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∗-Ricci tensor on three dimensional almost coKähler manifolds

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Abstract. In this paper, we obtain some classification results of three-dimensional non-coKähler almost coKähler manifold M whose Reeb vector field is strongly normal unit vector field with $\xi(\|\nabla_\xi h\|) = 0$, for which the ∗-Ricci tensor is of Codazzi-type or M satisfies the curvature condition $Q^* \cdot R = 0$.

1. Introduction

Corresponding to Ricci tensor, Tachibana in [22] introduced the concept of ∗-Ricci tensor. In [10] Hamada applied these ideas to real hypersurfaces in complex space form. The ∗-Ricci tensor S^* is defined by

$$S^*(X,Y) = \frac{1}{2} trace\{\varphi \circ R(X, \varphi Y)\}, \tag{1}$$

for all vector fields X, Y, where φ is a (1,1)-tensor field. If ∗-Ricci tensor is a constant multiple of g, then M is said to be ∗-Einstein manifold. Hamada gave a complete classification of ∗-Einstein hypersurfaces, and further Ivey and Ryan [12] updated and refined the work of Hamada [10]. It is important to note that Kaimakamis and Panagiotidou [13] introduced the concept of ∗-Ricci soliton in non-flat complex space form as a generalization of ∗-Einstein metric. Further, the idea of ∗-Ricci solitons in almost contact metric manifolds was extensively studied by many authors in [5, 7, 11, 23, 24].

As a special class of almost contact metric manifolds and analogy of Kähler manifolds, the geometry of (almost) coKähler manifolds was first introduced by Blair [1] and studied by Goldberg and Yano [8] and Olszak [18]. Such manifolds are actually the almost cosymplectic manifolds studied in the above literature. Due to Li's [14] work, recently many authors in their papers adopted this new terminology. From Li's work we are aware that the coKähler manifolds are really odd dimensional analogues of Kähler manifolds. In a recent survey [3], the authors collected some new results concerning (almost) coKähler manifolds both from geometrical and topological point of view. Perrone [20, 21] obtained a complete classification results of three-dimensional almost coKähler manifolds which are homogeneous or the Reeb vector field is minimal and also gave a local characterization of such manifolds.

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In recent years, many classification results on three-dimensional almost coKähler manifolds are emerged. For instance, Cho [4], studied Reeb flow symmetry (that is, the Ricci tensor is invariant along the Reeb flow) on three-dimensional almost coKähler manifolds. Moreover, the authors respectively in [6, 15, 26] considered local φ-symmetry, curvature and ball homogeneities in three-dimensional almost coKähler manifolds. Some other symmetry properties in terms of the Ricci operators, such as Codazzi-type, η-parallelism and transversal Killing on three-dimensional almost coKähler manifolds were also studied in [19, 27]. The authors in [11] studied contact metric generalized (κ,μ)-space form under some curvature condtion in terms of $*$-Ricci tensor, such as η-recurrent, $*$-Ricci semi-symmetry and globally φ-$*$-Ricci symmetry. Motivated by the above studies, in the present paper we start to study Codazzi-type $*$-Ricci tensor and curvature condtion $Q^* \cdot R = 0$ on three-dimensional almost coKähler manifolds under some reasonable conditions for the first time.

2. Almost coKähler three-manifolds

Let M be a smooth differentiable manifold of dimension $2n+1$. On M, if there exist a $(1,1)$-tensor field φ, a characterstic vector field ξ, a 1-form η and a Riemannian metric g such that

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,\\ &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{2}$$

for any vector fields X, Y, then we say that M admits an almost contact metric structure. We call ξ as a Reeb vector field. As a result of (2) we have $\varphi(\xi) = 0$, $\eta(\varphi) = 0$. One can define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, u\frac{d}{dt}\right) = \left(\varphi X - u\xi, \eta(X)\frac{d}{dt}\right),$$

where t is the coordinate of $\mathbb{R}$ and u is a smooth function. If the aforementioned structure J is integrable, then we say that an almost contact structure is normal, and this is equivalent to require

$$[\varphi,\varphi] = -2d\eta \otimes \xi,$$

where $[\varphi,\varphi]$ indicates the Nijenhuis tensor of φ.

