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Some separation axioms for partially ordered sets

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Abstract. The main goal of this paper is to introduce and study new classes of partial orderings on a set with Krull dimension at most 1. We show that these classes are related to door, submaximal and Whyburn T_0-spaces.

1. Introduction

A binary relation on a set X is said to be a quasi-order if it is reflexive and transitive. If, in addition, the quasi-order is antisymmetric, it will be a partial order. In the present paper, all binary relations will be considered orders. A poset $(X, \leq)$ is a couple of a non-empty set X and a partial order $\leq$.

Given a poset $(X, \leq)$ and an element $x \in X$, we denote by $(\downarrow x] = \{y \in X : y \leq x\}$ and $[x \uparrow) = \{y \in X : x \leq y\}$. More generally, given a subset A of X, we denote by $(\downarrow A] = \cup\{(\downarrow x] : x \in A\}$ and $[A \uparrow) = \cup\{[x \uparrow) : x \in A\}$. The subset $\{y \in (\downarrow x], y \neq x\}$ will be denoted simply by $(\downarrow x)$. Writing $x < y$, for some $x, y \in X$, means $x \leq y$ and $y \neq x$. A point $x \in X$ is called an isolated point if $[x \uparrow) = (\downarrow x] = \{x\}$. $Iso(X)$ will denote the family of all isolated points in $(X, \leq)$.

A collection $\{x_0, ..., x_n\}$ of elements of a poset $(X, \leq)$ is said to be a chain of length n if $x_0 < ... < x_n$. The supremum of lengths of all chains is called the Krull dimension of X and it is denoted by $dim_K(X)$ [7].

In this paper, we are interested in some new classes of partial orderings on a set X with Krull dimension at most 1. Our first aim in this manuscript is to We elaborate relations between those introduced classes illustrated by significant examples and counterexamples.

Before this let us introduce the following definitions.

Definition 1.1. A poset $(X, \leq)$ is called a T_{DD}-poset if and only if for every distinct points $x, y \in X, (\downarrow x) \cap (\downarrow y) = \emptyset$.

Definition 1.2. A poset $(X, \leq)$ is called a submaximal poset if and only if for every $x \in X$ and for every finite set $F \in X$ such that $x \notin F$ we have $[x \uparrow) \cap F = \emptyset$ or $\{x\} \cap [F \uparrow) = \emptyset$.

Definition 1.3. A poset $(X, \leq)$ is called a door poset if and only if for every two disjoint finite subsets F_1 and F_2 in X, we have $[F_1 \uparrow) \cap F_2 = \emptyset$ or $F_1 \cap [F_2 \uparrow) = \emptyset$.

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Definition 1.4. A poset $(X, \leq)$ is called a T_Y-poset if and only if for every distinct points $x, y \in X$, $|\ (\downarrow x] \cap (\downarrow y]\ | \leq 1$.

Definition 1.5. A poset $(X, \leq)$ is called a Whyburn poset if and only if for every $x \in X$, $|(\downarrow x]| \leq 2$.

In our works and when we study Alexandroff spaces, we can see that the most fundamental and needed property is that the category of all Alexandroff spaces is isomorphic to the category of all quasi-ordered sets. Taking a qoset (quasi-ordered set) $(X, \leq)$, the collection$\mathcal{B} = \{[x \uparrow)\ x \in X\}$ forms a basis of a topology on X, denoted by $\tau(\leq)$, called the Alexandroff topology on X defined by $\leq$. In this case the closure $\overline{\{x\}}^{\tau(\leq)}$, for every $x \in X$, is exactly $(\downarrow x]$ which means that $\tau(\leq)$ is an Alexandroff topology on X. Conversely, taking an Alexandroff space (X, τ), the binary relation $\leq_\tau$ defined by $x \leq_\tau y$ if and only if $x \in \overline{\{y\}}^{\tau}$ is a quasi-order on X. Now, the maps $\phi : \mathbf{Qos} \longrightarrow \mathbf{Alx}$ such that $\phi((X, \leq)) = (X, \tau(\leq))$ and $\psi : \mathbf{Alx} \longrightarrow \mathbf{Qos}$ such that $\psi((X, \tau)) = (X, \leq_\tau)$ are inverse one of the other which means that, considering an Alexandroff space is equivalent to consider a quasi-ordered set (For more information see [2], [1] and [5]).

