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Subgroups of products of para τ-discrete semitopological groups

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Abstract. In this article we define a new class of topological spaces called para τ-discrete spaces and give an internal characterization of a subgroup of product of para τ-discrete semitopological groups having character less than or equal to τ. Also we give a partial solution of an open problem posed by Sánchez [5, Problem 3.8].

1. Introduction

A semitopological group is a group endowed with a topology such that the right and left translations are continuous, and a paratopological group is a group endowed with a topology for which multiplication is continuous. If, additionally, the inversion in a paratopological group is continuous, then it is called a topological group.

A space X is regular if it is T_1 and for every closed subset $F \subseteq X$ and for every $x \notin F$, there exist disjoint open sets M, N such that $F \subseteq M$ and $x \in N$.

In [5], Sánchez obtained the following theorem: Let G be a regular semitopological group. Then G admits a homeomorphic embedding as a subgroup into a product of metrizable semitopological groups if and only if G has property (*) and countable index of regularity. A semitopological group G has property (*) if for every open neighborhood U of the identity e in G, the family $\{Ux : x \in G\}$ has an open basic refinement which is dominated by a countable family γ of open neighborhoods of e and σ-discrete with respect to γ. Also, a regular semitopological group G has a countable index of regularity, if for every neighborhood U of the identity e in G, there is a neighborhood V of e and a countable family γ of neighborhoods of e such that $\bigcap_{W\in\gamma} VW^{-1} \subseteq U$ (see[7]).

Notice that both the above notions require a countable family γ of neighborhoods of e. So the question still remains when a semitopological group satisfies weaker properties than the above, can G be embedded into a product of some semitopological groups?

For this purpose we introduce a new class of topological spaces called para τ-discrete spaces. This class contains the class of metric spaces. Then an analogue of Sánchez's theorem mentioned above is obtained in Section 3.

In Section 4, we answer partially a question [5, Problem 3.8] posed by Sánchez.

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2. Preliminaries

Let $\tau \geq \omega$ be a cardinal number and let γ be a family of subsets of a set X. Then γ is said to be τ-family if its cardinality $\left|\gamma\right|$ is less than or equal to τ. A family $\mathcal{U}$ of subsets of a set X is decomposable as a τ-union if there is an index set I and subfamilies $\mathcal{U}_i$, $i \in I$ of $\mathcal{U}$ such that $\mathcal{U} = \cup\{\mathcal{U}_i : i \in I\}$ and $|I| \leq \tau$.

Let G be a semitopological group with identity e. We denote by $\mathcal{N}(e)$ the family of open neighborhoods of e in G.

A family $\mathcal{U}$ of subsets of G is called τ-discrete if it is a τ-union of discrete families.

A family $\mathcal{U}$ of subsets of G is called discrete with respect to a family $\gamma \subseteq \mathcal{N}(e)$, if for each $x \in G$, there is a $V \in \gamma$ such that xV intersects at most one element of $\mathcal{U}$. Also, we say that a family $\mathcal{U}$ of subsets of G is τ-discrete with respect to a family $\gamma \subseteq \mathcal{N}(e)$, if $\mathcal{U}$ can be decomposed as a τ-union of families which are discrete with respect to γ. A family $\mathcal{U}$ of subsets of G is dominated by a family $\gamma \subseteq \mathcal{N}(e)$ if for each $U \in \mathcal{U}$ and $x \in U$ there is a $V \in \gamma$ such that $xV \subseteq U$.

A semitopological (paratopological) group G is called τ-balanced if for each $U \in \mathcal{N}(e)$, there exists a τ-family $\gamma \subseteq \mathcal{N}(e)$ such that for each $x \in G$ we can find $V \in \gamma$ with $xVx^{-1} \subseteq U$, such a family γ is usually called subordinated to U. A subset $U \subseteq G$ is called τ-good, if there is a τ-family $\gamma \subseteq \mathcal{N}(e)$ such that for each $x \in U$ there is a $V \in \gamma$ such that $xV \subseteq U$. It is clear that τ-good sets are open in G. Denote by $\mathcal{N}^*(e)$ the family of open neighborhoods of e in G that are τ-good. We say that G is locally τ-good if the family $\mathcal{N}^*(e)$ is a local base at e in G.

Let X be a topological space and $\mathcal{U}$ be a cover of it, then we say that a refinement $\mathcal{V}$ of $\mathcal{U}$ is basic if for every $U \in \mathcal{U}$ and $x \in U$ there exists a $V \in \mathcal{V}$ such that $x \in V \subseteq U$.

Definition 2.1. A topological space X is called para τ-discrete, if it has a base $\mathcal{B}$ which is τ-discrete, that is, $\mathcal{B} = \bigcup\{\mathcal{V}_i : i \in I\}$, where $|I| \leq \tau$ and each $\mathcal{V}_i$ is a discrete family, is a base for the topology of X.

Since each σ-discrete family is τ-discrete, it follows from the Bing metrization theorem that the class of metric spaces is contained in the class of para τ-discrete spaces.

