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Hypersurfaces of metallic Riemannian manifolds as k-almost Newton-Ricci solitons

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Abstract. This research investigates k-Almost Newton-Ricci solitons (k-ANRS) embedded in a metallic Riemannian manifold $\mathcal{M}^n$ having the potential function ψ. Furthermore, we prove geodesic and minimal conditions for hypersurfaces of metallic Riemannian manifolds. Beside this, we have explained some applications of metallic Riemannian manifold admitting k-Almost Newton-Ricci solitons.

1. Background

The Ricci flow theory was developed in 1982 by Richard S. Hamilton, which he presented in his groundbreaking work, was an exploration of a Riemannian manifold (M, g) [36]

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)), \quad g(0) = g_0,$$

in this equation, Ric indicates the Ricci tensor, while t indicates time. It is helpful in smoothing out singularities in a metric to deform it.

Assume V represents any vector field on M and identify the Lie derivative along V with the notation $\mathcal{L}_V$. Then, Ricci soliton on (M, g) is presented by (g, V, λ) and it can be viewed as an extension of Einstein metric and obeying

$$\frac{1}{2}\mathcal{L}_V g + Ric + \lambda g = 0, \tag{1}$$

λ is some real scalar. One also notices that a Ricci soliton becomes

- shrinking provided $\lambda < 0$,
- steady with $\lambda = 0$,

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- expanding for $\lambda > 0$.

Moreover, ψ describes any smooth function by $\psi : M \rightarrow \mathcal{R}$ and V be standing for the gradient of potential function $-\psi$. In this case, g will be termed as *gradient Ricci soliton*. Also, (1) reduces to

$$\nabla\nabla\psi = Ric + \lambda g, \tag{2}$$

wherein $\nabla\nabla\psi$ denotes the *Hessian* of ψ. The Einstein manifold with constant potential function [14] results in the trivial gradient Ricci soliton [23].

According to Pigola et al. [40], in (1), taking the constant λ and rewriting it as a smooth function in $\lambda \in C^\infty(M)$ produces almost Ricci soliton on manifold (M, g), which can be written as (g, V, λ). Cantino and Mazzieri ([15], [16]) reported it to be evolved from the Ricci-Bourguignon flow. Equation (1) can be used to define an almost Ricci soliton. Many geometers have conducted substantial research on the aforesaid solitons. Researchers in [9] investigated the properties of isometric immersions in solitons of this sort and Wylie [45] demonstrated compactness qualities. [22] examined the immersed almost Ricci soliton under P_k (Newton transformation) with second order differential operator L_k for $0 \leq k \leq n$, referred as k-ANRS. In [42], Siddiqi also discussed Ricci-Bourguignon almost solitons. For further literature, we refer ([3],[15], [17]-[19],[25],[27], [43],[41]) and the references therein.

On the other side, the very initial work about golden structure on a Riemannian manifold was carried out in [13] and it gave birth to new ideas about golden mean. This notion was further extended to metallic means by [35] producing golden mean as particular case. In the recent past, a plenty of good results have been established by different researchers regarding metallic means family. For further study, one may refer to ([4],[6],[7], [21],[30]-[32]). Also, an extensive work on warped product manifolds endowed with metallic structure has been carried out in [5] (see also [33], [32]). The preceding literature served as inspiration for the current article. We investigate k-almost Newton-Ricci solitons on the hypersurface of metallic Riemannian manifolds in this framework.

2. Metallic Riemannian manifolds

([13], [28], [2]) For Riemannian manifold $(\overline{\mathcal{M}}^m, g)$ and real numbers $a_1, \ldots, a_n$, $(1,1)$-tensor field F produces a polynomial structure when $P(F) = 0$, in this case

$$P(V) := V^n + a_n V^{n-1} + \cdots + a_2 V + a_1 I, \tag{3}$$

with I being used for identity transformation defined on $\Gamma(T\overline{\mathcal{M}})$.

A $(1,1)$-tensor field φ defined on $\overline{\mathcal{M}}$ yields a metallic structure such that

$$\varphi^2 = p\varphi + qI, \forall p, q \in \mathbb{N}^*.$$

The following relation also holds

$$g(U, \varphi V) = g(\varphi U, V), \qquad \forall U, V \in \Gamma(T\overline{\mathcal{M}}). \tag{4}$$

A metallic Riemannian manifold $\overline{\mathcal{M}}$ satisfies (4).

Using φU in place of U

$$g(\varphi U, \varphi V) = pg(U, \varphi V) + qg(U, V).$$

It is noted that metallic structure reduces to golden when $p = q = 1$ ([13], [34]). Also, F describes an almost product structure on $(\overline{\mathcal{M}}^m, g)$ provided $F^2 = I$ with $F \neq \pm I$ [2] . Furthermore, $(\overline{\mathcal{M}}, d)$ becomes almost product Riemannian manifold provided

$$g(FU, V) = g(U, FV).$$

For further details on metallic structures, we refer [35].

Definition 2.1. *[4] (i) The linear connection ∇ on metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \varphi)$ is φ-connection if*

$$\nabla\varphi = 0. \tag{5}$$

(ii) $(\overline{\mathcal{M}}, g, \varphi)$ represents locally metallic Riemannian manifold provided the Levi-Civita connection of g denoted by $\overline{\nabla}$ satisfies (5).