In this paper, by an almost coKähler manifold we mean an almost contact metric manifold (M,φ,ξ,η,g) in which η and Φ are closed, where the fundamental 2-form Φ of almost contact metric manifold M is defined by $\Phi(X,Y) = g(X,\varphi Y)$, for all vector fields X and Y. An almost coKähler manifold is said to be coKähler manifold (see [14]) if the associated almost contact structure is normal, which is also equivalent to

$$\nabla\varphi = 0, \quad (\nabla\Phi = 0).$$

On almost coKähler manifold, we set three (1,1)-type tensor fields $h = \frac{1}{2}\pounds_\xi g$, where $\pounds$ is the Lie differentiation, Jacobi operator $\ell = R(\cdot,\xi)\xi$ generated by ξ and $h' = h \circ \varphi$, where R is the Riemannian curvature tensor. From [2, 18], we are aware that ℓ, h and h' are symmetric and satisfy

$$h\xi = \ell\xi = 0, \quad tr(h) = tr(h') = 0, \tag{3}$$

$$h\varphi + \varphi h = 0, \quad \nabla\xi = h', \quad div\ \xi = 0, \tag{4}$$

$$\nabla_\xi h = -h^2\varphi - \varphi\ell, \quad \varphi\ell\varphi - \ell = 2h^2, \tag{5}$$

where tr and div indicates the trace and divergence operators, respectively. The well-known Ricci tensor S is defined by

$$S(X,Y) = g(QX,Y) = tr\{Z \to R(Z,X)Y\},$$

where Q denotes the Ricci operator. Note that a three-dimensional almost coKähler manifold is coKähler if and only if h vanishes. In this connection it is worth to note that (almost) coKähler manifold in fact is the (almost) cosymplectic manifold studied in [4, 20].

Let us recall some useful formula listed in [21]. Let $\mathcal{U}_1$ be the open subset of three-dimensional almost coKähler manifold M satisfying $h \neq 0$ and $\mathcal{U}_2$ be the open subset of M which is defined by $\mathcal{U}_2 = \{p \in M : h = 0 \text{ in a neighborhood of } p\}$. Consequently, $\mathcal{U}_1 \cup \mathcal{U}_2$ is open and dense in M and there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we set $h(e) = \lambda e$ and hence $h\varphi e = -\lambda\varphi e$, where λ is a positive function on $\mathcal{U}_1$. The eigenvalue function λ is continuous on M and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 2.1. *On $\mathcal{U}_1$, the Levi-Civita connection is given by*

$$\nabla_\xi e = f\varphi e, \quad \nabla_\xi \varphi e = -fe, \quad \nabla_e \xi = -\lambda\varphi e, \quad \nabla_{\varphi e}\xi = -\lambda e,$$
$$\nabla_e e = \frac{1}{2\lambda}(\varphi e(\lambda) + \sigma(e))\varphi e, \quad \nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}(e(\lambda) + \sigma(\varphi e))e,$$
$$\nabla_{\varphi e} e = \lambda\xi - \frac{1}{2\lambda}(e(\lambda) + \sigma(\varphi e))\varphi e, \quad \nabla_e \varphi e = \lambda\xi - \frac{1}{2\lambda}(\varphi e(\lambda) + \sigma(e))e,$$

where f is a smooth function and σ is the 1-form defined by $\sigma(\cdot) = S(\cdot, \xi)$.

As a result of above lemma, we have the following Poisson brackets:

$$\begin{aligned} &[\xi, e] = (\lambda + f)\varphi e, \quad [\xi, \varphi e] = (\lambda - f)e, \\ &[e, \varphi e] = \frac{1}{2\lambda}(e(\lambda) + \sigma(\varphi e))\varphi e - \frac{1}{2\lambda}(\varphi e(\lambda) + \sigma(e))e. \end{aligned} \tag{6}$$

Putting (6) into the well-known Jacobi identity $[[\xi, e], \varphi e] + [[e, \varphi e], \xi] + [[\varphi e, \xi], e] = 0$, we obtain

$$\begin{aligned} &e(\lambda - f) + \xi\left(\frac{\varphi e(\lambda) + \sigma(e)}{2\lambda}\right) + \frac{f - \lambda}{2\lambda}(e(\lambda) + \sigma(\varphi e)) = 0, \\ &\varphi e(\lambda + f) + \xi\left(\frac{e(\lambda) + \sigma(\varphi e)}{2\lambda}\right) - \frac{f + \lambda}{2\lambda}(\varphi e(\lambda) + \sigma(e)) = 0. \end{aligned} \tag{7}$$