Theorem 2.2. *Let X be a para τ-discrete space. Then every open cover of X has an open τ-discrete basic refinement.*

Proof. Let $\mathcal{U}$ be an open cover of X and $\mathcal{B}$ be a τ-discrete base of X. Put $\mathcal{V} = \{V \in \mathcal{B} : V \subseteq U$ for some $U \in \mathcal{U}\}$. Clearly, $\mathcal{V}$ is open τ-discrete basic refinement of $\mathcal{U}$. □

Theorem 2.3. *Let X be a topological space with character $\chi(X) \leq \tau$. If every open cover of X has an open τ-discrete basic refinement, then X is para τ-discrete space.*

Proof. For every $x \in X$, we can assume that $\{B_\alpha(x) : \alpha \in \tau\}$ is a local base at x. Put $\mathcal{U}_\alpha = \{B_\alpha(x) : x \in X\}$ for every $\alpha \in \tau$. Let $\mathcal{V}_\alpha$ be an open τ-discrete basic refinement of $\mathcal{U}_\alpha$, for all $\alpha \in \tau$. Then $\mathcal{B} = \bigcup_{\alpha\in\tau} \mathcal{V}_\alpha$ is τ-discrete. To show that $\mathcal{B}$ is a base for X, let O be an open set in X and $x \in O$. Then there exists an $\alpha \in \tau$ such that $x \in B_\alpha(x) \subseteq O$. Since $\mathcal{V}_\alpha$ is a basic refinement of $\mathcal{U}_\alpha$ and $x \in B_\alpha(x) \in \mathcal{U}_\alpha$, there exists a $V \in \mathcal{V}_\alpha$ such that $x \in V \subseteq B_\alpha(x)$. Thus $x \in V \subseteq O$ and $V \in \mathcal{B}$. Therefore, X is a para τ-discrete space. □

Definition 2.4. A semitopological group has property (τ^*) if for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic refinement which is dominated by a τ-family $\gamma \subseteq \mathcal{N}(e)$ and τ-discrete with respect to γ.

Theorem 2.5. *Let G be a semitopological group with character $\chi(G) \leq \tau$. Then G is para τ-discrete if and only if G has property (τ^*).*

Proof. Assume that G is para τ-discrete. Let $U \in \mathcal{N}(e)$. Then by Theorem 2.2, the family $\{Ux : x \in G\}$ has an open τ-discrete basic refinement $\mathcal{V}$. Since the character $\chi(G) \leq \tau$, there is a local base $\gamma \subseteq \mathcal{N}(e)$ at the identity e and this is a τ-family. Thus $\mathcal{V}$ is τ-discrete with respect to the τ-family γ and dominated by γ.

Conversely, assume that G has property (τ^*). Let $\{U_\alpha : \alpha \in \tau\}$ be a local base at the identity e in G. Then for each $\alpha \in \tau$, the family $\mathcal{U}_\alpha = \{U_\alpha x : x \in G\}$ has an open basic refinement $\mathcal{V}_\alpha$ which is dominated by a τ-family $\gamma_\alpha \subseteq \mathcal{N}(e)$ and τ-discrete with respect to γ_α. Put $\mathcal{B} = \bigcup_{\alpha\in\tau} \mathcal{V}_\alpha$. Then $\mathcal{B}$ is τ-discrete family with respect to $\bigcup_{\alpha\in\tau}\gamma_\alpha$. Since $\{U_\alpha x : x \in G, \alpha \in \tau\}$ is a base for G, $\mathcal{B}$ is a base for G. Indeed, let O be an open set in G and $x \in O$, then there is an $\alpha \in \tau$ such that $x \in U_\alpha x \subseteq O$. Since $\mathcal{V}_\alpha$ is a basic refinement of $\mathcal{U}_\alpha$, there is a $V \in \mathcal{V}_\alpha$ such that $x \in V \subseteq U_\alpha x$. This implies that $x \in V \subseteq O$ and $V \in \mathcal{B}$. Therefore, $\mathcal{B}$ is a τ-discrete base and hence G is a para τ-discrete space. □

Theorem 2.6. *If a semitopological group G has property (τ^*), then every subgroup of G has property (τ^*) as well.*

Proof. Let H be a subgroup of G and W be an open neighborhood of the identity e in H. Then there exists an open neighborhood U of e in G such that $W = U \cap H$. Since G has property (τ^*), the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{V}$ which is dominated by a τ-family $\gamma \subseteq \mathcal{N}(e)$ and τ-discrete with respect to γ.

Consider $\mathcal{V}_1 = \{O \in \mathcal{V} : O \subseteq Uh \text{ for some } h \in H\}$ and put $\mathcal{V}_H = \{O \cap H : O \in \mathcal{V}_1\}$. We claim that $\mathcal{V}_H$ is an open basic refinement of $\{Wh : h \in H\}$. Indeed, let $h_1 \in Wh = (U \cap H)h$ for some $h \in H$. Since $\mathcal{V}$ is a basic refinement of $\{Ux : x \in G\}$ and $h_1 \in Uh$, there exists an $O \in \mathcal{V}$ such that $h_1 \in O \subseteq Uh$. This implies that $O \in \mathcal{V}_1$ and $h_1 \in O \cap H \subseteq Uh \cap H = Uh \cap Hh = (U \cap H)h = Wh$. Thus $\mathcal{V}_H$ is open basic refinement of $\{Wh : h \in H\}$. To prove that, $\mathcal{V}_H$ is dominated by $\gamma_H = \{V \cap H : V \in \gamma\}$, let $h \in O \cap H \in \mathcal{V}_H$. Since $\mathcal{V}$ is dominated by γ, there is a $V \in \gamma$ such that $hV \subseteq O$. This implies that $h(V \cap H) \subseteq O \cap H$. Thus $\mathcal{V}_H$ is dominated by γ_H. Since $\mathcal{V}$ is τ-discrete with respect to γ. Therefore, $\mathcal{V}_H$ is τ-discrete with respect to γ_H. □

