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New characterizations for w-core inverses in rings with involution

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Abstract. Let R be a unital $*$-ring and let $a, b, w \in R$. In this paper, we give some new characterizations on w-core inverses in R. In particular, it is shown that a is w-core invertible if and only if it is $w(aw)^{n-1}$-core invertible for any positive integer n, in which case, the representations of the w-core inverse and the $w(aw)^{n-1}$-core inverse of a are both presented. We further characterize w-core inverses by Hermitian elements (or projections) and units.

1. Introduction

The notion of the core inverse, firstly introduced by Baksalary and Trenkler in complex matrices [1], and subsequently generalized by Rakić et al. to the case of elements in rings with involution [18], has been intensively investigated by a number of scholars. Further, the core inverse was extended to several new classes of generalized inverses such as the core-EP inverse of square complex matrices [17], the DMP inverse of square complex matrices [12], the pseudo core inverse of $*$-ring elements [9] and the e-core inverse of $*$-ring elements [15]. Moreover, their properties and characterizations have been studied (see, e.g., [7, 8, 21, 23, 24]). In 2022, Zhu et al. [25] introduced a new type of generalized inverses, called w-core inverses, extending Moore-Penrose inverses, core inverses and core-EP inverses.

The initial goal of this paper is to give several new characterizations for w-core inverses. The paper is organized as follows. In Section 2, several characterizations and expressions for w-core inverses are established. It is shown that the existence of the w-core inverse of a coincides with the existence of its $w(aw)^{n-1}$-core inverse for any positive integer n. We further characterize w-core inverses by properties of the left and right annihilators and ideals in a ring. As applications, the results on the core inverse in [11] and the Moore-Penrose inverse in [19] are special cases of Theorem 2.15. In Section 3, it is proved that a is w-core invertible if and only if there exists a unique Hermitian element (or projection) $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$ for all integers $n \geq 1$.

Let us now recall fundamental concepts on several well-known generalized inverses in rings. Let R be an associative ring with unity 1. An element $a \in R$ is called (von Neumann) regular if there exists some

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$x \in R$ such that $axa = a$. Such an x is called an inner inverse or a $\{1\}$-inverse of a, and is denoted by a^-. We denote by the symbol $a\{1\}$ the set of all inner inverses of a. An element $a \in R$ is called group invertible (see, e.g., [2]) if there exists some $x \in R$ such that $axa = a$, $xax = x$ and $ax = xa$. Such an x is called a group inverse of a. It is unique if it exists, and is denoted by $a^\#$. The symbols R^- and $R^\#$ will stand for the sets of all regular elements and group invertible elements in R.

Let $a, d \in R$. An element a is called left invertible along d [22] if there exists some $x \in R$ such that $xad = d$ and $x \in Rd$. Such an element x is called a left inverse of a along d, and is denoted by $a_l^{\|d}$. Dually, an element a is called right invertible along d if there exists some $x \in R$ such that $dax = d$ and $x \in dR$. Such an element x is called a right inverse of a along d, and is denoted by $a_r^{\|d}$. Specially, an element a is invertible along d if and only if it is left and right invertible along d [22], or equivalently, if there exists some $x \in R$ such that $xad = d = dax$ and $x \in dR \cap Rd$ [13]. Such an element x is called an inverse of a along d. It is unique if it exists, and is denoted by $a^{\|d}$. We denote by the symbols $R_l^{\|d}$, $R_r^{\|d}$ and $R^{\|d}$ the sets of all left invertible, right invertible and invertible elements along d in R, respectively. More results on the inverse along an element can be referred to [3, 4, 6].

Let R be a unital $*$-ring, that is a ring R with unity 1 and an involution $* : a \mapsto a^*$ satisfying $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. Throughout this article, any ring R considered is assumed to be a unital $*$-ring (unless otherwise noted).

An element $a \in R$ is said to be Moore-Penrose invertible [16] if there exists some $x \in R$ such that

$$\text{(i) } axa = a, \text{ (ii) } xax = x, \text{ (iii) } (ax)^* = ax, \text{ (iv) } (xa)^* = xa.$$

Such an x is called a Moore-Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$. Generally, if a and x satisfy the equations (i) and (iii), then x is called a $\{1,3\}$-inverse of a, and is denoted by $a^{(1,3)}$. If a and x satisfy the equations (i) and (iv), then x is called a $\{1,4\}$-inverse of a, and is denoted by $a^{(1,4)}$. We denote by the symbols $a\{1,3\}$ and $a\{1,4\}$ the sets of all $\{1,3\}$-inverses and $\{1,4\}$-inverses of a. In general, we denote by $R^{\{1,3\}}$, $R^{\{1,4\}}$ and $R^\dagger$ the sets of all $\{1,3\}$-invertible, $\{1,4\}$-invertible and Moore-Penrose invertible elements in R, respectively. It is well known that a is Moore-Penrose invertible if and only if it is both $\{1,3\}$-invertible and $\{1,4\}$-invertible.

Following [18], an element $a \in R$ is called core invertible if there exists some $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$. Such an x is called a core inverse of a. It is unique if it exists, and is denoted by $a^{\#\!\!\!\!\bigcirc}$. In [18], they also derived that the core inverse x of a satisfies the following five equations

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ xa^2 = a,\ (5)\ ax^2 = x.$$

As usual, we denote by $R^{\#\!\!\!\!\bigcirc}$ the set of all core invertible elements in R. It is shown in [20] that a is core invertible if and only if it is both group invertible and $\{1,3\}$-invertible. Various equivalent characterizations for the existence of core inverses in rings with involution can be found in [5, 18, 20].

