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Integral operators on grand Lebesgue spaces and related weights with properties

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Abstract. A number of properties for the classes B_p^{-1} and B_p^* have been proved. The class B_p^{-1} characterizes the L^p- inequality involving the averaging operator and the class B_p^* characterizes the L^p- inequality involving the adjoint averaging operator. The reverse inequalities involving the integral operators in $L_w^{p)}$ have also been studied.

1. Introduction

Let w be a weight which is positive and Lebesgue measurable function on $(0, \infty)$. The weight class B_p is due to Arino and Muckenhoupt [1] who used it to characterize the Hardy inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p v(x)dx \leq C \int_0^\infty f^p(x)w(x)dx \tag{1}$$

in the case $v = w$ for non-negative non-increasing functions f, and equivalently, to characterize the boundedness of the maximal operator between Lorentz spaces. The general case for different weights and for different indices p, q was proved by Sawyer [15]. The detailed information on the B_p-class weights can be found, e.g., in Cerda and Martin [2, 3], Kufner et al. [10], Maligranda [12], Sbordone and Wik [16] etc.

We say that $(v, w) \in B_p^{-1}$ if the following holds

$$\int_r^\infty \left(\frac{r}{x} \right)^p v(x)\, dx + \int_0^r v(x)\, dx \geq C \int_0^r w(x)dx, \quad r > 0 \tag{2}$$

In [13], Neugebauer used B_p^{-1} to characterize the reverse of the inequality (1). Precisely, the following was proved.

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Theorem 1.1. *Let $1 \leq p < \infty$ and v, w be weight functions defined on $(0, \infty)$. Then the reverse Hardy inequality*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p v(x)dx \geq C\int_0^\infty f^p(x)w(x)dx$$

holds for some constant $C > 0$ and for all non-negative, non-increasing measurable functions f if and only if $(v, w) \in B_p^{-1}$.

In this paper, we further investigate the class B_p^{-1} and prove a number of properties of weights belonging to this class. We also derive a number of properties of the weight class B_p^* which characterizes the Hardy inequality involving the conjugate Hardy operator

$$A^*f(x) = \frac{1}{x}\int_x^\infty f(t)dt$$

for non-increasing functions. In addition, we study the coresponding inequalities in grand Lebesgue spaces $L^{p)}$ which consist of all those measurable functions f for which

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1}\left(\epsilon\int_0^1 |f(x)|^{p-\epsilon}dx\right)^{\frac{1}{p-\epsilon}} < \infty, \quad p > 1.$$

These spaces were introduced by Iwaniec and Sbordone [8] and were further investigated by Fiorenza [4], Fiorenza and Karadzhov [5], Fiorenza and Rakotoson [6, 7]. Further, the weighted version of the space $L^{p)}$, denoted by $L_w^{p)}$ was introduced and the boundedness of the maximal operator was characterized in such spaces.

We show that the classes of weights characterising certain inequality in $L^{p)}$-spaces is essentially the same as that in $L_w^{p)}$-spaces. Finally, we shall discuss the inequality involving the conjugate averaging operator

$$A^*f(x) = \frac{1}{x}\int_x^\infty f(t)dt$$

in the framework of $L_w^{p)}$-spaces.

The rest of the paper is organized as follows. In Section 2, we study the class B_p^{-1} where we prove a number of properties of this class. The two weight class B_p^* has been investigated in Section 3 and finally in Section 4, we study reverse $L_w^{p)}$-inequalities for non-increasing functions involving averaging operator.

All the functions used in this paper are measurable and non-negative. The alphabet C has been used for a constant which may have a different value at different places but does not create any confusion whatsoever.

2. The class B_p^{-1}

For measurable function f, consider the modified Hardy averaging operator

$$(A_q f)(x) = \frac{1}{x^{1/q}}\int_0^x \frac{f(t)}{t^{1/q'}}dt, \quad q \geq 1, \quad q' = \frac{q}{q-1}.$$

Note that for $q = 1$, $A_q \equiv A$. In [14], Neugebauer proved the following.