Theorem 2.7. *Let $\{G_i : i \in I \text{ and } |I| \leq \tau\}$ be a family of semitopological groups with character $\chi(G_i) \leq \tau$. If each G_i has property (τ^*), then $G = \prod_{i\in I} G_i$ has property (τ^*) as well.*

Proof. Let U be an open basic set containing the identity e in G. Put $U = \prod_{i\in I} U_i$. There exists a finite subset J of I such that $U_i = G_i$ for every $i \notin J$. Put $J = \{i_1, i_2, ..., i_r\}$. Let $K = \{1, 2, ..., r\}$. First we show that the family $\mathcal{V} = \{Ug : g \in G\}$ has an open basic refinement which is τ-discrete with respect to a τ-family and also dominated by the τ-family. Since G_{i_k} has property (τ^*) for all $k \in K$, the family $\mathcal{V}_{i_k} = \{U_{i_k} x : x \in G_{i_k}\}$ of G_{i_k} has an open basic refinement $\mathcal{W}_{i_k}$, which is τ-discrete with respect to a τ- family $\gamma_{i_k} \subseteq \mathcal{N}(e_{i_k})$ and dominated by γ_{i_k}. For each $k \in K$, we can put $\mathcal{W}_{i_k} = \bigcup_{\alpha\in\tau} \mathcal{W}_{i_k}(\alpha)$, where $\mathcal{W}_{i_k}(\alpha)$ is discrete with respect to γ_{i_k}. Let $A \in \tau^r$. Put

$$\mathcal{W}(A) = \left\{p_J^{-1}(W_1 \times \cdots \times W_r) : W_1 \in \mathcal{W}_{i_1}(A(1)), ..., W_r \in \mathcal{W}_{i_r}(A(r))\right\}.$$

and

$$\gamma_U = \left\{p_J^{-1}(O_1 \times \cdots \times O_r) : O_1 \in \gamma_{i_1}, ..., O_r \in \gamma_{i_r}\right\}.$$

Here p_J is the projection map from G onto $G_J = \prod_{j\in J} G_j$. Clearly, γ_U is a τ-family. We claim that $\mathcal{W}(A)$ is discrete with respect to γ_U. Let $g = (g_i) \in G$. Since for each $k \in K$, the family $\mathcal{W}_{i_k}(A(k))$ is discrete with respect to γ_{i_k}. Therefore, there exists $O_k \in \gamma_{i_k}$ such that $g_{i_k} O_k$ meets at most one element of $\mathcal{W}_{i_k}(A(k))$. Let $V = p_J^{-1}(O_1 \times \cdots \times O_k)$. Then gV intersects at most one element of $\mathcal{W}(A)$. Let $\mathcal{W}_U = \bigcup_{A\in\tau^r} \mathcal{W}(A)$. Clearly, $\mathcal{W}_U$ is τ-discrete with respect to γ_U.

Let $g = (g_i) \in G$ and $y = (y_i) \in Ug$. Then $y_{i_k} \in U_{i_k} g_{i_k}$ for all $k \in K$. Since $\mathcal{W}_{i_k}$ is open basic refinement of the family $\mathcal{V}_{i_k}$ for each $k \in K$, there exists a $W_k \in \mathcal{W}_{i_k}$ such that $y_{i_k} \in W_k \subseteq U_{i_k} g_{i_k}$. Let $W = p_J^{-1}(W_1 \times \cdots \times W_r)$. Then $y \in W \subseteq \prod_{i\in I} U_i g_i$. This proves that the family $\mathcal{W}_U$ is an open basic refinement of the family $\mathcal{V}$.

Now we show that the family $\mathcal{W}_U$ is dominated by the family γ_U. Let $W_k \in \mathcal{W}_{i_k}$ for every $k \in K$ and $y = (y_i) \in p_J^{-1}(W_1\times, ..., \times W_r)$. Then $y_{i_k} \in W_k$ for each $k \in K$. Since $\mathcal{W}_{i_k}$ is dominated by γ_{i_k} for each $k \in K$, there is a $O_k \in \gamma_{i_k}$ such that $y_{i_k} O_k \subseteq W_k$. This implies that

$$(y_i)p_J^{-1}(O_1\times, ..., \times O_r) \subseteq p_J^{-1}(W_1\times, ..., \times W_r).$$