Recall that a topological space (X, τ) is called a T_0-space if and only if for any $x, y \in X$, we have $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$. It is clearly seen that an Alexandroff space (X, τ) is a T_0-space if and only if $(X, \leq_\tau)$ is a poset.

As an interesting class of Alexandroff spaces, there is the class of functional Alexandroff spaces called also primal spaces. Shirazi and Golestani [25] and Echi [10] , working independently, have explicitly introduced a class of Alexandroff spaces called by Echi primal spaces and called by Shirazi and Golestani functional Alexandroff spaces. In this paper, we will use the terminology "primal" to designate those spaces. Given a map $f : X \longrightarrow X$, we define the quasi-ordered set $(X, \leq_f)$ by $y \leq_f x$ if and only if $y = f^n(x)$ for $n \in \mathbb{N}$. Hence the corresponding topological space $(X, \tau(\leq_f))$ is exactly the primal space $(X, \mathcal{P}(f))$. Since their recent introduction, primal spaces have been further investigated in [8, 11–14, 17–20, 25].

The second goal of this paper is to applied the obtained new classes of partial orderings in the case of the corresponding specialization order $\leq_\tau$ of a T_0 topology τ. Consequently, the particular case $\leq_f$ of a given T_0-primal topology $(X, \mathcal{P}(f))$ is studied.

Hence, in the first section of this paper, we characterize T_{DD}, door, T_Y and submaximal posets. We prove that

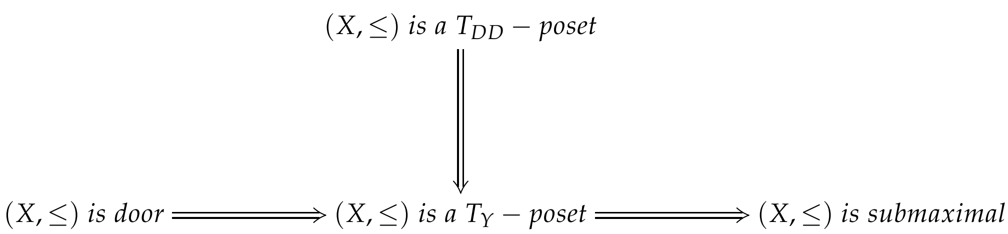


Figure 1

In the second section, we apply the results given in the first section in the particular case of the partial order $\leq_\tau$, for a given T_0 topology τ. We prove that the class of submaximal poset (resp, door poset) is isomorphic to the class of submaximal (resp, door) topological spaces.

In the third section we prove that in the particular case of $\leq_f$, the axioms T_Y, Whyburn and submaximal coincides. We find the following diagram.

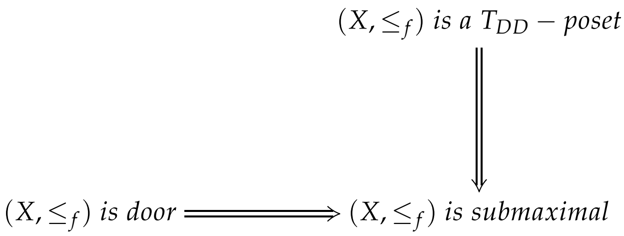


Figure 2

Along this paper, we use the Hass-diagram for orders. When the next situation express that $a < b$:

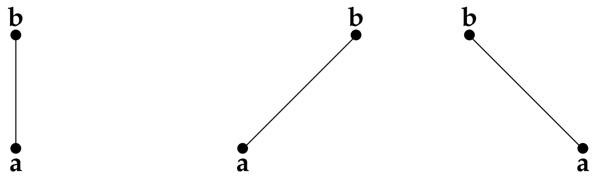


Figure 3

2. Posets of Krull dimension at most 1

Using the definition of T_{DD}-posets, the following characterization is immediate.