Theorem 2.1. *Let $1 \leq p, q < \infty$ and w be a weight function defined on $(0, \infty)$. Then the inequality*

$$\int_0^\infty \left(A_q f\right)^p (x)w(x)dx \leq C\int_0^\infty f^p(x)w(x)dx \tag{3}$$

holds for all non-increasing functions f if and only if $w \in B_{p/q}$.

Our first aim is to characterize the reverse of the inequality (3). In the definition of B_p^{-1}, if $v \equiv w$, we simply write $v \in B_p^{-1}$. In that case, the inequality (2) becomes

$$\int_r^\infty \left(\frac{r}{x}\right)^p v(x)\,dx \geq C \int_0^r v(x)dx, \quad r > 0.$$

The constant C in the above inequality is, of course, different than in (2).

Remark 2.2. *In (2), the value of the constant C is not specified. So, one could think that C could take the value* 1*. In that case $w \equiv v$ would imply that*

$$\int_r^\infty \left(\frac{r}{x}\right)^p v(x)\,dx \geq 0 \tag{4}$$

which seems to be true always and as such there seems to be no meaning of saying that $v \in B_p^{-1}$. But when we say that a particular inequality holds, it means that both the sides should exist finitely. In the present case, for $p = 4$ and $v(x) = x^5$, LHS of (4) is not finite, i.e., (4) does not hold.

Theorem 2.3. *Let $1 \leq p, q < \infty$ and w be a weight function defined on $(0, \infty)$. Then the inequality*

$$\int_0^\infty \left(A_q f\right)^p (x) w(x)dx \geq C \int_0^\infty f^p(x)w(x)dx \tag{5}$$

holds for all non-increasing functions f if and only if $w \in B_{p/q}^{-1}$.

Proof. We use the idea as in [14], Theorem 2.3). The necessity follows by using the function $f = \chi_{[0,r]}$ in ((5)) . For the sufficiency, by change of variable, we get

$$\frac{1}{x^{1/q}} \int_0^x \frac{f(u)}{u^{1/q'}} du = \frac{q}{x^{1/q}} \int_0^{x^{1/q}} f(z^q)dz,$$

so that

$$\begin{aligned} \int_0^\infty \left(A_q f\right)^p (x) w(x)dx &= q \int_0^\infty \left(\frac{1}{x^{1/q}} \int_0^{x^{1/q}} f(z^q)dz\right)^p w(x)dx \\ &= q^2 \int_0^\infty \left(\frac{1}{t} \int_0^t f(z^q)dz\right)^p w(t^q)t^{q-1}dt. \end{aligned}$$

Now, since $w \in B_{p/q}^{-1}$, we find that

$$\begin{aligned} \int_r^\infty \left(\frac{r}{x}\right)^p w(x^q)x^{q-1}dx &= \frac{1}{q} \int_{r^q}^\infty \left(\frac{r^q}{t}\right)^{p/q} w(t)dt \\ &\geq \frac{(C-1)}{q} \int_0^{r^q} w(t)dt \\ &= C \int_0^r w(x^q)x^{q-1}dx, \end{aligned}$$

which implies that $w(t^q)t^{q-1} \in B_p^{-1}$. Consequently, by Theorem 1.1 and applying some variable transformation, the inequality

$$\begin{aligned} \int_0^\infty \left(\frac{1}{t} \int_0^t f(z^q)dz\right)^p w(t^q)t^{q-1}dt &\geq C \int_0^\infty f(t^q)^p w(t^q)t^{q-1}dt \\ &= \frac{C}{q} \int_0^\infty f^p(x)w(x)dx \end{aligned}$$

i.e.

$$\int_0^\infty \left(A_q f\right)^p w(x)dx \geq C \int_0^\infty f^p(x)w(x)dx$$

holds, where we have used the constant C for qC. □

In [13], Neugebauer proved a number of properties for the weight class B_p. Here, we prove some similar properties as applicable for the weight class B_p^{-1}. We have

Theorem 2.4. *For* $1 < q < p < \infty$, *if* $w \in B_p^{-1}$ *then* $w(x^{q-1/p-1}) \in B_q^{-1}$.