The Ricci operator Q of three-dimensional almost coKähler manifold is expressed (see Proposition 4.1 in [21]) on $\mathcal{U}_1$ by

$$\begin{aligned} Q\xi &= -2\lambda^2\xi + \sigma(e)e + \sigma(\varphi e)\varphi e, \\ Qe &= \sigma(e)\xi + \frac{1}{2}(r + 2\lambda^2 - 4f\lambda)e + \xi(\lambda)\varphi e, \\ Q\varphi e &= \sigma(\varphi e)\xi + \xi(\lambda)e + \frac{1}{2}(r + 2\lambda^2 + 4f\lambda)\varphi e, \end{aligned} \tag{8}$$

with respect to the local basis $\{\xi, e, \varphi\}$, where r denotes the scalar curvature.

3. $*$-Ricci tensor on almost coKähler three-manifolds

In this section, first we classify three-dimensional almost coKähler manifolds whose $*$-Ricci tensor is of Codazzi-type, that is,

$$(\nabla_X Q^*)Y = (\nabla_Y Q^*)X, \tag{9}$$

for any vector fields X and Y.

Before giving our main results, we first find the expression of $*$-Ricci operator on non-coKähler almost coKähler three-manifold with respect to the local basis $\{\xi, e, \varphi e\}$.

Lemma 3.1. *The $*$-Ricci opearator Q^* of three-dimensional almost coKähler manifold is expressed on $\mathcal{U}_1$ by*

$$Q^*\xi = \sigma(e)e + \sigma(\varphi e)\varphi e, \quad Q^*e = \left(\frac{r}{2} + 2\lambda^2\right)e, \quad Q^*\varphi e = \left(\frac{r}{2} + 2\lambda^2\right)\varphi e, \tag{10}$$

with respect to $\{\xi, e, \varphi e\}$.

Proof. It is well known that the curvature tensor R of any three-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ &- \frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \end{aligned}$$

for any vector fields X, Y, Z. Applying (8), the curvature tensor R of a non-coKähler three-dimensional almost coKähler manifold M can be given as the following:

$$R(e,\xi)\xi = -\lambda(\lambda + 2f)e + \xi(\lambda)\varphi e, \tag{11}$$
$$R(\varphi e,\xi)\xi = \xi(\lambda)e - \lambda(\lambda - 2f)\varphi e, \tag{12}$$
$$R(e,\xi)e = \lambda(\lambda + 2f)\xi - \sigma(\varphi e)\varphi e, \tag{13}$$
$$R(e,\xi)\varphi e = -\xi(\lambda)\xi + \sigma(\varphi e)e, \tag{14}$$
$$R(\varphi e,\xi)e = -\xi(\lambda)\xi + \sigma(e)\varphi e, \tag{15}$$
$$R(\varphi e,\xi)\varphi e = \lambda(\lambda - 2f)\xi - \sigma(e)e, \tag{16}$$
$$R(e,\varphi e)\xi = \sigma(\varphi e)e - \sigma(e)\varphi e, \tag{17}$$
$$R(e,\varphi e)e = -\sigma(\varphi e)\xi - \left(\frac{r}{2} + 2\lambda^2\right)\varphi e, \tag{18}$$
$$R(e,\varphi e)\varphi e = \sigma(e)\xi + \left(\frac{r}{2} + 2\lambda^2\right)e. \tag{19}$$

By the definition of $*$-Ricci tensor, we have

$$\begin{aligned} S^*(X,Y) &= \frac{1}{2}\sum_{i=1}^{3} g(\varphi R(X,\varphi Y)e_i, e_i) \\ &= -\frac{1}{2}\sum_{i=1}^{3} g(R(e_i,\varphi e_i)X, \varphi Y) \\ &= \frac{1}{2}\sum_{i=1}^{3} g(\varphi R(e_i,\varphi e_i)X, Y), \end{aligned}$$

where $e_1 = \xi$, $e_2 = e$ and $e_3 = \varphi e$. In this sequel, we can write

$$\begin{aligned} Q^*X &= \frac{1}{2}\sum_{i=1}^{3} \varphi R(e_i,\varphi e_i)X \\ &= \frac{1}{2}\{\varphi R(e,\varphi e)X - \varphi R(\varphi e, e)X\}. \end{aligned} \tag{20}$$