Proposition 2.1. *Let $(X, \leq)$ be a poset. Then $(X, \leq)$ is a T_{DD}-poset set if and only if for every distinct points $x, y \in X$, $(\downarrow x] \cap (\downarrow y] \in \{\emptyset, \{x\}, \{y\}\}$.*

Now, we give another characterization of T_{DD}-posets.

Proposition 2.2. *Let $(X, \leq)$ be a poset. Then $(X, \leq)$ is a T_{DD}-poset if and only if for every $x \in X$, $\mid [x \uparrow) \mid \leq 2$.*

Proof. Assume $\mid [x \uparrow) \mid > 2$, then there exist $y \neq z$ such that $x < y$ and $x < z$. So $x \in (\downarrow y) \cap (\downarrow z)$, contradicting the fact that X is a T_{DD}-poset.

Conversely, suppose that for every $x \in X$, $\mid [x \uparrow) \mid \leq 2$. Let $a \neq b$ be in X. If there exists $c \in (\downarrow a) \cap (\downarrow b)$, then $a, b, c \in [c \uparrow)$, a contradiction. As a result $(\downarrow a) \cap (\downarrow b) = \emptyset$, and X is a T_{DD}-poset. $\square$

The graph of $X \setminus Iso(X)$ of a T_{DD}-poset is a disjoint union of components of type:

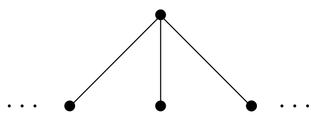


Figure 4

Proposition 2.3. *Let $(X, \leq)$ be a poset. Then, the following statements are equivalent.*

(i) $(X, \leq)$ *is a submaximal poset;*

(iii) $\dim_K(X, \leq) \leq 1$.

Proof. $(i) \implies (ii)$ Suppose $\dim_K(X, \leq) \geq 2$; then there exists a chain $x < y < z$ in X. Hence, taking $F = \{x, z\}$, we have $y \notin F, z \in [y \uparrow) \cap F$ and $y \in [F \uparrow)$, contradicting the fact that X is a submaximal poset.

$(ii) \implies (i)$ Assume that X is not a submaximal poset. Hence, there exist a finite subset F and $x \notin F$ with $x \in [F \uparrow)$ and $F \cap [x \uparrow) \neq \emptyset$. Thus there exist $y, z \in F$ such that $x < z$. So that $y < x < z$, contradicting that assumption $\dim_K(X, \leq) \leq 1$. □

Proposition 2.4. *A poset* $(X, \leq)$ *is a* T_Y*-poset if and only if it is submaximal and for every* x, y *in* X $|(\downarrow x) \cap (\downarrow y)| \leq 1$.

Proof. For the direct implication, using Proposition 2.3, it suffices to verify that $\dim_K(X, \leq) \leq 1$. Indeed, if there exists a chain $x < y < z$ in X, then $|(\downarrow z] \cap (\downarrow y]| \geq 2$, a contradiction.

For $x \neq y$ in X, as $(\downarrow x) \cap (\downarrow y) \subseteq (\downarrow x] \cap (\downarrow y]$, we deduce that $|(\downarrow x) \cap (\downarrow y)| \leq 1$.

Conversely, assume that $(X, \leq)$ is a submaximal poset and for every x, y in X, $|(\downarrow x) \cap (\downarrow y)| \leq 1$. We consider two cases.

Case 1. If x, y are comparable (for instance $x < y$), then in this case $(\downarrow x) \cap (\downarrow y) = (\downarrow x)$. But as $\dim_K(X, \leq) \leq 1$, we get $(\downarrow x) = \emptyset$. It follows that $(\downarrow x] \cap (\downarrow y] = \{x\}$.