Proof. By using change of variable, the fact that $w \in B_p^{-1}$ and again on using change of variable, we will obtain

$$\begin{aligned}
\int_r^\infty \left(\frac{r}{x}\right)^q w(x^{q-1/p-1})dx &= \alpha \int_{r^{1}/\alpha}^\infty \left(\frac{r}{u^\alpha}\right)^q \left(\frac{1}{u^{1-\alpha}}\right) w(u)du \\
&= \alpha r^{q-p/\alpha} \int_{r^{1}/\alpha}^\infty \left(\frac{r^{1/\alpha}}{u}\right)^p w(u)du \\
&\geq C\alpha r^{q-p/\alpha} \int_0^{r^{1}/\alpha} w(u)du \\
&= Cr^{1-1/\alpha} \int_0^r w(x^{1/\alpha})x^{1/\alpha-1}dx \\
&\geq Cr^{1-1/\alpha} r^{1/\alpha-1} \int_0^r w(x^{1/\alpha})dx \\
&= C \int_0^r w(x^{q-1/p-1})dx,
\end{aligned}$$

where $\alpha = \frac{p-1}{q-1}$, which proves the theorem. □

Theorem 2.5. *Let* $1 < q < p < \infty$. *If* $(v, w) \in B_p^{-1}$, *then* $(v, w) \in B_q^{-1}$.

Proof. In view of the monotonicity, we find that

$$\int_r^\infty \left(\frac{r}{x}\right)^q v(x)dx \geq \int_r^\infty \left(\frac{r}{x}\right)^p v(x)dx$$

and the result follows immediately. □

Theorem 2.6. *If* $w \in B_p^{-1}$, *then for all* $\epsilon > 0$, $x^\epsilon w(x^{1+\epsilon}) \in B_p^{-1}$.

Proof. It is clear that $w \in B_{p/1+\epsilon}^{-1}$ for all $\epsilon > 0$. The result now follows using this fact and some variable transformation. Indeed, we have

$$\begin{aligned}
\int_r^\infty \left(\frac{r}{x}\right)^p x^\epsilon w(x^{1+\epsilon})dx &= \frac{1}{(1+\epsilon)} \int_{r^{1+\epsilon}}^\infty (\frac{r^{1+\epsilon}}{u})^{\frac{p}{1+\epsilon}} w(u)du \\
&\geq \frac{C}{1+\epsilon} \int_0^{r^{1+\epsilon}} w(u)du \\
&= C \int_0^r x^\epsilon w(x^{1+\epsilon})dx,
\end{aligned}$$

and the theorem is proved. □

Theorem 2.7. *Let* $w \in B_1^{-1}$ *and* $\alpha \leq 1$. *Then* $w(x^\alpha) \in B_1^{-1}$.

Proof. By variable transformation and the fact that $w \in B_1^{-1}$, we have

$$\begin{aligned}\int_r^\infty \left(\frac{r}{x}\right) w(x^\alpha)dx &= \frac{1}{\alpha} r^{1-\alpha} \int_{r^\alpha}^\infty \left(\frac{r^\alpha}{u}\right) w(u)du \\ &\geq \frac{C}{\alpha} r^{1-\alpha} \int_0^{r^\alpha} w(u)du \\ &= Cr^{1-\alpha} \int_0^r w(x^\alpha)x^{\alpha-1}dx \\ &\geq C \int_0^r w(x^\alpha)dx,\end{aligned}$$

hence the theorem. □

Theorem 2.8. *Let* $1 < p < \infty$. *Then* $w \in B_p^{-1}$ *if and only if* $w(x) = u(x)x^{p-1}$ *with* $u(x^{\frac{1}{p}}) \in B_1^{-1}$.