By [2, Theorem 2.2.13], $\chi(G) \leq \tau$. Let $\mathcal{B}$ be a local base at the identity e in G with $|\mathcal{B}| \leq \tau$. Let O be any open in G containing the identity e. Let $\mathcal{W} = \bigcup\{\mathcal{W}_U : U \in \mathcal{B}\}$ and $\gamma = \bigcup\{\gamma_U : U \in \mathcal{B}\}$. Clearly, γ is a τ-family. Also $\mathcal{W}$ is open basic refinement of $\{Og : g \in G\}$ which is τ-discrete with respect to γ and dominated by γ. Thus, G has property (τ^*). □

Theorem 2.8. *Let G be a semitopological group with identity e. If for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic refinement dominated by a τ-family. Then G is τ-balanced and locally τ-good.*

Proof. Let $U \in \mathcal{N}(e)$ and the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{V}$ which is dominated by a τ-family $\gamma \subseteq \mathcal{N}(e)$. Let $g \in G$. Then there exists an $O \in \mathcal{V}$ such that $g \in O \subseteq Ug$. Since $\mathcal{V}$ is dominated by γ, there is a $V \in \gamma$ such that $gV \subseteq O$. Therefore, $gV \subseteq O \subseteq Ug$. Thus G is τ-balanced. Since $\mathcal{V}$ is a basic refinement, there exists a $V \in \mathcal{V}$ such that $e \in V \subseteq U$. Since $\mathcal{V}$ is dominated by the τ-family γ, the set V is τ-good. Thus, $\mathcal{V}$ is a local base at the identity e consisting of τ-good sets. Therefore, G is locally τ-good. □

Corollary 2.9. *Each semitopological group G with property (τ^*) is τ-balanced and locally τ-good.*

3. Characterizations of subgroups of products of para τ-discrete semitopological groups

Let $\mathcal{P}$ be a topological (or algebraic) property. Recall that a semitopological group G is *projectively-$\mathcal{P}$* or *range-$\mathcal{P}$* if for every neighborhood U of the identity element in G, there exists a continuous homomorphism $p : G \to H$ onto a semitopological group H with property $\mathcal{P}$ such that $p^{-1}(V) \subseteq U$, for some neighborhood V of the identity element of H (see[6, 7]).

Proposition 3.1. ([1, Theorem 3.4.21]) *Let $\mathcal{P}$ be a class of semitopological groups, τ an infinite cardinal number, and G a T_1 semitopological group, which is* range-$\mathcal{P}$ *and has a base $\mathcal{B}$ of open neighborhoods of identity element such that $|\mathcal{B}| \leq \tau$. Then G is topologically isomorphic to a subgroup of a product of a family $\{H_a : a \in \mathcal{A}\}$ of semitopological groups such that $H_a \in \mathcal{P}$, for each $a \in \mathcal{A}$, and $|\mathcal{A}| \leq \tau$.*

Lemma 3.2. ([4]) *Let G be a semitopological group with identity e. Suppose that a family $\gamma \subseteq \mathcal{N}(e)$ satisfies the following conditions:*

(a) for every $U \in \gamma$ and $x \in U$, there exists a $V \in \gamma$ such that $xV \subseteq U$;
(b) γ is subordinated to U, for each $U \in \gamma$.

Then the set $N = \bigcap\{U \cap U^{-1} : U \in \gamma\}$ is an invariant subgroup of G. Further, $UN = NU = U$ for each $U \in \gamma$.

Theorem 3.3. *Let G be a T_1 semitopological group with character $\chi(G) \leq \tau$. Then G admits a homeomorphic embedding as a subgroup into a product $\prod_{i\in I} H_i$, $|I| \leq \tau$, of para τ-discrete semitopological groups H_i with character less than or equal to τ if and only if G has property (τ^*).*

Proof. First suppose that G is topologically isomorphic to a subgroup of a product $H = \prod_{i\in I} H_i$, where $|I| \leq \tau$ and each H_i is para τ-discrete semitopological group having character $\chi(H_i) \leq \tau$. By Theorem 2.5, each H_i has property (τ^*). It follows from Theorems 2.6 and 2.7 that G has property (τ^*).

Conversely, assume that G has property (τ^*). By Proposition 3.1, it is sufficient to show that G is projectively para τ-discrete with character less than or equal to τ. Let $U_0 \in \mathcal{N}(e)$. We need to find a continuous homomorphism $p : G \to H$ onto a para τ-discrete semitopological group H with character $\chi(H) \leq \tau$ such that $p^{-1}(V_0) \subseteq U_0$ for some $V_0 \in \mathcal{N}(e_H)$.

We will construct by induction a sequence $\{\gamma_n : n \in \omega\}$ similarly as in Sánchez [5] such that for each $n \in \omega$:

(i) $\gamma_n \subseteq \mathcal{N}^*(e)$ and $|\gamma_n| \leq \tau$;
(ii) $\gamma_n \subseteq \gamma_{n+1}$;
(iii) γ_n is closed under finite intersections;
(iv) for every $U \in \gamma_n$ and $x \in U$, there exists a $V \in \gamma_{n+1}$ such that $xV \subseteq U$;
(v) the family γ_{n+1} is subordinated to U, for each $U \in \gamma_n$;
(vi) for each $U \in \gamma_n$, the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{V}_U$ which is τ-discrete with respect to γ_{n+1} and dominated by γ_{n+1}.