Let $a, w \in R$. An element a is called w-core invertible [25] if there exists some $x \in R$ such that $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$. Such an x is called a w-core inverse of a. It is unique if it exists, and is denoted by $a_w^{\#\!\!\!\!\bigcirc}$. We denote by $R_w^{\#\!\!\!\!\bigcirc}$ the set of all w-core invertible elements in R. It is proved that $a \in R_w^{\#\!\!\!\!\bigcirc}$ if and only if $w \in R^{\|a}$ and $a \in R^{\{1,3\}}$. Moreover, $a_w^{\#\!\!\!\!\bigcirc} = w^{\|a}a^{(1,3)}$. According to [25], the 1-core inverse is just the core inverse. It is also shown that the existence of the a^*-core inverse of a coincides with the existence of its Moore-Penrose inverse. More results on w-core inverses can be referred to [25, 26].

2. New characterizations for w-core inverses

In this section, we aim to give several characterizations for w-core inverses. In Proposition 2.2 below, we plan to characterize the w-core inverse by equations with higher powers. The following auxiliary lemma is given in order to derive the result.

Lemma 2.1. *Let $a, w \in R$ and let $n \geq 1$ be an integer. If $a \in R_w^{\#\!\!\!\!\bigcirc}$, then*

(i) $awa_w^{\#\!\!\!\!\bigcirc} = (aw)^n(a_w^{\#\!\!\!\!\bigcirc})^n$.

(ii) $a_w^{\#\!\!\!\!\bigcirc}aw = (a_w^{\#\!\!\!\!\bigcirc})^n(aw)^n$.

Proof. As $a \in R_w^{\textcircled{\#}}$, then $aw(a_w^{\textcircled{\#}})^2 = a_w^{\textcircled{\#}}$ and $a_w^{\textcircled{\#}}awa = a$.

(i) Since $aw(a_w^{\textcircled{\#}})^2 = a_w^{\textcircled{\#}}$, we have $awa_w^{\textcircled{\#}} = aw \cdot aw(a_w^{\textcircled{\#}})^2 = (aw)^2(a_w^{\textcircled{\#}})^2 = (aw)^2 \cdot aw(a_w^{\textcircled{\#}})^2 \cdot a_w^{\textcircled{\#}} = (aw)^3(a_w^{\textcircled{\#}})^3 = \cdots = (aw)^n(a_w^{\textcircled{\#}})^n$.

(ii) Since $a_w^{\textcircled{\#}}awa = a$, we have $a_w^{\textcircled{\#}}aw = a_w^{\textcircled{\#}} \cdot a_w^{\textcircled{\#}}awa \cdot w = (a_w^{\textcircled{\#}})^2(aw)^2 = (a_w^{\textcircled{\#}})^2 \cdot a_w^{\textcircled{\#}}awa \cdot waw = (a_w^{\textcircled{\#}})^3(aw)^3 = \cdots = (a_w^{\textcircled{\#}})^n(aw)^n$. □

Suppose $a \in R_w^{\textcircled{\#}}$. We remark the fact that $awa_w^{\textcircled{\#}} = aww^{\|a}a^{(1,3)} = aa^{(1,3)}$ is idempotent by [25, Theorem 2.9]. Hence, $awa_w^{\textcircled{\#}} = (awa_w^{\textcircled{\#}})^n$ for any positive integer n.

Proposition 2.2. *Let $a, w \in R$. Then the following conditions are equivalent*:

(i) $a \in R_w^{\textcircled{\#}}$.

(ii) *There exists some $x \in R$ such that $a = (awx)^n xawa$, $awx^2 = x$ and $aw(awx)^n x = (aw(awx)^n x)^*$ for any positive integer n.*

(iii) *There exists some $x \in R$ such that $a = (awx)^n xawa$, $awx^2 = x$ and $aw(awx)^n x = (aw(awx)^n x)^*$ for some positive integer n.*

(iv) *There exists some $y \in R$ such that $a = (aw)^n y^{n+1}awa$, $awy^2 = y$ and $(aw)^n y^n = ((aw)^n y^n)^*$ for any positive integer n.*

(v) *There exists some $y \in R$ such that $a = (aw)^n y^{n+1}awa$, $awy^2 = y$ and $(aw)^n y^n = ((aw)^n y^n)^*$ for some positive integer n.*

In this case, $a_w^{\textcircled{\#}} = (awx)^n x = (aw)^n y^{n+1}$.

Proof. To begin with, (ii) ⇒ (iii) and (iv) ⇒ (v) are obvious.

(i) ⇒ (ii) Suppose $a \in R_w^{\textcircled{\#}}$. Then there exists an x such that $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$, whence $x = awx^2 = (awx)^n x$ for any positive integer n. Also, we have $a = xawa = (awx)^n xawa$ and $aw(awx)^n x = awx = (aw(awx)^n x)^*$ for any positive integer n.

(iii) ⇒ (i) Set $r = (awx)^n x$, then

(1) $rawa = (awx)^n xawa = a$.

(2) $awr = aw(awx)^n x = (awr)^*$.

(3) Since $a = (awx)^n xawa$, we have $(awx)^n xawx = (awx)^n xaw(awx^2) = ((awx)^n xawa)wx^2 = awx^2 = x$, and hence $awr^2 = aw(awx)^n x(awx)^n x = aw \cdot (awx)^n xawx \cdot (awx)^{n-1}x = (awx)^n x = r$.

Hence, $a \in R_w^{\textcircled{\#}}$ and $a_w^{\textcircled{\#}} = (awx)^n x$.