Proof. Assume first that $w \in B_p^{-1}$. Then

$$\begin{aligned}\int_r^\infty \left(\frac{r}{x}\right) \frac{w(x^{1/p})}{x^{1/p'}}dx &= p \int_{r^{1/p}}^\infty \left(\frac{r^{1/p}}{t}\right)^p w(t)dt \\ &\geq pC \int_0^{r^{1/p}} w(t)dt \\ &= C \int_0^r \frac{w(x^{1/p})}{x^{1/p'}}dx.\end{aligned}$$

Thus, if we write

$$u(x^{\frac{1}{p}}) = \frac{w(x^{1/p})}{x^{1/p'}}, \tag{6}$$

then we have proved that $u(x^{\frac{1}{p}}) \in B_1^{-1}$. At the same time taking $x^{1/p} = t$ in 6, we find that $w(t) = u(t)t^{p-1}$ and the assertion follows. Conversely, assume that $w(x) = u(x)x^{p-1}$ with $u(x^{\frac{1}{p}}) \in B_1^{-1}$. We have

$$\begin{aligned}\int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx &= \int_r^\infty \left(\frac{r}{x}\right)^p u(x)x^{p-1}dx \\ &= \frac{1}{p} \int_{r^p}^\infty \left(\frac{r^p}{t}\right) u(t^{\frac{1}{p}})dt \\ &\geq \frac{C}{p} \int_0^{r^p} u(t^{\frac{1}{p}})dt \\ &= C \int_0^r u(x)x^{p-1}dx \\ &= C \int_0^r w(x)dx\end{aligned}$$

and the theorem is proved. □

3. The class B_p^*

On the similar lines, we prove some similar properties as applicable for the weight class B_p^*.

Theorem 3.1. *For $1 < q < p < \infty$, if $w \in B_q^*$, then $w(x^{p-1/q-1}) \in B_p^*$.*

Proof. By using change of variable, the fact that $w \in B_q^*$ and again using change of variable, we get

$$\begin{aligned}\int_0^r \left(\frac{r}{x}\right)^p w(x^\alpha)dx &= \frac{1}{\alpha}\int_0^{r^\alpha}\left(\frac{r}{u^{\frac{1}{\alpha}}}\right)^p\left(\frac{1}{u^{1-\frac{1}{\alpha}}}\right)w(u)du \\ &= \frac{1}{\alpha}r^{p-q\alpha}\int_0^{r^\alpha}\left(\frac{r^\alpha}{u}\right)^q w(u)du \\ &\leq \frac{C}{\alpha}r^{p-q\alpha}\int_0^{r^\alpha} w(u)du \\ &= Cr^{p-q\alpha}\int_0^r w(x^\alpha)x^{\alpha-1}dx \\ &\leq Cr^{1-\alpha}r^{\alpha-1}\int_0^r w(x^\alpha)dx \\ &= C\int_0^r w(x^\alpha)dx,\end{aligned}$$

where $\alpha = \frac{p-1}{q-1}$. □

Theorem 3.2. *Let $1 < q < p < \infty$. If $w \in B_p^*$, then $w \in B_q^*$.*

Proof. In view of the monotonicity, we find that

$$\int_0^r \left(\frac{r}{x}\right)^q w(x)dx \geq \int_0^r \left(\frac{r}{x}\right)^p w(x)dx$$

and the result follows immediately. □

Theorem 3.3. *If $w \in B_p^*$, then for all $\epsilon > 0$, $x^\epsilon w(x^{1+\epsilon}) \in B_p^*$.*

Proof. Using the previous theorem, we have $w \in B_{p/1+\epsilon}^*$ for all $\epsilon > 0$. The result now follows using this fact and some variable transformation. Indeed, we have

$$\begin{aligned}\int_0^r \left(\frac{r}{x}\right)^p x^\epsilon w(x^{1+\epsilon})dx &= \frac{1}{(1+\epsilon)}\int_0^{r^{1+\epsilon}}\left(\frac{r^{1+\epsilon}}{u}\right)r^{\frac{p}{1+\epsilon}}w(u)du \\ &\leq \frac{C}{1+\epsilon}\int_0^{r^{1+\epsilon}} w(u)du \\ &= C\int_0^r x^\epsilon w(x^{1+\epsilon})dx,\end{aligned}$$

