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Note on the Banach Problem 1 of condensations of Banach spaces onto compacta

Alexander V. Osipov[a]

[a] *Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, Yekaterinburg, Russia*

Abstract. It is consistent with any possible value of the continuum $\mathfrak{c}$ that every infinite-dimensional Banach space of density $\leq \mathfrak{c}$ condenses onto the Hilbert cube.

Let $\mu < \mathfrak{c}$ be a cardinal of uncountable cofinality. It is consistent that the continuum be arbitrary large, no Banach space X of density γ, $\mu < \gamma < \mathfrak{c}$, condenses onto a compact metric space, but any Banach space of density μ admits a condensation onto a compact metric space. In particular, for $\mu = \omega_1$, it is consistent that $\mathfrak{c}$ is arbitrarily large, no Banach space of density γ, $\omega_1 < \gamma < \mathfrak{c}$, condenses onto a compact metric space.

These results imply a complete answer to the Problem 1 in the Scottish Book for Banach spaces:

When does a Banach space X admit a bijective continuous mapping onto a compact metric space?

1. Introduction

The following problem is a reformulation of the well-known problem of Stefan Banach from the Scottish Book:

Banach Problem. When does a metric (possibly Banach) space X admit a condensation (i.e. a bijective continuous mapping) onto a compactum (= compact metric space) ?

M. Katetov [6] was one of the first who attacked the Banach problem. He proved that: a countable regular space has a condensation onto a compactum if, and only if, it is scattered (a space is said to be *scattered* if every nonempty subset of it has an isolated point).

Recall that a topological space is *Polish* if X is homeomorphic to a separable complete metric space and a topological space X is *σ-compact* if X is a countable union of compact subsets.

In 1941, A.S. Parhomenko [9] constructed a example of a σ-compact Polish space X such that X does not have a condensation onto a compact space.

Recall that a space X is called *absolute Borel*, if X is homeomorphic to a Borel subset of some complete metrizable space.

In 1976, E.G. Pytkeev [10] proved the following remarkable theorem for separable absolute Borel non-σ-compact spaces.

Theorem 1.1. *Every separable absolute Borel space X condenses onto the Hilbert cube, whenever X is not σ-compact.*

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Email address: OAB@list.ru (Alexander V. Osipov)

Pytkeev's result implies that every separable complete non-σ-compact metric space condenses onto the Hilbert cube. Thus every infinite-dimensional separable complete linear metric space (and, hence, each infinite-dimensional Banach space) admits a condensation onto a compactum.

It is well known that any locally compact space admits a condensation onto a compact space (Parhomenko's Theorem) [9]. Hence, each separable metrizable locally compact space (and thus each finite-dimensional Banach space) condenses onto a compactum. Thus every separable Banach space admits a condensation onto a compactum.

The *density* $d(X)$ of a topological space X is the smallest cardinality of a dense subset of X. Since metrizable compact spaces have cardinality at most continuum, every metric space admitting a condensation onto a compactum has density at most continuum.

T.Banakh and A.Plichko [1] proved the following interesting result.

Theorem 1.2. *Every Banach space X of density $\aleph_1$ or $\mathfrak{c}$ admits a condensation onto the Hilbert cube.*

Question 1.3. *What about intermediate densities between $\aleph_1$ and $\mathfrak{c}$?*

It is clear that this question cannot be answered from ZFC alone: its status depends on one's model of set theory.

In [2], T. Banakh announces the following results:

(1) It is consistent that the continuum is arbitrarily large and every infinite-dimensional Banach space of density $\leq \mathfrak{c}$ condenses onto the Hilbert cube $[0,1]^\omega$.

(2) It is consistent that the continuum is arbitrarily large and **no** Banach space of density $\aleph_1 < d(X) < \mathfrak{c}$ condenses onto a compact metric space.

In this paper we give an independent proof of these results.

2. Main results

Theorem 2.1. ([2]) *If for some infinite cardinal κ there is a partition of real line by κ many Borel sets, then any Banach space of density κ condenses onto the Hilbert cube.*

In [4] (Theorem 3.8), W.R. Brian and A.W. Miller proved the following result.

Theorem 2.2. *It is consistent with any possible value of $\mathfrak{c}$ that for every $\kappa \leq \mathfrak{c}$ there is a partition of 2^ω into κ closed sets.*

The following theorem is mathematical folklore. It is a corollary of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *It is consistent with any possible value of $\mathfrak{c}$ that every infinite-dimensional Banach space of density $\leq \mathfrak{c}$ condenses onto the Hilbert cube.*

Proof. Because ω^ω can be identified with a co-countable subset of 2^ω, the model in Theorem 2.2 has, for every $\kappa < \mathfrak{c}$, a partition of ω^ω (and hence the real line, identifying ω^ω with irrational numbers) into κ Borel sets. It remain to apply Theorem 2.1. $\square$

In fact the proof of Theorem 2.3 (using the Brian-Miller model of set theory) is a minor modification of the proof of Main Theorem from [1].