By Corollary 2.9, G is locally τ-good, that is, the family $\mathcal{N}^*(e)$ is a local base at the identity e in G. Then there exists a $U_0^* \in \mathcal{N}^*(e)$ such that $U_0^* \subseteq U_0$. Put $\gamma_0 = \{U_0^*\}$. Suppose that for some $n \in \omega$, we have defined families $\gamma_0, \dots \gamma_n$ satisfying (i)-(vi). By Corollary 2.9, G is τ-balanced. Since γ_n is a τ-family, we can find a τ-family $\lambda_{n,1} \subseteq \mathcal{N}^*(e)$ subordinated to every $U \in \gamma_n$. As $\gamma_n \subseteq \mathcal{N}^*(e)$, there exists a τ-family $\lambda_{n,2} \subseteq \mathcal{N}^*(e)$ such that for each $U \in \gamma_n$ and $x \in U$, there exists a $V \in \lambda_{n,2}$ satisfying $xV \subseteq U$. Since G has property (τ^*), so for every $U \in \gamma_n$, the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{V}_U$ which is τ-discrete with respect to a τ-family $\lambda_U \subseteq \mathcal{N}^*(e)$ and dominated by λ_U. Let γ_{n+1} be the minimal family contained in $\mathcal{N}^*(e)$ and containing $\gamma_n \cup \bigcup_{i=1}^2 \lambda_{n,i} \cup \bigcup_{U\in\gamma_n} \lambda_U$ and closed under finite intersections. Clearly, γ_{n+1} satisfies (i)-(vi). This finishes our construction.

Put $\gamma = \bigcup_{n\in\omega} \gamma_n$. Clearly, γ is a τ-family. By (vi), for each $U \in \gamma$, the family $\mathcal{V}_U$ is an open basic refinement of $\{Ug : g \in G\}$ which is dominated by γ and τ-discrete with respect to γ. By Lemma 3.2, $N = \bigcap\{V \cap V^{-1} : V \in \gamma\}$ is an invariant subgroup of G. Consider the abstract group G/N. Let $p : G \to G/N$ be the canonical homomorphism. Put $\mathcal{B} = \{p(V) : V \in \gamma\}$. Then the family $\mathcal{B}$ satisfies the following properties:

1. for each $O, P \in \mathcal{B}$, there exists a $Q \in \mathcal{B}$ such that $Q \subseteq O \cap P$;
2. for all $O \in \mathcal{B}$ and $o \in O$, there exists a $P \in \mathcal{B}$ such that $oP \subseteq O$;
3. $\mathcal{B}$ is subordinated to each $O \in \mathcal{B}$.

Indeed, the statement (1) follows from the fact that γ is closed under finite intersections. The statements (2) and (3) follow from (iv) and (v), respectively.

Put $H = G/N$. The statements (1)-(3) imply that there is a topology $\mathcal{T}$ on H such that $(H, \mathcal{T})$ is a semitopological group and the family $\mathcal{B}$ is a local base at e_H, where e_H is the identity of H. Since γ is a τ-family. So $\mathcal{B}$ is a τ-family. This implies that the character $\chi(H) \leq \tau$.

Let us show that H has property (τ^*). Let $p(U) \in \mathcal{B}, (U \in \gamma)$ be a basic open set in H. It is sufficient to show that the cover $\{p(U)h : h \in H\}$ has an open basic refinement which is dominated by $\mathcal{B}$ and τ-discrete with respect to $\mathcal{B}$. By (vi), the cover $\{Ug : g \in G\}$ has an open basic refinement $\mathcal{V}_U$ which is dominated by γ and τ-discrete with respect to γ. We claim that the family $p(\mathcal{V}_U) = \{p(P) : P \in \mathcal{V}_U\}$ is an open basic refinement of the family $\{p(U)h : h \in H\}$ and it is dominated by $\mathcal{B} = p(\gamma)$. Let $P \in \mathcal{V}_U$ and $y \in p(P)$. Then we have an $x \in P$ such that $p(x) = y$. Since $\mathcal{V}_U$ is dominated by γ, there is a $V \in \gamma$ such that $xV \subseteq P$. Therefore, $y \in yp(V) \subseteq p(P)$. Hence $p(P)$ is open in H. Let $p(U)h_1 \in \{p(U)h : h \in H\}$ and $y \in p(U)h_1$. Choose $g_1, x \in G$ such that $p(U)h_1 = p(Ug_1)$ and $y = p(x)$ with $x \in Ug_1$. Since $\mathcal{V}_U$ is an open basic refinement of $\{Ug : g \in G\}$, there is a $P \in \mathcal{V}_U$ such that $x \in P \subseteq Ug_1$. Therefore, $y \in p(P) \subseteq p(U)h_1$. Hence $p(\mathcal{V}_U)$ is an open basic refinement of $\{p(U)h : h \in H\}$.

Now we will show that $p(\mathcal{V}_U)$ is dominated by $\mathcal{B}$. Let $P \in \mathcal{V}_U$ and $y \in p(P)$. Then we can find an $x \in P$ such that $p(x) = y$. Since $\mathcal{V}_U$ is dominated by γ, so there is a $V \in \gamma$ such that $xV \subseteq P$. Hence, $yp(V) \subseteq p(P)$. Therefore, $p(\mathcal{V}_U)$ is dominated by $\mathcal{B}$.