which proves the result. □

Theorem 3.4. *Let $w \in B_1^*$ and $\alpha > 1$. Then $w(x^\alpha) \in B_1^*$.*

Proof. By variable transformation and the fact that $w \in B_1^*$, we have

$$\int_0^r \left(\frac{r}{x}\right) w(x^\alpha)dx = \frac{1}{\alpha} r^{1-\alpha} \int_0^{r^\alpha} \left(\frac{r^\alpha}{u}\right) w(u)du \le \frac{C}{\alpha} r^{1-\alpha} \int_0^{r^\alpha} w(u)du$$
$$= Cr^{1-\alpha} \int_0^r w(x^\alpha)x^{\alpha-1}dx$$
$$\le C \int_0^r w(x^\alpha)dx,$$

and the result is proved. □

Theorem 3.5. *Let $1 < p < \infty$. Then $w \in B_p^*$ if and only if $w(x) = u(x)x^{p-1}$ with $u(x^{\frac{1}{p}}) \in B_1^*$.*

Proof. Assume first that $w \in B_p^*$. Then

$$\int_0^r \left(\frac{r}{x}\right) \frac{w(x^{1/p})}{x^{1/p'}}dx = p \int_0^{r^{1/p}} \left(\frac{r^{1/p}}{t}\right)^p w(t)dt$$
$$\le pC \int_0^{r^{1/p}} w(t)dt$$
$$= C \int_0^r \frac{w(x^{1/p})}{x^{1/p'}}dx.$$

Thus, if we write

$$u(x^{\frac{1}{p}}) = \frac{w(x^{1/p})}{x^{1/p'}}, \tag{7}$$

then we have proved that $u(x^{\frac{1}{p}}) \in B_1^*$. At the same time taking $x^{1/p} = t$ in ((7)) , we find that $w(t) = u(t)t^{p-1}$ and the assertion follows. Conversely, assume that $w(x) = u(x)x^{p-1}$ with $u(x^{\frac{1}{p}}) \in B_1^*$. We have

$$\int_0^r \left(\frac{r}{x}\right)^p w(x)dx = \int_0^r \left(\frac{r}{x}\right)^p u(x)x^{p-1}dx = \frac{1}{p} \int_0^{r^p} \left(\frac{r^p}{t}\right) u(t^{\frac{1}{p}})dt$$
$$\le \frac{C}{p} \int_0^{r^p} u(t^{\frac{1}{p}})dt$$
$$= C \int_0^r u(x)x^{p-1}dx$$
$$= C \int_0^r w(x)dx$$

and the theorem is proved. □

4. Applications to Grand Lebesgue Spaces

In this section, we shall study some inequalities in the framework of weighted grand Lebesgue spaces $L_w^{p)}$: These spaces consist of all those measurable functions f for which

$$\|f\|_{p),w} := \sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 |f(x)|^{p-\epsilon} w(x)dx\right)^{\frac{1}{p-\epsilon}} < \infty, \quad p > 1.$$

Jain and Kumari [9] proved that the averaging operator A is bounded between $L^{p)}_w$ spaces for non-increasing functions if and only if $w \in B_p$. In other words, it was proved that $L^{p)}_w$-boundedness and L^p_w-boundedness of A are equivalent, where L^p_w is used to denote weighted L^p-space. The equivalence of $L^{p)}_w$-boundedness and L^p_w-boundedness of the maximal operator has been proved in terms of the famous A_p-condition.

In this section, we investigate the corresponding result of Theorem 1.1 in the context of $L^{p)}_w$ spaces. These spaces require that the functions should be defined on bounded intervals, say,$(0,1)$. Note that Theorem A is true for all functions which are non-negative and non-increasing. Among these functions, if we choose those which are supported in $(0,1)$, the result remains valid. However, in the corresponding two weighted B_p^{-1} condition, the integral $\int_r^\infty$ will be replaced by $\int_r^1$. In order to avoid any ambiguity, we shall denote this modified condition by $B_p^{-1}(0,1)$. Thus, we have the following modification of Theorem 1.1.