Emplyoing $X = \xi$ in above equation, recalling (17) we obtain

$$\begin{aligned} Q^*\xi &= \varphi R(e,\varphi e)\xi \\ &= \sigma(e)e + \sigma(\varphi e)\varphi e. \end{aligned}$$

Simillarly, setting X by e and φe separately in (20), utilization of (18) and (19) gives second and third term of (10) respectively. □

Proposition 3.2. *The $*$-Ricci tensor of three-dimensional almost coKähler manifold is symmetric if and only if Reeb vector field is an eigenvector field of the Ricci operator.*

Proof. As a result of Lemma 3.1, we have

$$S^*(\xi, e) = g(Q^*\xi, e) = \sigma(e), \quad S^*(e, \xi) = g(Q^*e, \xi) = 0,$$
$$S^*(e, \varphi e) = g(Q^*e, \varphi e = 0, \quad S^*(\xi, \varphi e) = g(Q^*\xi, \varphi e) = \sigma(\varphi e),$$
$$S^*(\varphi e, \xi) = g(Q^*\varphi e, \xi) = 0, \quad S^*(\varphi e, e) = g(Q^*\varphi e, e) = 0$$

Above relations enables us to conclude that S^* is symmetric if and only if $\sigma(e) = \sigma(\varphi e) = 0$, that is, Reeb vector field is an eigenvector field of the Ricci operator. □

Remark 3.3. *It is worth to remark that the $*$-Ricci tensor is not symmetric for three-dimensional almost coKähler manifolds. But, our Proposition 3.2 gives a necessary and sufficient condition for the $*$-Ricci tensor to be symmetric.*

Lemma 3.4. *The $*$-Ricci operator of three-dimensional non-coKähler almost coKähler manifold is of Codazzi type if and only if Reeb vector field is an eigenvector field of the Ricci operator and $r = -4\lambda^2$.*

Proof. On $\mathcal{U}_1$ by applying Lemma 2.1 and relation (10) we obtain the following equations:

$$(\nabla_\xi Q^*)\xi = (\xi(\sigma(e)) - f\sigma(\varphi e))e + (\xi(\sigma(\varphi e)) + f\sigma(e))\varphi e, \tag{21}$$

$$(\nabla_\xi Q^*)e = \xi\left(\frac{r}{2} + 2\lambda^2\right)e, \quad (\nabla_\xi Q^*)\varphi e = \xi\left(\frac{r}{2} + 2\lambda^2\right)\varphi e, \tag{22}$$

$$(\nabla_e Q^*)e = e\left(\frac{r}{2} + 2\lambda^2\right)e, \quad (\nabla_{\varphi e} Q^*)\varphi e = \varphi e\left(\frac{r}{2} + 2\lambda^2\right)\varphi e, \tag{23}$$

$$(\nabla_e Q^*)\varphi e = \lambda\left(\frac{r}{2} + 2\lambda^2\right)\xi - \lambda\sigma(e)e + (e\left(\frac{r}{2} + 2\lambda^2\right) - \lambda\sigma(\varphi e))\varphi e, \tag{24}$$

$$(\nabla_{\varphi e} Q^*)e = \lambda\left(\frac{r}{2} + 2\lambda^2\right)\xi + (\varphi e\left(\frac{r}{2} + 2\lambda^2\right) - \lambda\sigma(e))e - \lambda\sigma(\varphi e)\varphi e, \tag{25}$$

$$\begin{aligned}(\nabla_e Q^*)\xi = \lambda\sigma(\varphi e)\xi &+ \left\{e(\sigma(e)) - \frac{\sigma(\varphi e)}{2\lambda}(\varphi e(\lambda) + \sigma(e))\right\}e \\ &\left\{\lambda\left(\frac{r}{2} + 2\lambda^2\right) + e(\sigma(\varphi e)) + \frac{\sigma(e)}{2\lambda}(\varphi e(\lambda) + \sigma(e))\right\}\varphi e,\end{aligned} \tag{26}$$

$$\begin{aligned}(\nabla_{\varphi e} Q^*)\xi = \lambda\sigma(e)\xi &+ \left\{\lambda\left(\frac{r}{2} + 2\lambda^2\right) + \varphi e(\sigma(e)) + \frac{\sigma(\varphi e)}{2\lambda}(e(\lambda) + \sigma(\varphi e))\right\}e \\ &\left\{\varphi e(\sigma(\varphi e)) - \frac{\sigma(e)}{2\lambda}(e(\lambda) + \sigma(\varphi e))\right\}\varphi e.\end{aligned} \tag{27}$$