Case 2. If x, y are incomparable, then in this case $(\downarrow x] \cap (\downarrow y] = (\downarrow x) \cap (\downarrow y)$, and we are done.

Therefore $(X, \leq)$ is a T_Y-poset. □

The following diagrams represent the possible components of $X \setminus Iso(X)$ of a T_Y-poset:

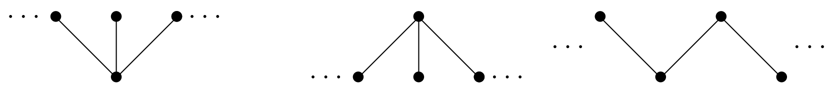


Figure 5

Proposition 2.5. *For a poset* $(X, \leq)$*, the following diagram for implications holds.*

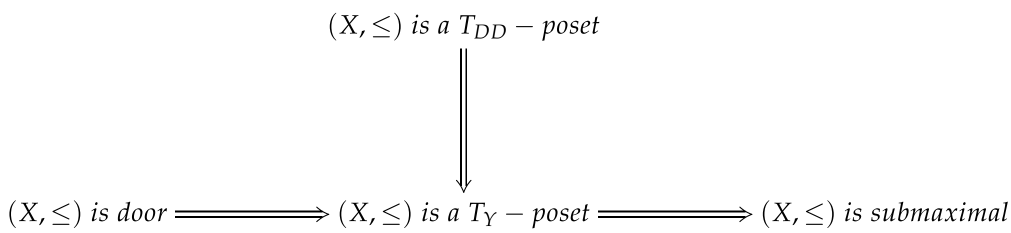


Figure 6

Proof. According to Proposition 2.4, only two implications deserve proofs.

(i)]$(X, \leq)$ is door $\implies (X, \leq)$ is a T_Y poset.

Let $(X, \leq)$ be a door poset and suppose that X is not T_Y. Let x, y be two distinct elements such that $|(\downarrow x] \cap (\downarrow y]| \geq 2$. Let z and t be two distinct points in X with $\{z, t\} \subseteq (\downarrow x] \cap (\downarrow y]$. Since x and y are distinct, then $\{z, t\} \neq \{x, y\}$. There are two cases:

First case Exactly one of z, t is equal to x or y. Hence in this case we can suppose that $(\downarrow x] \cap (\downarrow y]$ contain $\{x, z\}$. So let $F_1 = \{x\}$ and $F_2 = \{z, y\}$. Clearly, F_1 and F_2 are two disjoint subsets of X but $[F_1 \uparrow) \cap F_2 = \{y\}$ and $[F_2 \uparrow) \cap F_1 = \{x\}$, contradiction.

Second case x, y, z, t are mutually distinct points. So let $F_1 = \{z, y\}$ and $F_2 = \{x, t\}$. Clearly, F_1 and F_2 are two disjoint subsets of X but $[F_1 \uparrow) \cap F_2 = \{x\}$ and $[F_2 \uparrow) \cap F_1$ contain at least y, contradiction.

(ii)] $(X, \leq)$ is $T_{DD} \Longrightarrow (X, \leq)$ is a T_Y.

Let $(X, \leq)$ be a T_{DD}-poset. By Proposition 2.1 any distinct points x and y satisfies $(\downarrow x] \cap (\downarrow y] \in \{\emptyset, \{x\}, \{y\}\}$ and thus $| (\downarrow x] \cap (\downarrow y] | \leq 1$. Therefore $(X, \leq)$ is a T_Y-poset. □

Remark 2.6. One may check easily that none of the implications in the previous proposition is reversible.

Let us close this section by the following remark.
=

Remark 2.7. Let $(X, \leq) \in \{T_1, T_{DD}, door, T_Y, submaximal\}$, then $(X, \leq)$ is a submaximal poset and thus by Proposition 2.3, $\dim_K(X, \leq) \leq 1$. Therefore, the graph of $(X, \leq)$ has no chain of length greater than 2.