Theorem 4.1. *Let $1 \le p < \infty$ and v, w be weight functions defined on $(0,\infty)$. Then the reverse Hardy inequality*

$$\int_0^1 \left(\frac{1}{x}\int_0^x f(t)dt\right)^p v(x)dx \ge C\int_0^1 f^p(x)w(x)dx \tag{8}$$

holds for some constant $C > 0$ and for all non-negative, non-increasing measurable functions f if and only if $(v,w) \in B_p^{-1}(0,1)$, i.e.,

$$\int_r^1 \left(\frac{r}{x}\right)v(x)dx + \int_0^1 v(x)dx \ge C\int_0^1 w(x)dx, \quad 0 < r < 1.$$

Remark 4.2. *The result of Theorem 2.5 is valid for the class $(v,w) \in B_p^{-1}(0,1)$ too, i.e., $(v,w) \in B_p^{-1}(0,1)$ implies $(v,w) \in B_q^{-1}(0,1)$ for $1 < q < p < \infty$. Indeed, the implication follows by monotonicity.*

We now prove the following.

Theorem 4.3. *Let $1 < p < \infty$ and v, w be weight functions defined on $(0,1)$. The necessary condition for the inequality*

$$||Af||_{p),v} \ge ||f||_{p),w} \tag{9}$$

to hold for all non-negative and non-increasing functions f is $(v,w) \in B_p^{-1}(0,1)$.

Proof. Let $(v,w) \in B_p^{-1}(0,1)$ and $0 < \sigma < p-1$. We have

$$\begin{aligned}
||f||_{p),w} &= \max\left\{\sup_{0<\epsilon<\sigma}\left(\epsilon\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\epsilon}}, \sup_{\sigma\le\epsilon<p-1}\left(\epsilon\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\epsilon}}\right\}\\
&\le \max\left\{\sup_{0<\epsilon<\sigma}\left(\epsilon\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\epsilon}}, \sup_{\sigma\le\epsilon<p-1}(\epsilon)^{\frac{1}{p-\epsilon}}\sigma^{-\frac{1}{p-\sigma}}\sigma^{\frac{1}{p-\sigma}}\left(\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\sigma}}\right\}\\
&\le (p-1)\sigma^{-\frac{1}{p-\sigma}}\sup_{0<\epsilon<\sigma}\left(\epsilon\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\epsilon}}.
\end{aligned} \tag{10}$$

Since $0 < \epsilon < \sigma$, therefore $p - \epsilon > 1$. In view of Remark 4.2 , $(v,w) \in B_{p-\epsilon}^{-1}(0,1)$. Then, in view of Theorem 4.1, the Inequality 8 with p replaced by $p-\epsilon$ holds. In the corresponding inequality, multiplying both sides by $\epsilon^{\frac{1}{p-\epsilon}}$, we obtain

$$C\left(\epsilon\int_0^1 [f(x)]^{p-\epsilon}w(x)dx\right)^{\frac{1}{p-\epsilon}} \le \left(\epsilon\int_0^1 [Af(x)]^{p-\epsilon}v(x)dx\right)^{\frac{1}{p-\epsilon}},$$

which on passing to the sup over $0 < \epsilon < \sigma$ gives

$$C \sup_{0<\epsilon<\sigma} \left(\epsilon \int_0^1 [f(x)]^{p-\epsilon} w(x)dx\right)^{\frac{1}{p-\epsilon}} \leq \sup_{0<\epsilon<\sigma} \left(\epsilon \int_0^1 [Af(x)]^{p-\epsilon} v(x)dx\right)^{\frac{1}{p-\epsilon}} \leq \|Af\|_{p),v}.$$

Combining the last estimate with (10), we get

$$\|Af\|_{p),v} \geq \frac{C}{p-1} \sigma^{\frac{1}{p-\sigma}} \|f\|_{p),w},$$

which is true for all $\sigma \in (0, p-1)$. Therefore, we have

$$\|Af\|_{p),v} \geq C(p, v, w) \|f\|_{p),w}$$

with

$$C(p, v, w) = \frac{C}{p-1} \sup_{0<\epsilon<\sigma} \sigma^{\frac{1}{p-\sigma}},$$

and the result is proved. □

For the converse of the above theorem, we have the following.