Let us suppose that the $*$-Ricci operator of M is of Codazzi-type. Then switching $X = e$ and $Y = \xi$ into (9) we obtain $(\nabla_e Q^*)\xi - (\nabla_\xi Q^*)e = 0$. In this relation, applying (26) and first term of (22) we get

$$\begin{aligned}&\lambda\sigma(\varphi e) = 0, \\ &e(\sigma(e)) - \frac{\sigma(\varphi e)}{2\lambda}(\varphi e(\lambda) + \sigma(e)) - \xi\left(\frac{r}{2} + 2\lambda^2\right) = 0, \\ &\lambda\left(\frac{r}{2} + 2\lambda^2\right) + e(\sigma(\varphi e)) + \frac{\sigma(e)}{2\lambda}(\varphi e(\lambda) + \sigma(e)) = 0.\end{aligned} \tag{28}$$

Similarly, setting $X = \varphi e$ and $Y = \xi$ into (9) we have $(\nabla_{\varphi e} Q^*)\xi - (\nabla_\xi Q^*)\varphi e = 0$. In this relation, using (27) and second term of (22) we obtain

$$\begin{aligned}&\lambda\sigma(e) = 0, \\ &\lambda\left(\frac{r}{2} + 2\lambda^2\right) + \varphi e(\sigma(e)) + \frac{\sigma(\varphi e)}{2\lambda}(e(\lambda) + \sigma(\varphi e)) = 0, \\ &\varphi e(\sigma(\varphi e)) - \frac{\sigma(e)}{2\lambda}(e(\lambda) + \sigma(\varphi e)) - \xi\left(\frac{r}{2} + 2\lambda^2\right) = 0.\end{aligned} \tag{29}$$

Employing $X = e$ and $Y = \varphi e$ into (9) we obtain $(\nabla_e Q^*)\varphi e - (\nabla_{\varphi e} Q^*)e = 0$. In this relation, applying (24) and (25) we get

$$e\left(\frac{r}{2} + 2\lambda^2\right) = 0, \quad \varphi e\left(\frac{r}{2} + 2\lambda^2\right) = 0. \tag{30}$$

In view of λ is positive function on $\mathcal{U}_1$, it follows from first terms of (28) and (29) that $\sigma(e) = \sigma(\varphi e) = 0$, that is, Reeb vector field is an eigenvector field of the Ricci operator. This together with second term of (29) enables us to claim that $r = -4\lambda^2$. Conversely, suppose that Reeb vector field is an eigenvector field of the Ricci operator and the relation $r = -4\lambda^2$ holds, one can check directly that (9) holds trivially for any vector fields X, Y. □

As a consequence of above lemma, we state the following:

Proposition 3.5. *If ∗-Ricci operator of three-dimensional non-coKähler almost coKähler manifold is of Codazzi-type, then the ∗-Ricci tensor vanishes.*

In [9], the authors introduced the notion of strongly normal unit vector field. A unit vector field V on a Riemannian manifold is called strongly normal if

$$g((\nabla_X \nabla V)Y, Z) = 0, \quad \text{for any } X, Y, Z \perp V.$$

Many geometers studied three-dimensional almost coKähler manifold under the condition $\nabla_\xi h = 0$ (see [28]). In this paper we consider the condition $\xi(\|\nabla_\xi h\|) = 0$, which is weaker than $\nabla_\xi h = 0$. Applying this with Lemma 3.4, we obtain the following outcome:

Theorem 3.6. *Let M be a three-dimensional non-coKähler almost coKähler manifold whose Reeb vector field ξ is strongly normal unit vector field with $\xi(\|\nabla_\xi h\|) = 0$. Then ∗-Ricci operator is of Codazzi-type if and only if it is locally isometric to a simply connected unimodular Lie group equipped with a left invariant almost coKähler structure. More precisely, we have the following classification:*

- *In case $f = 0$, then M is locally isometric to the group $E(1,1)$ of rigid motions of the Minkowski 2-space.*
- *In case $f > 0$, then M is locally isometric to either the universal covering $\widetilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f > \lambda$, the Heisenberg group H^3 if $f = \lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f < \lambda$.*
- *In case $f < 0$, then M is locally isometric to either the universal covering $\widetilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f < -\lambda$, the Heisenberg group H^3 if $f = -\lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f > -\lambda$.*