(i) ⇒ (iv) Suppose $a \in R_w^{\textcircled{\#}}$. We have $awy^2 = y$, $yawa = a$ and $(awy)^* = awy$ for some $y \in R$. It follows from Lemma 2.1 that $(aw)^n y^n = ((aw)^n y^n)^*$ and $a = yawa = awy^2awa = awy \cdot yawa = (aw)^n y^n \cdot yawa = (aw)^n y^{n+1}awa$ for any positive integer n.

(v) ⇒ (i) It is clear for the case of $n = 1$. For the case of $n \geq 2$, set $z = (aw)^n y^{n+1}$, then

(1) $zawa = (aw)^n y^{n+1}awa = a$.

(2) $awz = aw(aw)^n y^{n+1} = (aw)^n \cdot awy^2 \cdot y^{n-1} = (aw)^n y^n = (awz)^*$.

(3) As $a = (aw)^n y^{n+1}awa$, then $(aw)^{n-1} = (aw)^n y^{n+1}(aw)^n$. Thus, $awz^2 = aw \cdot (aw)^n y^{n+1}(aw)^n \cdot y^{n+1} = (aw)^n y^{n+1} = z$.

Hence, $a \in R_w^{\textcircled{\#}}$ and $a_w^{\textcircled{\#}} = (aw)^n y^{n+1}$. □

Lemma 2.3. *Let $a, d \in R$. Then*

(i) [22, Theorem 2.3] $a \in R_l^{\|d}$ *if and only if $d \in Rdad$. In this case, $a_l^{\|d} = sd$, where $s \in R$ satisfies $d = sdad$.*

(ii) [22, Theorem 2.4] $a \in R_r^{\|d}$ *if and only if $d \in dadR$. In this case, $a_r^{\|d} = dt$, where $t \in R$ satisfies $d = dadt$.*

(iii) [14, Theorem 2.2] $a \in R^{\|d}$ *if and only if $d \in dadR \cap Rdad$. In this case, $a^{\|d} = dt = sd$, where $t, s \in R$ satisfy $d = dadt = sdad$.*

(iv) [28, Lemma 3.3 (4)] *a is invertible along d with inverse y if and only if a is right invertible along d with a right inverse x and a is left invertible along d with a left inverse z. In this case, $y = x = z$.*

Lemma 2.4. [27, Lemma 2.2] *Let $a \in R$. We have the following results*:

(i) $a \in R^{\{1,3\}}$ *if and only if $a \in Ra^*a$. In particular, if $xa^*a = a$ for some $x \in R$, then x^* is a $\{1,3\}$-inverse of a.*

(ii) $a \in R^{\{1,4\}}$ *if and only if $a \in aa^*R$. In particular, if $aa^*y = a$ for some $y \in R$, then y^* is a $\{1,4\}$-inverse of a.*

Lemma 2.5. [14, Theorem 2.1] *Let $a, w \in R$. Then the following conditions are equivalent*:

(i) $w \in R^{\|a}$.

(ii) $a \in awR$ *and* $aw \in R^{\#}$.

(iii) $a \in Rwa$ *and* $wa \in R^{\#}$.

In this case, $w^{\|a} = (aw)^{\#}a = a(wa)^{\#}$.

Lemma 2.6. [25, Lemma 2.2] *For any $a, w \in R$, if $x \in R$ is the w-core inverse of a, then $awxa = a$ and $xawx = x$.*

In [25, Theorem 2.9], Zhu et al. showed that $a \in R_w^{\text{\textcircled{\#}}}$ implies that $w^{\|a}$ and $a^{(1,3)}$ both exist. We also claim that if $a \in R_w^{\text{\textcircled{\#}}}$, then $w^{\|a}$ and $(aw)^{(1,3)}$ both exist. Indeed, $a \in R_w^{\text{\textcircled{\#}}}$ gives $a \in R^{\{1,3\}}$ and $a \in awaR$. Consequently, it follows from Lemma 2.4 that $a \in Ra^*a$, and hence $aw \in Ra^*aw \subseteq R(aw)^*aw$, i.e., $aw \in R^{\{1,3\}}$. One may ask if the converse statement holds. The following theorem shows the accuracy of this assumption, and gives more existence characterizations on the w-core inverse.

Theorem 2.7. *Let $a, w \in R$. Then the following conditions are equivalent*:

(i) $a \in R_w^{\text{\textcircled{\#}}}$.

(ii) $w^{\|a}$ *exists and* $a \in R^{\{1,3\}}$.

(iii) $w^{\|a}$ *exists and* $aw \in R^{\{1,3\}}$.

(iv) $w^{\|a}$ *exists and* $awa \in R^{\{1,3\}}$.

(v) $w^{\|a}$ *exists and* $w^{\|a}w \in R^{\{1,3\}}$.

In this case, $a_w^{\text{\textcircled{\#}}} = w^{\|a}a^{(1,3)} = w^{\|a}w(aw)^{(1,3)} = a(awa)^{(1,3)} = (aw)^{\#}(w^{\|a}w)^{(1,3)}$.

Proof. (i) $\Leftrightarrow$ (ii) follows from [25, Theorem 2.9].

(ii) $\Rightarrow$ (iii) by the implication above and (i) $\Leftrightarrow$ (ii).

(iii) $\Rightarrow$ (ii) As $w^{\|a}$ exists, then $a \in awaR$ by Lemma 2.3, and therefore, $a = awat$ for some $t \in R$. It follows from $aw \in R^{\{1,3\}}$ that $aw \in R(aw)^*aw$, whence $a = awat \in R(aw)^*awat = R(aw)^*a = Rw^*a^*a \subseteq Ra^*a$. This gives $a \in R^{\{1,3\}}$ by Lemma 2.4.