We claim that $p(\mathcal{V}_U)$ is τ-discrete with respect to $\mathcal{B}$. Indeed, since $\mathcal{V}_U$ is τ-discrete with respect to γ, we can put $\mathcal{V}_U = \bigcup_{\alpha\in\tau} \mathcal{V}_\alpha$, where each $\mathcal{V}_\alpha$ is discrete with respect to γ. Fix $\alpha \in \tau$. Let $y \in H$. Then we have $x \in G$ such that $p(x) = y$. Since $\mathcal{V}_\alpha$ is discrete with respect to γ, there is a $V \in \gamma$ such that xV meets at most one element of $\mathcal{V}_\alpha$. Suppose that for some $P \in \mathcal{V}_\alpha$ we have that $p(P) \cap yp(V) \neq \emptyset$. Let $h \in p(P) \cap yp(V) \neq \emptyset$. Then we can find a $g \in P$ such that $p(g) = h$. Since $\mathcal{V}_\alpha$ is dominated by γ, there is a $W \in \gamma$ such that $gW \subseteq P$. Therefore, $h \in hp(W) \subseteq p(P)$ and $hp(W) \cap yp(V) \neq \emptyset$. So $gWN \cap xVN \neq \emptyset$. By Lemma 3.2, $gW \cap xV \neq \emptyset$. Thus $P \cap xV \neq \emptyset$. Since xV meets at most one element of $\mathcal{V}_\alpha$. Hence $yp(V)$ meets at most one element of $p(\mathcal{V}_\alpha) = \{p(P) : P \in \mathcal{V}_\alpha\}$. Thus we have proved $p(\mathcal{V}_\alpha)$ is discrete with respect to $\mathcal{B}$. Therefore, $p(\mathcal{V}_U) = \bigcup_{\alpha\in\tau} p(\mathcal{V}_\alpha)$ is τ-discrete with respect to $\mathcal{B}$.

Let $W \in \mathcal{N}(e_H)$. Consider the cover $\mathcal{W} = \{Wh : h \in H\}$. Let $y \in Wh$ for some $h \in H$. Then we can find a $U \in \gamma$ such that $y \in p(U)y \subseteq Wh$. Since $p(\mathcal{V}_U)$ is an open basic refinement of $\{p(U)h : h \in H\}$, there exists a $P \in \mathcal{V}_U$ such that $y \in p(P) \subseteq p(U)y$. Hence $y \in p(P) \subseteq Wh$. Let us take $\mathcal{V} = \bigcup_{U\in\gamma} p(\mathcal{V}_U)$. Then it follows that $\mathcal{V}$ is an open basic refinement of $\mathcal{W}$. Also $\mathcal{V}$ is dominated by $\mathcal{B}$ and τ-discrete with respect to $\mathcal{B}$. Thus we have proved that H is a semitopological group with character $\chi(H) \leq \tau$ having property (τ^*). By Theorem 2.5, H is para τ-discrete.

Finally, $V_0 = p(U_0^*)$ is an open neighborhood of e_H in H and $p^{-1}(V_0) = U_0^* \subseteq U_0$. Therefore, G is a projectively para τ-discrete with character less than or equal to τ. □

The index of regularity $Ir(G)$ of a regular semitopological group is defined as a minimum cardinal number κ such that for every neighborhood U of the identity e in G, one can find a neighborhood V of e and a family γ of neighborhoods of e in G such that $\bigcap_{W\in\gamma} VW^{-1} \subseteq U$ and $|\gamma| \leq \kappa$ ([7]).

Proposition 3.4. *Let G be a regular semitopological group with character $\chi(G) \leq \tau$, then $Ir(G) \leq \tau$.*

Proof. Let γ be a local base at the identity e of G with $|\gamma| \leq \tau$. Let $U \in \mathcal{N}(e)$. By regularity of G, there will be $V \in \mathcal{N}(e)$ such that $\overline{V} \subseteq U$. Let $x \in G \setminus U$. By regularity of G, there exist disjoint open sets M and N in G such that $\overline{V} \subseteq M$ and $x \in N$. Thus $V \cap N = \emptyset$ and $e \in x^{-1}N \cap V$. Then there is a $W \in \gamma$ such that $e \in W \subseteq x^{-1}N \cap V$. This implies that $xW \subseteq N$ and $xW \cap V = \emptyset$. Thus, $x \notin VW^{-1}$, which gives $\bigcap_{V\in\gamma} VW^{-1} \subseteq U$. □

Theorem 3.5. *Let G be a regular semitopological group with character $\chi(G) \leq \tau$. Then G admits a homeomorphic embedding as a subgroup into a product $\prod_{i\in I} H_i$, $|I| \leq \tau$, of regular para τ-discrete semitopological groups H_i with character less than or equal to τ if and only if G has property (τ^*).*

Proof. Suppose that G is topologically isomorphic to a subgroup of a product $H = \prod_{i\in I} H_i$, where $|I| \leq \tau$ and each H_i is regular para τ-discrete semitopological group having character $\chi(H_i) \leq \tau$. By Theorem 2.5, each H_i has property (τ^*). It follows from Theorems 2.6 and 2.7 that G has property (τ^*).