Proof. As a result of Lemma 2.1 we find

$$\begin{aligned}(\nabla_e \nabla \xi)e &= -\lambda^2 \xi + \varphi e(\lambda)e - e(\lambda)\varphi e,\\ (\nabla_e \nabla \xi)\varphi e = (\nabla_{\varphi e} \nabla \xi)e &= -e(\lambda)e - \varphi e(\lambda)\varphi e,\\ (\nabla_{\varphi e} \nabla \xi)\varphi e &= -(\nabla_e \nabla \xi)e - 2\lambda^2 \xi,\end{aligned}$$

and so ξ is strongly normal implies $e(\lambda) = \varphi e(\lambda) = 0$. Suppose that M has a Codazzi-type ∗-Ricci tensor, then Lemma 3.4 is applicable. Switching $r = -4\lambda^2$ into (8), recalling $\sigma(e) = \sigma(\varphi e) = 0$ yields

$$Q\xi = -2\lambda^2 \xi, \quad Qe = -\lambda(\lambda + 2f)e + \xi(\lambda)\varphi e, \quad Q\varphi e = \xi(\lambda)e + \lambda(2f - \lambda)\varphi e. \tag{31}$$

Applying Lemma 2.1 and (31), by a direct calculation, we have

$$\begin{aligned}(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi, \quad (\nabla_e Q)e &= \lambda\xi(\lambda)\xi - 2\lambda e(f)e + e(\xi(\lambda))\varphi e,\\ (\nabla_{\varphi e} Q)\varphi e &= \lambda\xi(\lambda)\xi + \varphi e(\xi(\lambda))e + 2\lambda\varphi e(f)\varphi e,\end{aligned}$$

where we utilized $X(trh^2) = 0$ for any $X \in Ker\eta$. Applying aforementioned three equations in the well-known formula $div\, Q = \frac{1}{2} grad\, r$ we see that the following relation holds on $\mathcal{U}_1$:

$$\frac{1}{2} grad\, r = -2\lambda\xi(\lambda)\xi + (\varphi e(\xi(\lambda)) - 2\lambda e(f))e + (2\lambda\varphi e(f) + e(\xi(\lambda)))\varphi e. \tag{32}$$

In view of $\lambda > 0$, taking inner product of above equation with ξ we obtain that $\xi(\lambda) = 0$. Utilization of this in $X(trh^2) = 0$ for any $X \in Ker\eta$ shows that λ is a positive constant and the scalar curvature r is also constant. Again, take inner product of (32) with e and φ respectively to obtain $e(f) = \varphi e(f) = 0$, that is, $X(f) = 0$ for any $X \in Ker\, \eta$. Utilization of Lemma 2.1, a simple calculation, gives

$$\nabla_\xi h = \frac{1}{\lambda}\xi(\lambda)h + 2f\varphi h.$$

Since ξ is minimal and λ is constant, we obtain from above equation that $\|\nabla_\xi h\|^2 = 8\lambda^2 f^2$. We know that $e(f) = \varphi e(f) = 0$ and hence, since $\xi(\|\nabla_\xi h\|) = 0$ gives $\xi(f) = 0$, so that f is constant.

Next, we shall separate our discussions into two cases as follows.

Case 1. $f = 0$. In this context, we obtain from Poisson brackets (6) that

$$[\xi, e] = \lambda\varphi e, \quad [\varphi e, \xi] = -\lambda e, \quad [e, \varphi e] = 0.$$

According to Milnor [16] and the abovementioned relations, it can be easily seen that the manifold is locally isometric to the group $E(1, 1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost coKähler structure.

Case 2. $f \neq 0$. We obtain from Poisson brackets (6) that

$$[\xi, e] = (\lambda + f)\varphi e, \quad [\xi, \varphi e] = (\lambda - f)e, \quad [e, \varphi e] = 0.$$

Now, we consider the following invariant

$$p = \|\nabla_\xi h\| - \sqrt{2}\|h\|^2,$$

which is defined by Perrone in [21]. From the relation $\nabla_\xi h = 2f\varphi h$ with $f \in \mathbb{R}$ and using simple computation we obtain that

$$\bar{p} = 2\sqrt{2}\lambda(f - \lambda), \quad \text{if } f > 0,$$
$$\bar{p} = -2\sqrt{2}\lambda(f + \lambda), \quad \text{if } f < 0.$$

We know that Reeb vector field is minimal and also note that both $\|\nabla_\xi h\|$ and $\|h\|$ are constants. From Theorem 4.4 of Perrone [21] we conclude that M is locally isometric to a simply connected unimodular Lie group G equipped with a left invariant almost coKähler structure. More precisely, G is the universal covering $\widetilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $\bar{p} > 0$, the Heisenberg group H^3 if $\bar{p} = 0$ or the group $E(1, 1)$ of rigid motions of the Minkowski 2-space if $\bar{p} < 0$.