(ii) $\Leftrightarrow$ (iv) is analogous to (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (v) Let $x = awa_w^{\text{\textcircled{\#}}}$. Then x is a $\{1,3\}$-inverse of $w^{\|a}w$. Indeed, we have

(1) $w^{\|a}wx = w^{\|a}wawa_w^{\text{\textcircled{\#}}} = awa_w^{\text{\textcircled{\#}}} = (w^{\|a}wx)^*$.

(2) Note that $w^{\|a} \in aR$. Then there exists some $z \in R$ such that $w^{\|a} = az$, and hence $w^{\|a}wxw^{\|a}w = awa_w^{\text{\textcircled{\#}}}w^{\|a}w = awa_w^{\text{\textcircled{\#}}}azw = azw = w^{\|a}w$ by Lemma 2.6.

(v) $\Rightarrow$ (i) By Lemma 2.5, one knows that $w \in R^{\|a}$ implies $aw \in R^{\#}$. Suppose $y = (aw)^{\#}(w^{\|a}w)^{(1,3)}$. Then, by Lemma 2.5, $y = (aw)^{\#}((aw)^{\#}aw)^{(1,3)}$. We next show that y is the w-core inverse of a.

We have

$$\begin{aligned}
(1)\ yawa &= (aw)^{\#}((aw)^{\#}aw)^{(1,3)}awa \\
&= (aw)^{\#}(aw)^{\#}aw((aw)^{\#}aw)^{(1,3)}(aw)^{\#}(aw)^2a \\
&= (aw)^{\#}((aw)^{\#}aw((aw)^{\#}aw)^{(1,3)}(aw)^{\#}aw)awa \\
&= (aw)^{\#}(aw)^{\#}awawa \\
&= (aw)^{\#}awa \\
&= w^{\|a}wa \\
&= a.
\end{aligned}$$

$$\begin{aligned}
(2)\ awy &= aw(aw)^{\#}((aw)^{\#}aw)^{(1,3)} \\
&= ((aw)^{\#}aw)((aw)^{\#}aw)^{(1,3)} \\
&= (awy)^*.
\end{aligned}$$

$$\begin{aligned}(3)\ awy^2 &= aw(aw)^{\#}((aw)^{\#}aw)^{(1,3)}(aw)^{\#}((aw)^{\#}aw)^{(1,3)}\\ &= (aw)^{\#}aw((aw)^{\#}aw)^{(1,3)}(aw)^{\#}aw(aw)^{\#}((aw)^{\#}aw)^{(1,3)}\\ &= (aw)^{\#}((aw)^{\#}aw)^{(1,3)}\\ &= y.\end{aligned}$$

We next show that $w(aw)^{(1,3)} \in a\{1,3\}$ for any $(aw)^{(1,3)} \in (aw)\{1,3\}$. Indeed, $aw(aw)^{(1,3)} = (aw(aw)^{(1,3)})^*$, and $aw(aw)^{(1,3)}a = aw(aw)^{(1,3)}awat = awat = a$ by (iii) ⇒ (ii). Similarly, $wa(awa)^{(1,3)} \in a\{1,3\}$ for any $(awa)^{(1,3)} \in (awa)\{1,3\}$.

Hence, $a_w^{\text{\textcircled{\#}}} = w^{\|a}a^{(1,3)} = w^{\|a}w(aw)^{(1,3)} = w^{\|a}wa(awa)^{(1,3)} = a(awa)^{(1,3)} = (aw)^{\#}(w^{\|a}w)^{(1,3)}$. □

Suppose $a \in R_w^{\text{\textcircled{\#}}}$. By Theorem 2.7, we have $awa \in R^{\{1,3\}}$, and hence $awa \in R^-$. Applying Theorem 2.7, the following representation of the w-core inverse in R can be obtained.

Proposition 2.8. *Let $a, w \in R$ with $a \in R_w^{\text{\textcircled{\#}}}$. Then $a_w^{\text{\textcircled{\#}}} = a(awa)^-aa^{(1,3)}$.*

Proof. It follows from $a \in R_w^{\text{\textcircled{\#}}}$ that $awa = awa(awa)^-awa$ for all $(awa)^- \in (awa)\{1\}$ by the illustration above. Again by $a \in R_w^{\text{\textcircled{\#}}}$, then $w^{\|a}$ exists by Theorem 2.7 (i) ⇒ (ii). So, $a \in awaR$, and $a = awat$ for some $t \in R$. Post-multiplying the equation $awa = awa(awa)^-awa$ by t yields $a = awa(awa)^-a$, which gives $w^{\|a} = a(awa)^-a$ by Lemma 2.3 (iv). We hence get $a_w^{\text{\textcircled{\#}}} = w^{\|a}a^{(1,3)} = a(awa)^-aa^{(1,3)}$ by Theorem 2.7. □

We conclude that $a \in R_w^{\text{\textcircled{\#}}}$ implies that $awa \in R^-$ and $a \in R^{\{1,3\}}$ from Proposition 2.8. However, the converse does not hold in general. A counterexample is given as follows.

Example 2.9. Let R be the ring of all 2×2 complex matrices and let the involution $*$ be the transpose. Take $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$. Then $AWA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So, $AWA \in R^-$. We also have $A^*A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, and whence $\text{rank}(A^*A) = \text{rank}(A)$. Note that $C(A^*A) \subseteq C(A)$ ($C(A)$ denotes the row space of A). Then $C(A^*A) = C(A)$, so that $A \in R^{\{1,3\}}$. But $A \notin AWAR \cap RAWA$. It follows that $W \notin R^{\|A}$, and consequently $A \notin R_W^{\text{\textcircled{\#}}}$.