Conversely, assume that G has property (τ^*). By Proposition 3.1, it is sufficient to show that G is projectively regular para τ-discrete with character less than or equal to τ. Let $U_0 \in \mathcal{N}(e)$. We need to find a continuous homomorphism $p : G \to H$ onto a regular para τ-discrete semitopological group H with character $\chi(H) \leq \tau$ such that $p^{-1}(V_0) \subseteq U_0$ for some $V_0 \in \mathcal{N}(e_H)$.

We will construct by induction a sequence $\{\gamma_n : n \in \omega\}$ similarly as in Theorem 3.3 satisfying the properties (i) to (vi) and the following:

(vii) for each $U \in \gamma_n$, there exists a $V \in \gamma_{n+1}$ such that $\bigcap_{W\in\gamma_{n+1}} VW^{-1} \subseteq U$;

By Corollary 2.9, G is locally τ-good, that is, the family $\mathcal{N}^*(e)$ is a local base at the identity e in G. Then there exists a $U_0^* \in \mathcal{N}^*(e)$ such that $U_0^* \subseteq U_0$. Put $\gamma_0 = \{U_0^*\}$. Suppose that for some $n \in \omega$, we have defined families $\gamma_0, \dots \gamma_n$ satisfying (i)-(vii). By Corollary 2.9, G is τ-balanced. Since γ_n is a τ-family, we can find a τ-family $\lambda_{n,1} \subseteq \mathcal{N}^*(e)$ subordinated to every $U \in \gamma_n$. As $\gamma_n \subseteq \mathcal{N}^*(e)$, there exists a τ-family $\lambda_{n,2} \subseteq \mathcal{N}^*(e)$ such that for each $U \in \gamma_n$ and $x \in U$, there exists a $V \in \lambda_{n,2}$ satisfying $xV \subseteq U$. Since G has property (τ^*), so for every $U \in \gamma_n$, the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{V}_U$ which is τ-discrete with respect to a τ-family $\lambda_U \subseteq \mathcal{N}^*(e)$ and dominated by λ_U. By Proposition 3.4, $Ir(G) \leq \tau$, we can find a τ-family $\lambda_{n,3} \subseteq \mathcal{N}^*(e)$ such that for every $U \in \gamma_n$, there exists a $V \in \lambda_{n,3}$ satisfying $\bigcap_{W\in\lambda_{n,3}} VW^{-1} \subseteq U$. Let γ_{n+1} be the minimal family contained in $\mathcal{N}^*(e)$ and containing $\gamma_n \cup \bigcup_{i=1}^{3} \lambda_{n,i} \cup \bigcup_{U\in\gamma_n} \lambda_U$ and closed under finite intersections. Clearly, γ_{n+1} satisfies (i)-(vii). This finishes our construction.

Put $\gamma = \bigcup_{n\in\omega} \gamma_n$. Clearly, γ is a τ-family. By (vi), for each $U \in \gamma$, the family $\mathcal{V}_U$ is an open basic refinement of $\{Ug : g \in G\}$ which is dominated by γ and τ-discrete with respect to γ. By Lemma 3.2, $N = \bigcap\{V \cap V^{-1} : V \in \gamma\}$ is an invariant subgroup of G. By (vii) we have that for each $U \in \gamma$, there is a $V \in \gamma$ such that $\bigcap_{W\in\gamma} VW^{-1} \subseteq U$. It follows from (iii) and (vii) that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$, for every $U \in \gamma$. So that $N = \bigcap_{V\in\gamma} VV^{-1}$. Consider the abstract group G/N. Let $p : G \to G/N$ be the canonical homomorphism. Put $\mathcal{B} = \{p(V) : V \in \gamma\}$. Then the family $\mathcal{B}$ as in preceding Theorem 3.3 generates a topology $\mathcal{T}$ on $H = G/N$ such that $(H, \mathcal{T})$ is a semitopological group and the family $\mathcal{B}$ is a local base at the identity e_H, and character $\chi(H) \leq \tau$.

We show that the semitopological group $(H, \mathcal{T})$ is Hausdorff. Take $y \in H \setminus \{e_H\}$. Then there is a $x \in G \setminus N$ such that $p(x) = y$. Since $x \notin N = \bigcap_{V\in\gamma} VV^{-1}$, there exists a $V \in \gamma$ such that $x \notin VV^{-1}$, Equivalently, $xV \cap V = \emptyset$. It follows from Lemma 3.2 that $xVN \cap VN = \emptyset$, so that $yp(V) \cap p(V) = \emptyset$. Since H is a homogeneous space. Thus H is Hausdorff.