Conversely, on non-coKähler almost coKähler structures defined on the above Lie groups, from Perrone [20] one can easily check that r is constant and hence equation (9) holds true. This completes the proof. □

Now, we give the coKähler version of Theorem 3.6 as follows:

Theorem 3.7. *The $*$-Ricci operator of three-dimensional coKähler manifold is of Codazzi-type if and only if the manifold is locally isometric to the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c ($c = 0$ means that M is locally the flat Euclidean space $\mathbb{R}^3$).*

Proof. The authors in [17], gave the expression of $*$-Ricci operator Q^* on three-dimensional coKähler manifold in the following form:

$$Q^*X = \frac{r}{2}X - \frac{r}{2}\eta(X)\xi.$$

But, we know that the expression of Ricci operator is of the form $QX = \frac{r}{2}X - \frac{r}{2}\eta(X)\xi$. This together with above equation shows that $Q^* = Q$. Consequently, M becomes a manifold whose Ricci operator is of Codazzi-type (Riemannian curvature tensor is harmonic). According to Theorem 5.1 of Wang [25], we state that the manifold M is locally isometric to the product space $\mathbb{R}\times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c ($c = 0$ means that M is locally the flat Euclidean space $\mathbb{R}^3$). The converse part can be proved easily. □

Now, we characterize three-dimensional almost coKähler manifold whose $*$-Ricci operator satisfy $Q^*\cdot R = 0$ and this curvature condition is defined by

$$\begin{aligned}(Q^* \cdot R)(X, Y)Z = Q^*(R(X, Y)Z) - R(Q^*X, Y)Z \\ -R(X, Q^*Y)Z - R(X, Y)Q^*Z,\end{aligned} \tag{33}$$

for any vector fields X, Y, Z.

We prove the following outcome.

Lemma 3.8. *A three-dimensional non-coKähler almost coKähler manifold M satisfies the curvature condition $Q^*\cdot R = 0$ if and only if Reeb vector field is an eigenvector field of the Ricci operator and the scalar curvature $r = -4\lambda^2$.*

Proof. Let us suppose that M satisfies the curvature condition $Q^* \cdot R = 0$, then setting $X = Z = e$ and $Y = \varphi e$ into (33), recalling (10) and (18) gives

$$\sigma(e)\sigma(\varphi e) = 0, \quad (\frac{r}{2} + 2\lambda^2)\sigma(\varphi e) = 0, \quad 2\left(\frac{r}{2} + 2\lambda^2\right)^2 - (\sigma(\varphi e))^2 = 0. \tag{34}$$

Similarly, taking $X = e$ and $Y = Z = \varphi e$ into (33), applying (10) and (19) we obtain

$$\left(\frac{r}{2} + 2\lambda^2\right)\sigma(e) = 0, \quad (\sigma(e))^2 - 2\left(\frac{r}{2} + 2\lambda^2\right)^2 = 0, \quad \sigma(\varphi e)\sigma(e) = 0. \tag{35}$$

Setting $X = e$, $Y = \varphi e$ and $Z = \xi$ into (33), according to (10) and (18) one can get

$$\left(\frac{r}{2} + 2\lambda^2\right)\sigma(e) = 0, \quad \left(\frac{r}{2} + 2\lambda^2\right)\sigma(\varphi e) = 0. \tag{36}$$

Substituting $X = Z = e$ and $Y = \xi$ into (33), as a result of (10), (13) and (18) gives

$$\begin{aligned}(\sigma(\varphi e))^2 - 2\lambda(\lambda + 2f)\left(\frac{r}{2} + 2\lambda^2\right) = 0, \quad \lambda(\lambda + 2f)\sigma(e) = 0, \\ \lambda(\lambda + 2f)\sigma(\varphi e) + 2\left(\frac{r}{2} + 2\lambda^2\right)\sigma(\varphi e) = 0.\end{aligned} \tag{37}$$