3. Door and submaximal Alexandroff spaces

A topological space X is called submaximal if every dense subspace of X is open in X. Some authors add the condition that X has no isolated points to the definition of such spaces. Hewitt [15] calls submaximal spaces without isolated points MI-spaces. The significance of considering submaximal spaces is provided by the theory of maximal spaces. A topological space X is called maximal if it is dense-in-itself and no larger topology on the set X is dense-in-itself.

In [20] and [1, Proposition 2.2] the authors give a characterization of submaximal spaces in the class of Alexandroff spaces as follows.

Theorem 3.1. *Let $(X, \leq)$ be a poset; then $(X, \tau(\leq))$ is a submaximal space if and only if $\dim_K(X, \leq) \leq 1$.*

Combining Theorem 3.1 and Proposition 2.3, we obtain the following result, justifying the introduction of submaximal posets.

Theorem 3.2. *Let $(X, \leq)$ be a poset. Then, the following statements are equivalent:*

(i) $(X, \leq)$ *is a submaximal poset;*

(ii) $(X, \tau(\leq))$ *is a submaximal space.*

A topological space is a door space if and only if every set is either open or closed. Check that a door space is submaximal. Considering Alexandroff door spaces, the following Theorem was proved in [20].

Theorem 3.3. *Let $(X, \leq)$ be a poset; then $(X, \tau(\leq))$ is a door space if and only if $\dim_K(X, \leq) \leq 1$ and the poset $(Y = X \setminus Iso(X))$ is either empty or has one of the following forms:*

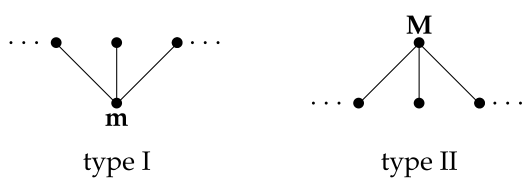


Figure 7

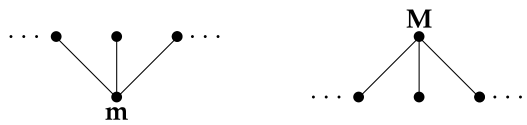
The following result gives a a relation between door spaces and door posets.

Proposition 3.4. *Let $(X, \leq)$ be a poset. The following statements are equivalent:*

(i) $(X, \tau(\leq))$ *is a door space;*

(ii) $(X, \leq)$ *is a door poset.*

Proof. $(i) \Longrightarrow (ii)$ Suppose that $(X, \tau(\leq))$ is a door space. Then by Theorem 3.3, $Y = X \setminus Iso(X)$ has one of the following forms:



type I type II

Figure 8

Now, let F_1 and F_2 be two disjoint subsets. Then, for $i = 1, 2$ $F_i = (F_i \cap Iso(X)) \cup (F_i \cap Y)$ and $[F_i \uparrow) = (F_i \cap Iso(X)) \cup ([F_i \uparrow) \cap Y)$. So, $[F_1 \uparrow) \cap F_2 = \emptyset$ (resp, $[F_2 \uparrow) \cap F_1 = \emptyset$) means that $([F_1 \uparrow) \cap F_2) \cap Y = \emptyset$ (resp, $([F_2 \uparrow) \cap F_1) \cap Y = \emptyset$). Thus, we can suppose that X is without isolated point.

If X is of type I, then for every subset F, we have $[F \uparrow) = F$ if $m \notin F$ and $[F \uparrow) = X$ if not. Since F_1 and F_2 are disjoint subsets, then either $[F_1 \uparrow) \cap F_2 = F_1 \cap F_2$, or $[F_2 \uparrow) \cap F_1 = F_2 \cap F_1$ and thus one of them is empty.

If X is of type II, then $[F \uparrow) = F$ if M belongs to F and $[F \uparrow) = F \cup \{M\}$ if not. Hence suppose that $[F_1 \uparrow) \cap F_2 \neq \emptyset$. Then $(F_1 \cup \{M\}) \cap F_2 \neq \emptyset$ and thus $\{M\} \cap F_2 \neq \emptyset$ which implies that $M \in F_2$. In this case $[F_2 \uparrow) = F_2$ and consequently, $[F_2 \uparrow) \cap F_1 = \emptyset$.

