a)^i.
\end{aligned}$$

According to the representation in (4.1), the equation (4.3) can be obtained. □

Acknowledgements

The authors would like to thank the referees for their valuable suggestions and comments. The very detailed comments improve many proofs of the paper, e.g., Theorem 3.1, Theorem 4.6 and Corollary 4.8.

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