Suppose that $a \in R_w^{\text{\textcircled{\#}}}$ and n is a positive integer. Then $(aw)^n \in R^{\{1,3\}}$. Indeed, from $a \in R_w^{\text{\textcircled{\#}}}$ with the w-core inverse y, it follows that $(aw)^ny^n = ((aw)^ny^n)^*$ by Proposition 2.2. Also, $a = awya = (aw)^ny^na$ in terms of Lemmas 2.1 and 2.6. This guarantees $(aw)^n = (aw)^ny^n(aw)^n$. A natural question is under what conditions $(aw)^n \in R^{\{1,3\}}$ can imply $a \in R_w^{\text{\textcircled{\#}}}$.

Theorem 2.10. *Let $a, w \in R$. Then the following conditions are equivalent:*

(i) $a \in R_w^{\text{\textcircled{\#}}}$.

(ii) $a \in R^{\{1,3\}}$, $a \in (aw)^naR \cap Ra(wa)^n$ *for any positive integer* n.

(iii) $a \in R^{\{1,3\}}$, $a \in (aw)^naR \cap Ra(wa)^n$ *for some positive integer* n.

(iv) $(aw)^n \in R^{\{1,3\}}$, $a \in (aw)^naR \cap Ra(wa)^n$ *for any positive integer* n.

(v) $(aw)^n \in R^{\{1,3\}}$, $a \in (aw)^naR \cap Ra(wa)^n$ *for some positive integer* n.

Proof. (ii) ⇒ (iii) and (iv) ⇒ (v) are clear.

(i) ⇒ (ii) Since $a \in R_w^{\text{\textcircled{\#}}}$, we have $w \in R^{\|a}$ and $a \in R^{\{1,3\}}$ by Theorem 2.7, thus, $a \in awaR \cap Rawa$ by Lemma 2.3. Given $a \in awaR$, then there exists some $t \in R$ such that $a = awat = aw(awat)t = (aw)^2at^2 = (aw)^2(awat)t^2 = (aw)^3at^3 = \cdots = (aw)^nat^n \in (aw)^naR$ for any positive integer n. Dually, as $a \in Rawa$, then there exists some $s \in R$ such that $a = sawa = s(sawa)wa = s^2a(wa)^2 = s^2(sawa)(wa)^2 = s^3a(wa)^3 = \cdots = s^na(wa)^n \in Ra(wa)^n$ for any positive integer n. So, $a \in (aw)^naR \cap Ra(wa)^n$ for any positive integer n.

(iii) ⇒ (iv) Given $a \in (aw)^naR \subseteq awaR$, then $a \in (aw)^naR$ for any positive integer n by the implication (i) ⇒ (ii). Therefore, there exists some $b \in R$ such that $a = (aw)^nab$ for any positive integer n. As $a \in R^{\{1,3\}}$, then $a \in Ra^*a$ by Lemma 2.4, and whence $(aw)^n \in Ra^*(aw)^n$ for any positive integer n. Hence, $(aw)^n \in R((aw)^nab)^*(aw)^n = R(ab)^*((aw)^n)^*(aw)^n \subseteq R((aw)^n)^*(aw)^n$, which implies $(aw)^n \in R^{\{1,3\}}$. Similarly, we can also obtain $a \in Ra(wa)^n$ for any positive integer n.

(v) ⇒ (i) To prove $a \in R_w^{\oplus}$, it is sufficient to show that $w^{\|a}$ exists and $a \in R^{\{1,3\}}$ by Theorem 2.7. Suppose $a \in (aw)^n aR \cap Ra(wa)^n$ for some positive integer n. Then $a \in awaR \cap Rawa$, i.e., $w^{\|a}$ exists, and $a = (aw)^n ab$ for some $b \in R$. By $(aw)^n \in R^{\{1,3\}}$, then $(aw)^n \in R((aw)^n)^*(aw)^n$. These imply $a = (aw)^n ab \in R((aw)^n)^*(aw)^n ab = R((aw)^n)^* a = R(w(aw)^{n-1})^* a^* a \subseteq Ra^* a$, namely $a \in R^{\{1,3\}}$. As a consequence, $a \in R_w^{\oplus}$. □

Next, several basic properties of group inverses are presented.

Lemma 2.11. *Let $a \in R^{\#}$ and let n be a positive integer. Then*
(i) $a^{\#} = a^{n-1}(a^{\#})^n$.
(ii) $a^n \in R^{\#}$. *In this case,* $(a^n)^{\#} = (a^{\#})^n$.

For any positive integer n, if $a \in (aw)^n aR \cap Ra(wa)^n$, then we have $a \in a(w(aw)^{n-1})aR \cap Ra(w(aw)^{n-1})a$. This ensures $w(aw)^{n-1} \in R^{\|a}$ by Lemma 2.3.

Corollary 2.12. *Let $a, w \in R$ and let n be a positive integer. Then the following conditions are equivalent*:
(i) $a \in R_w^{\oplus}$.
(ii) $a \in R^{\oplus}_{w(aw)^{n-1}}$.
(iii) $a \in R^{\{1,3\}}$, $w(aw)^{n-1} \in R^{\|a}$.
(iv) $(aw)^n \in R^{\{1,3\}}$, $w(aw)^{n-1} \in R^{\|a}$.
In this case, $a_w^{\oplus} = (aw)^{n-1} a^{\oplus}_{w(aw)^{n-1}}$, $a^{\oplus}_{w(aw)^{n-1}} = ((aw)^{\#})^{n-1} a_w^{\oplus}$.