$(ii) \Longrightarrow (i)$ Let $(X, \leq)$ be a door poset. By Remark 2.7, the graph of the specialization order $\leq$ has no chains of length greater than 2. Assume, the existence of two chains of length 2 without common point. That is, there exist four pairwise distinct points a, b, c, d such that the unique non trivial relations are $b < a$ and $d < c$. Hence if we set $F_1 = \{a, d\}$ and $F_2 = \{b, c\}$, then neither $[F_1 \uparrow) \cap F_2 = \emptyset$ nor $[F_2 \uparrow) \cap F_1 =]\emptyset$. Therefore all chains of length 2 contain a common point, which is necessarily a maximal point or a minimal point. □

A topological space X is called a Whyburn space [23] if for every non-closed subset A of X and for every $x \in \overline{A} \setminus A$, there exists $B \subseteq A$ such that $\overline{B} \setminus A = \{x\}$. It is called weakly Whyburn [24] if for every non-closed subset A of X there exists $B \subseteq A$ such that $\overline{B} \setminus A$ is a one point set. Clearly, every Whyburn space is weakly Whyburn. The characterization of Whyburn spaces and weakly Whyburn spaces in the class of Alexandroff spaces was given by [20] as follow.

Theorem 3.5. *Let* (X, τ) *be an Alexandroff space. Then the following statements are equivalent:*

(i) X *is Whyburn;*

(ii) X *is weakly Whyburn;*

(iii) The closure of each element of X *has at most* 2 *points, that is,* $|(\downarrow x]| \leq 2$.

Remark 3.6. Regarding Theorem 3.5, every connected component in a Whyburn space is either a 2-cycle (i.e: a pair (a, b), $a \neq b$, $a \leq_\tau b$ and $b \leq_\tau a$), a single point or a component of type I of Figure 8 (we can consider single point as a particular case of type I, when there is no points below m).

Theorem 3.5 enables us to introduce the following concept.

Definition 3.7. We say that $(X, \leq)$ is a Whyburn poset if for every $x \in X$, $|(\downarrow x]| \leq 2$.

Remark 3.8. Let $(X, \leq)$ be a poset.

(1) $(X, \leq)$ is a Whyburn poset if and only if $(X, \tau(\leq))$ is a Whyburn space.

(2) Since $(X, \leq)$ is a poset, then if it is Whyburn there is no 2-cycles and every connected component is of type:

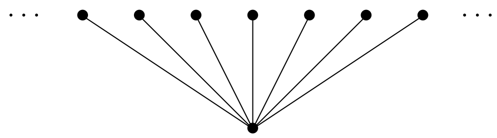


Figure 9

Proposition 3.9. *Every Whyburn poset is a T_Y-poset.*

Proof. Let $(X, \leq)$ be a Whyburn poset. For any distinct points x and y, $|(\downarrow x]| \leq 2$ and $|(\downarrow y]| \leq 2$, then $|(\downarrow x] \cap (\downarrow y]| \leq 2$ and it is exactly 2 if and only if $(\downarrow x] = (\downarrow y]$ and thus $x = y$ which is not the case. therefore $|\ (\downarrow x] \cap (\downarrow y]\ | \leq 1$. □

Remark 3.10. (1) A Whyburn poset need not be a door poset.

(2) A Whyburn poset need not be a T_{DD}-poset.

(3) A door poset (resp, T_{DD}-poset) need not be a Whyburn poset.