Let $FIN(\kappa, 2)$ be the partial order of finite partial functions from κ to 2, i.e., Cohen forcing.

Proposition 2.4. (Corollary 3.13 in [4]) *Suppose M is a countable transitive model of ZFC + GCH. Let κ be any cardinal of M of uncountable cofinality which is not the successor of a cardinal of countable cofinality. Suppose that G is $FIN(\kappa, 2)$-generic over M, then in $M[G]$ the continuum is κ and for every uncountable $\gamma < \kappa$ if $F : \gamma^\omega \to \omega^\omega$ is continuous and onto, then there exists a $Q \in [\gamma]^{\omega_1}$ such that $F(Q^\omega) = \omega^\omega$.*

Note that trivial modifications to the proof of Proposition 3.14 in [4] allow us to replace ω_2 with any cardinal μ of uncountably cofinality.

Proposition 2.5. *Assume that μ is a cardinal of uncountably cofinality. It is consistent that the continuum be arbitrary large, ω^ω can be partitioned into μ Borel sets, and ω^ω is not a condensation of κ^ω whenever $\mu < \kappa < \mathfrak{c}$.*

Theorem 2.6. *Suppose μ is a cardinal of uncountable cofinality. It is consistent that the continuum be arbitrary large, no Banach space X of density γ, $\mu < \gamma < \mathfrak{c}$, condenses onto a compactum, but any Banach space of density μ admit a condensation onto a compactum.*

Proof. Suppose M is a countable transitive model of $ZFC + GCH$. Let $\kappa > \mu$ be any cardinal of M of uncountable cofinality which is not the successor of a cardinal of countable cofinality. Suppose that G is $FIN(\kappa, 2)$-generic over M, then in $M[G]$ the continuum is κ and for every uncountable $\gamma < \kappa$ if $F : \gamma^\omega \to \omega^\omega$ is continuous and onto, then there exists a $Q \in [\gamma]^\mu$ such that $F(Q^\omega) = \omega^\omega$ (Proposition 2.4 (Corollary 3.13 in [4]) with replacement ω_1 with any cardinal $\mu < \kappa$ of uncountably cofinality).

By Proposition 2.5, ω^ω can be partitioned into μ Borel sets. By Theorem 2.1, any Banach space of density μ admits a condensation onto the Hilbert cube $[0, 1]^\omega$.

The proof of Theorem 3.7 in [8] uses Cohen reals, but the same idea shows that this generic extension has the property that

($\star$) for every family $\mathcal{F}$ of Borel subsets of ω^ω with size $\mu < |\mathcal{F}| < \mathfrak{c}$, if $\bigcup \mathcal{F} = \omega^\omega$ then there exists $\mathcal{F}_0 \in [\mathcal{F}]^\mu$ with $\bigcup \mathcal{F}_0 = \omega^\omega$ (see Proposition 3.14 in [4]).

Let $\mu < \gamma < \mathfrak{c}$. It suffices to note that any Banach space of density γ is homeomorphic to $J(\gamma)^\omega$ where $J(\gamma)$ is hedgehog of weight γ (Theorem 5.1, Remark and Theorem 6.1 in [11]).

Let f be a condensation from γ^ω onto $J(\gamma)^\omega$ [10]. Suppose that g is a condensation of $J(\gamma)^\omega$ onto a compact metric space K. Then we have the condensation $h = g \circ f : \gamma^\omega \to K$ of γ^ω onto K.

Let $\sum = [\gamma]^\omega \cap M$. Note that $|\sum| < \mathfrak{c}$ since in M $|\gamma^\omega| > \gamma$ if and only if γ has cofinality ω, but in that case $|\gamma^\omega| = |\gamma^+| < \mathfrak{c}$. Since the forcing is c.c.c.

$M[G] \models \gamma^\omega = \bigcup\{Y^\omega : Y \in \sum\}$.

For any $Y \in \sum$ the continuous image $h(Y^\omega)$ is an analytic set (a Σ^1_1 set) and, hence the union of ω_1 Borel sets in K (see Ch.3, § 39, Corollary 3 in [7]), i.e., $h(Y^\omega) = \bigcup\{B(Y, \beta) : \beta < \omega_1\}$ where $B(Y, \beta)$ is a Borel subset of K for each $\beta < \omega_1$. Note that $|\{B(Y, \beta) : Y \in \sum, \beta < \omega_1\}| \leq |\sum| \cdot \aleph_1 = |\sum|$.