Proof. The equivalences follow from Theorems 2.7 and 2.10. From Lemmas 2.5, 2.11 and Theorem 2.7, we get that $a_w^{\oplus} = w^{\|a} a^{(1,3)} = (aw)^{\#} aa^{(1,3)}$ and $a^{\oplus}_{w(aw)^{n-1}} = (w(aw)^{n-1})^{\|a} a^{(1,3)} = ((aw)^n)^{\#} aa^{(1,3)} = ((aw)^{\#})^n aa^{(1,3)}$. Consequently, $a_w^{\oplus} = (aw)^{n-1}((aw)^{\#})^n aa^{(1,3)} = (aw)^{n-1} a^{\oplus}_{w(aw)^{n-1}}$ and $a^{\oplus}_{w(aw)^{n-1}} = ((aw)^{\#})^{n-1}(aw)^{\#} aa^{(1,3)} = ((aw)^{\#})^{n-1} a_w^{\oplus}$. □

Several notations are presented as follows:

$$a^0 = \{x \in R \mid ax = 0\} \text{ and } {}^0a = \{x \in R \mid xa = 0\}.$$

Existence criteria for several types of generalized inverses, such as group inverses, Moore-Penrose inverses, $\{1,3\}$-inverses, $\{1,4\}$-inverses and core inverses are given in terms of properties of annihilators and ideals of certain elements, which have been widely concerned by scholars. In 1976, Hartwig [10] obtained that $a \in R^{\#}$ if and only if $R = aR \oplus a^0$ if and only if $R = Ra \oplus {}^0a$. Also, he showed that $a \in R^{\{1,3\}}$ if and only if $R = aR \oplus (a^*)^0$ if and only if $R = Ra^* \oplus {}^0a$. Dually, $a \in R^{\{1,4\}}$ if and only if $R = a^*R \oplus a^0$ if and only if $R = Ra \oplus {}^0(a^*)$. Note that $a \in R^{\dagger}$ if and only if $a \in R^{\{1,3\}} \cap R^{\{1,4\}}$. Accordingly, $a \in R^{\dagger}$ if and only if $R = aR \oplus (a^*)^0 = a^*R \oplus a^0$. Xu et al. [20] gave that $a \in R^{\oplus}$ if and only if $a \in R^{\{1,3\}} \cap R^{\#}$. Hence, he derived that $a \in R^{\oplus}$ if and only if $R = aR \oplus (a^*)^0 = aR \oplus a^0$ according to the aforementioned results. Motivated by these, we consider whether the w-core inverse can also be described by annihilators and ideals in a ring.

Lemma 2.13. [26, Proposition 2.4] *Let $a, w \in R$. Then $a \in awaR \cap Ra^*a$ if and only if $a \in R(awa)^*a$.*

Theorem 2.14. *Let $a, w \in R$. Then the following conditions are equivalent*:
(i) $a \in R_w^{\oplus}$.
(ii) $a \in R(awa)^* a \cap Rawa$.
(iii) $R = R(awa)^* \oplus {}^0a = Raw \oplus {}^0a$.
(iv) $R = R(awa)^* + {}^0a = Raw + {}^0a$.

Proof. (i) ⇔ (ii) follows directly from Lemmas 2.3 (iii), 2.4, 2.13 and Theorem 2.7.

(ii) ⇒ (iii) As $a \in R(awa)^*a$, then there exists some $h \in R$ such that $a = h(awa)^*a$, which gives $(1-h(awa)^*)a = 0$, i.e., $1-h(awa)^* \in {}^0a$. For any $r \in R$, we write $r = rh(awa)^* + r(1-h(awa)^*) \in R(awa)^* + {}^0a$. Let $y \in R(awa)^* \cap {}^0a$. Then $ya = 0$ and $y = l(awa)^*$ for some $l \in R$. Hence, $y = l(wa)^*a^* = l(wa)^*(h(awa)^*a)^* = l(wa)^*a^*awah^* = yawah^* = 0$. Therefore, $R = R(awa)^* \oplus {}^0a$.

Given $a \in Rawa$, then $a = sawa$ for some $s \in R$, which implies $(1 - saw)a = 0$, i.e., $1 - saw \in {}^0a$. For any $r' \in R$, then r' can be written as $r' = r'saw + r'(1 - saw) \in Raw + {}^0a$. Since $a \in R(awa)^*a$, we have

$a = h(awa)^*a = h(wa)^*a^*a \in Ra^*a$ by the implication above, which guarantees that $a \in R^{\{1,3\}}$ and $a^{(1,3)} = wah^*$ by Lemma 2.4. These give $a = a(wah^*)a = awah^*a$. Let $y' \in Raw \cap {}^0a$. Then $y'a = 0$ and $y' = l'aw$ for some $l' \in R$, so that $y' = l'(awah^*a)w = (l'aw)ah^*aw = y'ah^*aw = 0$. As a consequence, $R = Raw \oplus {}^0a$.

(iii) ⇒ (iv) is obvious.

(iv) ⇒ (ii) It follows from $R = R(awa)^* + {}^0a$ that $Ra = R(awa)^*a$. Similarly, we have $Ra = Rawa$ since $R = Raw + {}^0a$. So, $a \in R(awa)^*a \cap Rawa$. □

Let $n \geq 2$ be an integer. It was proved in [25, Theorem 2.26] that $a \in R_w^{\#}$ if and only if $a \in R((aw)^*)^n a \cap R(aw)^{n-1}a$. Applying this, we give another characterization of the w-core inverse based on properties of annihilators and ideals of certain elements as follows.

Theorem 2.15. *Let $a, w \in R$ and let $n \geq 2$ be an integer. Then the following conditions are equivalent*:

(i) $a \in R_w^{\#}$.