Let $p(U) \in \mathcal{B}, (U \in \gamma)$ be a local neighborhood of the identity e_H in H. Then there is a $V \in \gamma$ such that $\bigcap_{W\in\gamma} VW^{-1} \subseteq U$. We claim that $\overline{p(V)} \subseteq p(U)$. Let $y \in H \setminus p(U)$. Then there is a $x \in G \setminus U$ such that $p(x) = y$. Since $x \notin U$, there is a $W \in \gamma$ such that $x \notin VW^{-1}$. Thus $xW \cap V = \emptyset$. By Lemma 3.2, $xWN \cap VN = \emptyset$. Hence

$yp(W) \cap p(V) = \emptyset$. This implies that $y \notin \overline{p(V)}$. Therefore $\overline{p(V)} \subseteq p(U)$. Since H is a homogeneous space, we conclude that H is regular.

We have shown that H has property (τ^*) in Theorem 3.3. By Theorem 2.5, H is para τ-discrete semitopological group. Thus we have proved that H is a regular para τ-discrete semitopological group with character $\chi(H) \leq \tau$.

Finally, $V_0 = p(U_0^*)$ is an open neighborhood of e_H in H and $p^{-1}(V_0) = U_0^* \subseteq U_0$. Therefore, G is a projectively regular para τ-discrete with character less than or equal to τ. □

A semitopological group G is called τ-narrow if for every neighborhood U of the identity e in G, there exists a subset $A \subseteq G$ with $|A| \leq \tau$ such that $AU = G = UA$ (see [1, p. 286]).

Proposition 3.6. *If G is projectively τ-narrow semitopological group, then G is τ-narrow.*

Proof. Let U be an open neighborhood of the identity e_G in G. Since G is projectively τ-narrow semitopological group, there exists a continuous homomorphism f from G onto a τ-narrow semitopological group H such that $f^{-1}(V) \subseteq U$, for some open neighborhood V of the identity e_H in H. Since H is τ-narrow and V is an open neighborhood of e_H, there is a subset $D \subseteq H$ with $|D| \leq \tau$ such that $DV = H = VD$. Choose a subset $A \subseteq G$ such that $f(A) = D$ and $|A| \leq \tau$.

We show that $AU = G = UA$. Clearly, $AU \subseteq G$. Let $g \in G$. Then there is a $d \in D$ and a $v \in V$ such that $f(g) = dv$. We can find an $a \in A$ with $f(a) = d$. Thus $f(g) = f(a)v$, or equivalently, $f(a^{-1}g) = v$. Thus $a^{-1}g \in f^{-1}(V) \subseteq U$. This implies that $g \in aU \subseteq AU$, So $G \subseteq AU$. Thus we have proved that $G = AU$. Similarly, $G = UA$. □

Example 3.7. The Sorgenfrey line is a para $\mathfrak{c}$-discrete semitopological group. Since $\mathcal{B} = \{[x, x+r) : x \in \mathbf{R}, r > 0\} = \bigcup\{\mathcal{V}_{x,r} : x \in \mathbf{R} \text{ and } r > 0\}$ where $\mathcal{V}_{x,r} = \{[x, x+r)\}$, is a $\mathfrak{c}$-discrete base for Sorgenfrey line. Therefore, Sorgenfrey line is para $\mathfrak{c}$-discrete. Let $U \in \mathcal{N}(0)$. By Theorem 2.2, the family $\{U + x : x \in \mathbf{R}\}$ has an open $\mathfrak{c}$-discrete basic refinement $\mathcal{V}$ (say). Since $\gamma = \{[0, r) : r > 0\}$ is a local base at the identity 0 of cardinality $\mathfrak{c}$. Thus $\mathcal{V}$ is $\mathfrak{c}$-discrete with respect to $\mathfrak{c}$-family γ. Let $V \in \mathcal{V}$ and $v \in V$. Then there exists $W \in \gamma$ such that $v + W \subseteq V$. Hence $\mathcal{V}$ is dominated by $\mathfrak{c}$-family γ. Therefore, Sorgenfrey line has property $(\mathfrak{c}^*)$, but it does not have property (ω^*) [5, Example 3.5]. Note that, the property (*)[5] and property (ω^*) are same.

In other way, since Sorgenfrey line is para $\mathfrak{c}$-discrete semitopological group. Thus, by Theorem 2.5, it has property $(\mathfrak{c}^*)$. If Sorgenfrey line has property (ω^*). By Theorem 2.5, Sorgenfrey line is para ω-discrete. Thus it has a σ-discrete base, which is not possible. Therefore, Sorgenfrey line does not have property (ω^*).

4. On problem of Sánchez

We refer the reader to [5] for basic notations of semitopological group with property (*).

Definition 4.1. ([5, Definition 2.3]) A semitopological group G has property (*) if for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic refinement which is dominated by a countable family $\gamma \subseteq \mathcal{N}(e)$ and σ-discrete with respect to γ.

In 2017, Sánchez posed the following question.

Question 4.2. ([5, Problem 3.8]) *Let G be a regular paratopological group such that for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{U}$ which is σ-discrete with respect to countable family γ. Does G have property (*)? What if, additionally, G is ω-balanced?*

The following theorem answers the above Question partially.

Theorem 4.3. *Let G be a first-countable paratopological group such that for every $U \in \mathcal{N}(e)$, the family $\{Ux : x \in G\}$ has an open basic refinement $\mathcal{U}$ which is σ-discrete with respect to countable family γ. Then G has property (*).*

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