4. Primal posets

Let X be a set and $f : X \longrightarrow X$ be a map. We define the quasi-ordered set $(X, \leq_f)$ (called primal quasi-ordered set) by for any $x, y \in X$, $y \leq_f x$ if and only if $y = f^n(x)$ for some $n \in \mathbb{N}$. Hence, the corresponding topological space $(X, \tau_{\leq_f})$ is exactly the primal space $(X, \mathcal{P}(f))$. Recall that an element $x \in X$ is said to be a periodic point if $f^n(x) = x$ for some positive integer $n > 1$. It is called a fixed point if $f(x) = x$. The following result is an immediate consequence of [10, Proposition 2.5].

Lemma 4.1. *Let $(X, \leq_f)$ be a primal quasi-ordered set.*

(i) $(X, \leq_f)$ is an equality poset if and only if f is the identity map.

(ii) $(X, \leq_f)$ is a poset if and only if f is without periodic point.

Now, since in our study all given binary relations are partial orders, then all considered functions f are without periodic points.

Let us start by characterizing T_{DD}-poset.

Proposition 4.2. *A primal poset $(X, \leq_f)$ is a T_{DD}-poset if and only if the following properties hold.*

(i) $f^2 = f$.

(ii) For all $a \neq b \in X$, if $f(a) = f(b)$ then a or b is a fixed point.

Proof. Suppose that there exists $x \in X$ such that $f(f(x)) \neq f(x)$, then $f^2(x) <_f f(x) <_f x$ and consequently $f^2(x) \in (\downarrow x) \cap (\downarrow f(x))$ contradicting the fact that $(X, \leq_f)$ is a T_{DD} poset.

Conversely, assume the properties (i) and (ii) hold. Let $x \neq y$ in X. If we suppose that $(\downarrow x) \cap (\downarrow y) \neq \emptyset$, then there exists $z \in X$ such that $z <_f x$ and $z <_f y$. As $f^2 = f$, we have $f(x) = z = f(y)$. Hence x or y is a fixed point, this leads to $x = z$ or $y = z$, a contradiction. □

Remark 4.3. A T_{DD}-poset need not be a primal poset.

The following proposition follows immediately from combining [9, Proposition 4.1] and Theorem 3.2.

Proposition 4.4. *Let $(X, \leq_f)$ be a primal poset. Then, the following statements are equivalent:*

(i) $(X, \leq_f)$ *is a submaximal poset;*

(ii) $(X, \mathcal{P}(f))$ *is a submaximal topological space;*

(iii) $f^2 = f$.

Remark 4.5. A submaximal poset need not be primal.

The following proposition is an immediate consequence of [9, Proposition 4.3] and Proposition 2.3.

Proposition 4.6. *Let $(X, \leq_f)$ be a primal poset. Then, the following statements are equivalent.*

(i) $(X, \leq_f)$ *is a door poset;*

(ii) $(X, \mathcal{P}(f))$ *is a door topological space;*

(iii) *There exist at most one element $a \in X$ such that $|[\uparrow a)| > 1$.*

Remark 4.7. A door poset need not be primal.

Proposition 4.8. *Let $(X, \leq_f)$ be a primal poset.*

(i) $(X, \leq_f)$ *is a T_Y-primal poset if and only if $(X, \leq_f)$ is a submaximal poset.*

(ii) $(X, \leq_f)$ *is a Whyburn-primal poset if and only if $(X, \leq_f)$ is a submaximal poset.*

Proof. (i) Using Proposition 2.4 it is enough to see that every submaximal primal poset satisfies: for all $x, y \in X, x \neq y, |\ (\downarrow x) \cap (\downarrow y)\ | \leq 1$ is. So, Let x and y be two distinct points. If x or y is fixed, then $(\downarrow x) \cap (\downarrow y)$ is empty and thus $|\ (\downarrow x) \cap (\downarrow y)\ | = 0 \leq 1$. If not $(\downarrow x) \cap (\downarrow y) = \{f(x)\} \cap \{f(y)\}$ and also $|\ (\downarrow x) \cap (\downarrow y)\ | \leq 1$.