(ii) $a \in R((aw)^*)^n a \cap R(aw)^{n-1}a$.

(iii) $R = R((aw)^*)^n \oplus {}^0a = R(aw)^{n-1} \oplus {}^0a$.

(iv) $R = R((aw)^*)^n + {}^0a = R(aw)^{n-1} + {}^0a$.

In this case, $a_w^{\#} = (aw)^{n-1}x^*$, where $x \in R$ satisfies $a = x((aw)^*)^n a$.

Proof. (i) ⇔ (ii) by [25, Theorem 2.26].

The equivalences of the conditions (ii)-(iv) are similar to the equivalences of the conditions (ii)-(iv) in Theorem 2.14.

Next, we give the representation of $a_w^{\#}$. Given $a \in R((aw)^*)^n a \cap R(aw)^{n-1}a$, then $a = x((aw)^*)^n a = x((aw)^{n-1})^* w^* a^* a \in Ra^*a$ for some $x \in R$. Consequently, it follows from Lemma 2.4 that $a \in R^{\{1,3\}}$ and $w(aw)^{n-1}x^* \in a\{1,3\}$. Using Theorem 2.7, we get $a_w^{\#} = w^{\|a} a^{(1,3)} = w^{\|a} w(aw)^{n-1}x^* = w^{\|a} waw(aw)^{n-2}x^* = (aw)^{n-1}x^*$. □

Observe that Theorem 2.15 is not valid in general for the case of $n = 1$. One can see the counterexample in [25, Remark 2.27].

Remark 2.16. The representation of $a_w^{\#}$ can be expressed by another way. It follows from Theorem 2.15 (i) ⇒ (ii) that $a \in R((aw)^*)^n a \cap R(aw)^{n-1}a$. Hence, there is some $x \in R$ such that $a = x((aw)^*)^n a$ and $aw = x((aw)^*)^n aw$. Note also that $aw \in R((aw)^*)^n aw \cap R(aw)^n$. Then, by [11, Theorem 2.10], $aw \in R^{\#}$ and $(aw)^{\#} = (aw)^{n-1}x^*$. Therefore, $a_w^{\#} = (aw)^{\#} = (aw)^{n-1}x^*$ by [25, Theorem 2.26].

As shown in [25] that $a \in R^{\dagger}$ if and only if $a \in R_{a^*}^{\#}$, which is equivalent to $a \in R(aa^*)^n a \cap R(aa^*)^{n-1}a$ for all integers $n \geq 2$ by Theorem 2.15, i.e., $a \in R(aa^*)^{n+1}a \cap R(aa^*)^n a$ for all integers $n \geq 1$. We state that $a \in R(aa^*)^{n+1}a \cap R(aa^*)^n a$ can be reduced to $a \in R(aa^*)^n a$. Indeed, $a \in R(aa^*)^n a$ implies that there exists some $c \in R$ such that $a = c(aa^*)^n a = caa^*(aa^*)^{n-1}a = c(c(aa^*)^n a)a^*(aa^*)^{n-1}a = (c^2(aa^*)^{n-1})(aa^*)^{n+1}a \in R(aa^*)^{n+1}a$. In another word, $a \in R^{\dagger}$ if and only if $a \in R(aa^*)^n a$.

Set $w = 1$ and $w = a^*$ in Theorem 2.15, respectively, then several corollaries for the core inverse and the Moore-Penrose inverse can be obtained in a ring R.

Corollary 2.17. [11, Proposition 2.9 and Theorem 2.10] *Let $a, w \in R$ and let $n \geq 2$ be an integer. Then the following conditions are equivalent*:

(i) $a \in R^{\#}$.

(ii) $a \in R(a^*)^n a \cap Ra^n$.

(iii) $R = R(a^*)^n \oplus {}^0a = Ra^{n-1} \oplus {}^0a$.

(iv) $R = R(a^*)^n + {}^0a = Ra^{n-1} + {}^0a$.

Corollary 2.18. [19, Theorems 3.1 and 3.11] *Let $a, w \in R$ and let $n \geq 1$ be an integer. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$.

(ii) $a \in R(aa^*)^n a$.

(iii) $R = R(aa^*)^n \oplus {}^0a$.

(iv) $R = R(aa^*)^n + {}^0a$.

3. Characterizations for w-core inverses by Hermitian elements and units in a ring

An element $p \in R$ is called Hermitian if $p^* = p$. In addition, we call p a projection if p also satisfies $p = p^2$. We call $a \in R$ invertible if there exists an $x \in R$ such that $ax = xa = 1$. Such an x is called an inverse of a. It is unique if it exists, and is denoted by a^{-1}. By the symbol R^{-1} we denote the set of all invertible elements (or units) in R.

Li and Chen [11] derived the characterization for core inverses by Hermitian elements or projections in a ring, that is, $a \in R^{\#}$ if and only if there exists a Hermitian element (or a projection) $q \in R$ such that $qa = 0$ and $a^n + q \in R^{-1}$ for all integers $n \geq 1$.

Recently, Zhu et al. [25, Theorem 2.30] showed the characterization for w-core inverses, namely, $a \in R_w^{\#}$ if and only if there exists a (unique) Hermitian element (or a projection) $p \in R$ such that $pa = 0$ and $aw + p \in R^{-1}$. A natural question is that whether the characterization above holds if the index of aw extends from 1 to an arbitrary positive integer n. The following theorem shows that the hypothesis is valid.

Theorem 3.1. *Let $a, w \in R$ and let $n \geq 2$ be an integer. Then the following conditions are equivalent*:

(i) $a \in R_w^{\#}$.