Theorem 4.4. *Let $1 < p < \infty$, $\sigma \in (0, p-1)$ and v, w be weight functions defined on $(0, 1)$. The sufficient condition for the inequality $\|Af\|_{p),v} \geq \|f\|_{p),w}$ for non-negative and non-increasing function f to hold is $(v, w) \in B^{-1}_{p-\sigma}(0, 1)$.*

Proof. Let $\|Af\|_{p),v} \geq \|f\|_{p),w}$ i.e.

$$\sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-\epsilon} v(x)dx\right)^{\frac{1}{p-\epsilon}} \geq C \sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 (f(x))^{p-\epsilon} w(x)dx\right)^{\frac{1}{p-\epsilon}}$$

hold. Then there exists a $\sigma \in (0, p-1)$ such that the inequality

$$\sigma\Big(\int_0^1 \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-\sigma} v(x)dx\Big)^{\frac{1}{p-\sigma}} \geq C \sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 (f(x))^{p-\epsilon} w(x)dx\right)^{\frac{1}{p-\epsilon}}$$

holds. This implies that the LHS dominates the RHS for every $\epsilon \in (0, p-1)$ and in particular, for $\epsilon = \sigma$. Consequently, the inequality

$$\int_0^1 \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-\sigma} v(x)dx \geq C^{p-\sigma} \int_0^1 (f(x))^{p-\sigma} w(x)dx$$

holds. Now, consider the function $f = (0, r)$ for a fixed $0 < r < 1$, which is a non-negative and non-increasing function. With this f the last inequality becomes

$$\int_r^1 \left(\frac{r}{x}\right)^{p-\sigma} v(x)dx + \int_0^r v(x)dx \geq C^{p-\sigma} \int_0^r w(x)dx$$

which means that $(v, w) \in B^{-1}_{p-\sigma}(0, 1)$ and we are done. □

As regards a kind of converse of Theorem 2.5, we believe that the following should be true.

Conjecture 4.5. *Let $1 < p < \infty$ and v, w be weight functions defined on $(0, \infty)$. If $(v, w) \in B^{-1}_p$, then there exists $\epsilon > 0$ such that $(v, w) \in B^{-1}_{p+\epsilon}$.*

References

[1] M. Arino and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.
[2] J. Cerda and J. Martin, *Conjugate Hardy's inequalities with decreasing weights*, Proc. Amer. Math. Soc. **126** (1998), 2341–2344.
[3] J. Cerda and J. Martin, *Weighted Hardy inequalities and Hardy transforms of weights*, Studia Math. **132** (2000), 189–196.
[4] A. Fiorenza, *Duality and reflexivity in grand Lebesgue spaces*, Collect. Math. **51** (2000), 131–148.
[5] A. Fiorenza and G. E. Karadzhov, *Grand and small Lebesgue spaces and their analogs*, Zeit. fur Anal. Anwen. **23** (2004), 657–681.
[6] A. Fiorenza and J. M. Rakotoson, *New properties of small Lebesgue spaces and their applications*, Math. Ann. **326** (2003), 543–561.
[7] A. Fiorenza and J. M. Rakotoson, *Compactness, interpolation inequalities for small Lebesgue-Sobolev spaces and applications*, Calc. Var. Partial Differential Equations **25** (2006), 187–203.
[8] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rat. Mach. Anal. **119** (1992), 129–143.
[9] P. Jain and S. Kumari, *On grand Lorentz spaces and the maximal operator*, Georgian Math. J. **19** (2012), 235–246.
[10] A. Kufner, L. Maligranda and L. E. Persson, *The Hardy Inequality. About Its History and Some Related Results*, Pilson, 2007.
[11] A. Kufner and L. E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[12] L. Maligranda, *Weighted estimates of integral operators for decreasing functions*, In: Proc. of Internat. Conference Boundary Value Problems, Special Functions and Fractional Calculus (Belarus, Minsk, Feb. 16–20, 1996), Belgosuniversitet, Minsk (1996), 226–236.
[13] C. J. Neugebauer, *Weighted norm inequalities for averaging operators of monotone functions*, Pub. Mat. **35** (1991), 429–447.
[14] C. J. Neugebauer, *Some classical operators on Lorentz space*, Forum Math. **4** (1992), 135–146.
[15] E. T. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
[16] C. Sbordone and I. Wik, *Maximal functions and related weight classes*, Publ. Mat. **38** (1994), 127–155.