Setting $X = e$ and $Y = Z = \xi$ into (33), utilization of (10) and (11)-(19) yields

$$\sigma(\varphi e)\xi(\lambda) - \lambda(\lambda + 2f)\sigma(e) = 0, \quad (\sigma(\varphi e))^2 = 0, \quad \sigma(e)\sigma(\varphi e) = 0. \tag{38}$$

Taking $X = e$, $Y = \xi$ and $Z = \varphi e$ into (33), applying (10) and (11)-(19) we obtain

$$\begin{aligned}2\left(\frac{r}{2} + 2\lambda^2\right)\xi(\lambda) - \sigma(e)\sigma(\varphi e) = 0, \\ \sigma(e)\xi(\lambda) + 2\left(\frac{r}{2} + 2\lambda^2\right)\sigma(\varphi e) = 0, \quad \sigma(\varphi e)\xi(\lambda) = 0.\end{aligned} \tag{39}$$

Substituting $X = \varphi e$, $Y = \xi$ and $Z = e$ into (33), recalling (10) and (11)-(19) gives

$$\begin{aligned}2\xi(\lambda)\left(\frac{r}{2} + 2\lambda^2\right) - \sigma(e)\sigma(\varphi e) = 0, \quad \sigma(e)\xi(\lambda) = 0, \\ \sigma(\varphi e)\xi(\lambda) + 2\left(\frac{r}{2} + 2\lambda^2\right)\sigma(e) = 0.\end{aligned} \tag{40}$$

Switching $X = Z = \varphi e$ and $Y = \xi$ into (33) and making use of (10) and (11)-(19) we obtain

$$\begin{aligned}&(\sigma(e))^2 - 2\lambda(\lambda - 2f)\left(\frac{r}{2} + 2\lambda^2\right) = 0, \quad \lambda(\lambda - 2f)\sigma(\varphi e) = 0,\\ &\qquad \lambda(\lambda - 2f)\sigma(e) + 2\sigma(e)\left(\frac{r}{2} + 2\lambda^2\right) = 0.\end{aligned} \tag{41}$$

Setting $X = \varphi e$ and $Y = Z = \xi$ into (33), utilization of (10) and (11)-(19) we have

$$\sigma(e)\xi(\lambda) - \lambda(\lambda - 2f)\sigma(\varphi e) = 0, \quad \sigma(e)\sigma(\varphi e) = 0, \quad (\sigma(e))^2 = 0. \tag{42}$$

The relation $\sigma(e) = \sigma(\varphi e) = 0$ follows directly from third term of (42) and second term of (38). This together with second term of equation (35) shows that the scalar curvature $r = -4\lambda^2$. Convesely, if the conditions $r = -4\lambda^2$ and $\sigma(e) = \sigma(\varphi e) = 0$ holds, then it is not hard to show that M satisfies $Q^* \cdot R = 0$. □

Proposition 3.9. *If three-dimensional non-coKähler almost coKähler manifold M satisfies the curvature condition $Q^* \cdot R = 0$, then the $*$-Ricci tensor vanishes.*

Theorem 3.10. *Let M be a three-dimensional non-coKähler almost coKähler manifold whose Reeb vector field ξ is strongly normal unit vector field with $\xi(\|\nabla_\xi h\|) = 0$. Then M satisfies the curvature condition $Q^* \cdot R = 0$ if and only if it is locally isometric to a simply connected unimodular Lie group equipped with a left invariant almost coKähler structure. More precisely, we have the following classifications:*

- *In case $f = 0$, then M is locally isometric to the group $E(1,1)$ of rigid motions of the Minkowski 2-space.*
- *In case $f > 0$, then M is locally isometric to either the universal covering $\widetilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f > \lambda$, the Heisenberg group H^3 if $f = \lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f < \lambda$.*
- *In case $f < 0$, then M is locally isometric to either the universal covering $\widetilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f < -\lambda$, the Heisenberg group H^3 if $f = -\lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f > -\lambda$.*

Proof. The proof of this theorem follows the same steps and arguments as followed in Theorem 3.6. □

Remark 3.11. *From Lemma 3.4 and Lemma 3.8, we can state that in a three-dimensional non-coKähler almost coKähler manifold M the following conditions are equivalent:*

- *$*$-Ricci operator is Codazzi-type.*
- *M satisfies $Q^* \cdot R = 0$.*
- *Reeb vector field is an eigenvector field of the Ricci operator and the scalar curvature $r = -4\lambda^2$.*

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