(ii) *There exists a unique projection $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

(iii) *There exists a projection $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

(iv) *There exists a Hermitian element $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

In this case, $a_w^{\#} = (aw)^{n-1}u^{-1}$.

Proof. (ii) ⇒ (iii) and (iii) ⇒ (iv) are clear.

(i) ⇒ (ii) As $a \in R_w^{\#}$, then $awa_w^{\#}a = a$ by Lemma 2.6, and hence $(1 - awa_w^{\#})a = 0$. Set $p = 1 - awa_w^{\#}$, then $p^2 = p = p^*$ and $pa = 0$. By Lemma 2.1, we have

$$\begin{aligned} u((a_w^{\#})^n + 1 - a_w^{\#}aw) &= ((aw)^n + 1 - awa_w^{\#})((a_w^{\#})^n + 1 - a_w^{\#}aw) \\ &= (aw)^n(a_w^{\#})^n + (aw)^n(1 - a_w^{\#}aw) + (1 - awa_w^{\#})(a_w^{\#})^n + (1 - awa_w^{\#})(1 - a_w^{\#}aw) \\ &= awa_w^{\#} + 0 + 0 + 1 - awa_w^{\#} \\ &= 1. \end{aligned}$$

Similarly, it is easy to check $((a_w^{\#})^n + 1 - a_w^{\#}aw)u = 1$, and whence $u = (aw)^n + p \in R^{-1}$.

Next, we show that such p is unique. Let p, q satisfy $pa = 0 = qa$, $(aw)^n + p \in R^{-1}$ and $(aw)^n + q \in R^{-1}$. Since $(1 - p)((aw)^n + p) = (aw)^n$, we have $1 - p = (aw)^n((aw)^n + p)^{-1}$. Thus, $q(1 - p) = q(aw)^n((aw)^n + p)^{-1} = 0$, which implies $q = qp$. Similarly, we can get $p = pq$. Consequently, $p = p^* = (pq)^* = q^*p^* = qp = q$.

(iv) ⇒ (i) Suppose that there exists a Hermitian element $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$. Then $u^* = ((aw)^n)^* + p \in R^{-1}$. Post-multiplying the equation $u = (aw)^n + p$ by a yields $ua = (aw)^n a$. Then $a = u^{-1}(aw)^n a \in R(aw)^{n-1}a$. Again, post-multiplying the equation $u^* = ((aw)^n)^* + p$ by a yields $u^*a = ((aw)^n)^*a = ((aw)^*)^n a$. Then $a = (u^*)^{-1}((aw)^*)^n a \in R((aw)^*)^n a$, so that, $a \in R((aw)^*)^n a \cap R(aw)^{n-1}a$. By Theorem 2.15, we get $a \in R_w^{\#}$ and $a_w^{\#} = (aw)^{n-1}u^{-1}$. □

Remark 3.2. We give another representation of $a_w^{\#}$ in Theorem 3.1. Assume that there exists a projection $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$ in Theorem 3.1. Then $(1 - p)u = (aw)^n$, and we hence get $1 - p = (aw)^n u^{-1}$. According to Theorem 3.1 (iv) ⇒ (i), one gets that $w^{\|a} = u^{-1}(aw)^{n-1}a$ by Lemma 2.3, and $w(aw)^{n-1}u^{-1} \in a\{1, 3\}$. This in turn gives $a_w^{\#} = w^{\|a}a^{(1,3)} = (u^{-1}(aw)^{n-1}a)(w(aw)^{n-1}u^{-1}) = u^{-1}(aw)^{n-1}((aw)^n u^{-1}) = u^{-1}(aw)^{n-1}(1 - p)$ by Theorem 2.7, that is, $a_w^{\#} = u^{-1}(aw)^{n-1}(1 - p)$.

Applying Theorem 3.1, Remark 3.2 and [25, Theorem 2.30], one can get the following corollary.

Corollary 3.3. *Let $a, w \in R$ and let $n \geq 1$ be an integer. Then the following conditions are equivalent*:

(i) $a \in R_w^{\#}$.

(ii) *There exists a unique projection $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

(iii) *There exists a projection $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

(iv) *There exists a Hermitian element $p \in R$ such that $pa = 0$ and $u = (aw)^n + p \in R^{-1}$.*

For the case of $n = 1$, $a_w^{\#} = u^{-1}awu^{-1} = u^{-1}(1 - p)$.

For the case of $n \geq 2$, $a_w^{\#} = (aw)^{n-1}u^{-1} = u^{-1}(aw)^{n-1}(1 - p)$.

Proof. It suffices to prove $a_w^{\circledS} = u^{-1}awu^{-1}$ for the case of $n = 1$. Suppose $pa = 0$ and $u = aw + p \in R^{-1}$. Then $paw = 0$, and therefore, $aw \in R^{\circledS}$ in terms of [11, Theorem 3.4], which implies $aw \in R^{\#}$. Besides, we also obtain $ua = awa$ and $u^*a = (aw)^*a$. These ensure $a = u^{-1}awa = u^{-1}awaw(aw)^{\#}a = aw(aw)^{\#}a = awaw(aw)^{\#}(aw)^{\#}a \in awaR \cap Rawa$ and $a = (u^*)^{-1}(aw)^*a = (u^*)^{-1}w^*a^*a \in Ra^*a$ since $u \in R^{-1}$. Consequently, from Lemmas 2.3 and 2.4, it follows that $w \in R^{\|a}$ and $a \in R^{\{1,3\}}$. Moreover, $w^{\|a} = u^{-1}a$ and $a^{(1,3)} = wu^{-1}$. So, $a_w^{\circledS} = w^{\|a}a^{(1,3)} = u^{-1}awu^{-1}$ by Theorem 2.7. □

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