(ii) Let $(X, \leq_f)$ be a primal poset. By definition $(X, \leq_f)$ is a Whyburn-primal poset if and only $(X, \mathcal{P}(f))$ is a Whyburn space. Using [22, Corollary 4], $(X, \mathcal{P}(f))$ is a Whyburn space if and only if $f^2(x) \in \{x, f(x)\}$.But $\leq_f$ is an order, then f is without periodic point and thus $f^2(x) = f(x)$ for every x. Finally, Proposition 4.4 complete the proof. □

Corollary 4.9. *Let $(X, \leq_f)$ be a primal poset. Then the following statements are equivalent:*

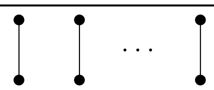
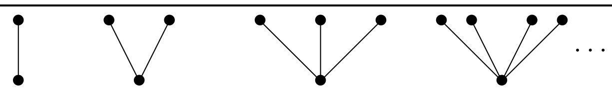
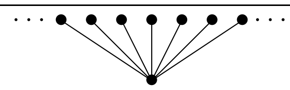
(i) $(X, \leq_f)$ *is a submaximal primal poset;*

(ii) $(X, \leq_f)$ *is a T_Y-primal poset;*

(iii) $(X, \leq_f)$ *is a Whyburn primal poset;*

(iv) $f^2 = f$.

The following table give the possible diagrams of $X \setminus Iso(X)$ for our primal posets:

Primal poset	Diagram
T_{DD}-primal	
submaximal primal, T_Y-primal, Whyburn primal	
door primal	 (One component)

The following chart shows the ordering relations between our main separation axioms.

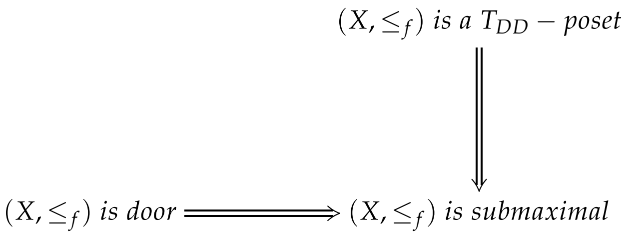


Figure 10

Example 4.10. (i) Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by, for every $n \in \mathbb{Z}, f(n) = (-1)^n$. It is clear that f is without periodic point and Thus $(X, \leq_f)$ is a primal poset. But we have the sequence $-1 < 1 < 0$, which implies that $(X, \leq_f)$ is not a submaximal primal poset.

Such situation can be illustrated by the following figure.

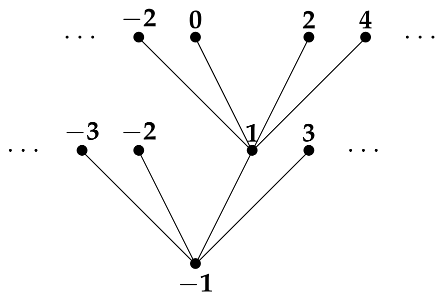


Figure 11

(ii)] Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by, for every $n \in \mathbb{Z}, f(n) =\mid n \mid$. It is clear that $(X, \leq_f)$ is a submaximal primal poset which is not door. This example show also that this set is T_{DD} which is not door. Such situation can be illustrated by the following figure.

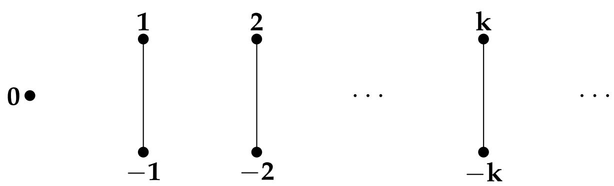


Figure 12

(iii) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ the constant map zero. $(X, \leq_f)$ is a door primal poset which is not T_{DD}. This example show also that this poset is submaximal which is not T_{DD}. Such situation can be illustrated by the following figure.

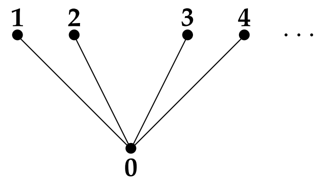


Figure 13

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