Assume that $\theta = |\{B(Y, \beta) : Y \in \sum, \beta < \omega_1\}| < \gamma$. Consider a function $\phi : \{B(Y, \beta) : Y \in \sum, \beta < \omega_1\} \to \sum$ such that $\phi(B(Y, \beta)) = Y_\xi \in \sum$ where $h(Y^\omega_\xi)$ contains in decomposition $B(Y, \beta)$ (Y_ξ may be the same for different $B(Y_1, \beta_1)$ and $B(Y_2, \beta_2)$). Then $\bigcup\{Y_\xi : \xi \in \theta\} \in [\gamma]^{\leq\theta}$ and $\gamma^\omega = \bigcup\{Y^\omega_\xi : \xi \in \theta\}$ is a contradiction. Thus, $\gamma \leq \theta \leq |\sum| < \mathfrak{c}$.

Since K is Polish, there is a continuous surjection $p : \omega^\omega \to K$. Given a family $\mathcal{F} = \{p^{-1}(B(Y, \beta)) : Y \in \sum, \beta < \omega_1\}$ of θ-many Borel sets ($\mu < \theta < \mathfrak{c}$) whose union is ω^ω. By property ($\star$), there is a subfamily $\mathcal{F}_0 = \{F_\alpha : F_\alpha = p^{-1}(B(Y_\alpha, \beta_\alpha)), \alpha < \mu\}$ of size μ whose union is ω^ω. Then the family $\{h(Y^\omega_\alpha) : \alpha < \mu\}$ of size μ whose union is K. Let $Q = \bigcup\{Y_\alpha : \alpha < \mu\}$. Then $Q \in [\gamma]^\mu$ and $h(Q^\omega) = K$. Since $\mu < \gamma$, we obtain a contradiction with injectivity of the mapping h. □

By Theorem 2.6 for $\mu = \omega_1$ we have the following result.

Theorem 2.7. *Suppose M is a countable transitive model of $ZFC + GCH$. Suppose that G is $FIN(\mathfrak{c}, 2)$-generic over M. No Banach space X of density γ, $\aleph_1 < \gamma < \mathfrak{c}$ condenses onto a compact metric space.*

In [3], W. Brian proved the following result.

Theorem 2.8. *Let $\kappa < \aleph_\omega$, let $f : Y \to X$ be a condensation of a topological space Y onto a Banach space X of density κ. Then there is a partition of Y into κ Borel sets.*

Theorem 2.9. *Let $n < \omega$. The following assertions are equivalent:*

1. *Any Banach space X of density $\aleph_n$ condenses onto the Hilbert cube;*
2. *ω^ω can be partitioned into $\aleph_n$ Borel sets;*
3. *ω^ω is a condensation of ω_n^ω.*

Proof. (2) ⇒ (1) Since there is a partition of ω^ω into $\aleph_n$ Borel sets then, by Theorem 2.1 , any Banach space of weight $\aleph_n$ admit a condensation onto the Hilbert cube.

(1) ⇒ (2) Since $J(\aleph_n)^\omega$ is a condensation of ω_n^ω and $J(\aleph_n)^\omega$ admit a condensation onto the Hilbert cube $[0,1]^\omega$ then $[0,1]^\omega$ can be partitioned into $\aleph_n$ Polish sets B_α (Theorem 2.8). Since $[0,1]^\omega$ is Polish there is a continuous surjection $p : \omega^\omega \to [0,1]^\omega$. Hence, ω^ω can be partitioned into $\aleph_n$ Borel sets $p^{-1}(B_\alpha)$.

(2) ⇔ (3) By Theorem 3.6 in [4]. □

By Theorems 2.6 and 2.9 and Theorem 3.2 (and Corollaries 3.3 and 3.4) in [5] we have the following results for $\aleph_0 < \kappa \leq \mathfrak{c}$.

Corollary 2.10. *Given any $A \subseteq \mathbb{N}$, there is a forcing extension in which*

1. *Any Banach space X of density $\kappa \in \{\aleph_n : n \in A\} \cup \{\aleph_1, \aleph_\omega, \aleph_{\omega+1} = \mathfrak{c}\}$ condenses onto the Hilbert cube;*
2. *No Banach space X of density $\kappa \notin \{\aleph_n : n \in A\} \cup \{\aleph_1, \aleph_\omega, \aleph_{\omega+1} = \mathfrak{c}\}$ condenses onto a compact metric space.*

Corollary 2.11. *Given any finite $A \subseteq \mathbb{N}$, there is a forcing extension in which*

1. *Any Banach space X of density $\kappa \in \{\aleph_n : n \in A\} \cup \{\aleph_1\}$ condenses onto the Hilbert cube;*
2. *No Banach space X of density $\kappa \notin \{\aleph_n : n \in A\} \cup \{\aleph_1\}$ condenses onto a compact metric space.*

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