One has the decomposition

$$T_x\overline{\mathcal{M}} = T_x\mathcal{M} \oplus T_x^{\perp}\mathcal{M}, \;\; x \in \mathcal{M}.$$

When $(\overline{\mathcal{M}} = \mathcal{M}_1 \times \mathcal{M}_2, F)$ stands for locally Riemannian product manifold, $\mathcal{M}_1$ and $\mathcal{M}_2$ possessing constant sectional curvatures c_1 and c_2, resp. One can write [4]

$$\begin{aligned}\mathcal{R}(v_1, v_2)v_3 &= \frac{1}{4}(c_1 + c_2)[g(v_2, v_3)v_1 - g(v_1, v_3)v_2 + g(Fv_2, v_3)Fv_1 \\ &- g(Fv_1, v_3)Fv_2] + \frac{1}{4}(c_1 - c_2)[g(Fv_2, v_3)v_1 \\ &- g(Fv_1, v_3)v_2 + g(v_2, v_3)Fv_1 - g(v_1, v_3)Fv_2].\end{aligned} \tag{6}$$

In view of almost product structure and (6), we achieve [20]

$$\begin{aligned}\mathcal{R}(v_1, v_2)v_3 &= \frac{1}{4}(c_1 + c_2)[g(v_2, v_3)v_1 - g(v_1, v_3)v_2] \\ &+ \frac{1}{4}(c_1 + c_2)\Big\{\frac{4}{(2\sigma_{p,q} - p)^2}[g(\varphi v_2, v_3)\varphi v_1 - g(\varphi v_1, v_3)\varphi v_2] \\ &+ \frac{p^2}{(2\sigma_{p,q} - p)^2}[g(v_2, v_3)v_1 - g(v_1, v_3)v_2] \\ &+ \frac{2p}{(2\sigma_{p,q} - p)^2}[g(\varphi v_1, v_3)v_2 + g(v_1, v_3)\varphi v_2 \\ &- g(\varphi v_2, v_3)v_1 - g(v_2, v_3)\varphi v_1]\Big\} \\ &\pm \frac{1}{2}(c_1 - c_2)\Big\{\frac{1}{2\sigma_{p,q} - p}[g(v_2, v_3)\varphi v_1 - g(v_1, v_3)\varphi v_2] \\ &+ \frac{1}{2\sigma_{p,q} - p}[g(\varphi v_2, v_3)v_1 - g(\varphi v_1, v_3)v_2] \\ &+ \frac{p}{2\sigma_{p,q} - p}[g(v_1, v_3)v_2 - g(v_2, v_3)v_1]\Big\}.\end{aligned} \tag{7}$$

Example 2.2. *(Clifford algebras) Assume that $\sum_{k=1}^{m}\left(\mu^k\right)^2$ stands for the positive definite form of $\mathbb{R}^m$ and $C^{\gamma}(n)$ be the real Clifford algebra of this positive definite form [38]. Then, taking view of Clifford product, one observes that standard base of $\mathbb{R}^m$ satisfies:*

$$\begin{cases} E_k^2 = 1 & , \;\; k = l \\ E_kE_l + E_lE_k = 0 & , \;\; k \neq l \end{cases}$$

Taking $\varphi_i = \frac{1}{2}\left(p + \sqrt{p^2 + 4q}E_i\right)$ produces

$$\begin{cases} \varphi_k, \;\; \text{metallic structure} & , \;\; k = l \\ \varphi_k\varphi_l + \varphi_l\varphi_k = p\left(\varphi_k + \varphi_l\right) - \frac{p^2}{2} & , \;\; k \neq l, \end{cases}$$

E_1 and E_2 are orthonormal basis vectors of $\mathbb{R}_2^2$ [38]:

$$1 = I_2 \;\;,\;\; E_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\;,\;\; E_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and thus we obtain

$$(i)\ \varphi_1 = \frac{1}{2}\left(p + \sqrt{p^2+4q}E_1\right) = \begin{pmatrix} \frac{p+\sqrt{p^2+4q}}{2} & 0 \\ 0 & \frac{p-\sqrt{p^2+4q}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{p,q} & 0 \\ 0 & p-\sigma_{p,q} \end{pmatrix}$$

$$(ii)\ \varphi_2 = \frac{1}{2}\left(p + \sqrt{p^2+4q}E_2\right) = \frac{1}{2}\begin{pmatrix} p & \sqrt{p^2+4q} \\ \sqrt{p^2+4q} & p \end{pmatrix}.$$

Example 2.3. *Assume* $(2n+m)$*-dimensional affine space* $\mathbb{R}^{2n+m}_n$ *equipped with* $(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_m)$*. Further, let* g *and* ϑ *be the semi-Riemannian metric and tensor field given by*

$$g = \begin{pmatrix} -\sigma_{p,q}\delta_{ij} & 0 & 0 \\ 0 & \sigma_{p,q}\delta_{ij} & 0 \\ 0 & 0 & (p-\sigma_{p,q})\delta_{ij} \end{pmatrix},$$

$$\vartheta = \frac{1}{2}\begin{pmatrix} p\delta_{ij} & (2\sigma_{p,q}-p)\delta_{ij} & 0 \\ (2\sigma_{p,q}-p)\delta_{ij} & p\delta_{ij} & 0 \\ 0 & 0 & \sigma_{p,q}\delta_{ij} \end{pmatrix},$$

$\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ *and* ϑ *defines a metallic structure on* $\mathbb{R}^{2n+m}_n$*.*

3. *k*-almost Newton-Ricci soliton

Let us identify by $\overline{\mathcal{M}}^{n+1}$, any metallic Riemannian manifold and immerse an oriented and connected hypersurface $f : \mathcal{M}^n \longrightarrow \overline{\mathcal{M}}^{n+1}$ into $\overline{\mathcal{M}}^{n+1}$. Then $\mathcal{M}^n$ represents an k-ANRS, for some $0 \le k \le m$, provided ([16], [22])

$$Ric + P_k \circ Hessian\psi = \lambda g, \tag{8}$$

where ψ and λ both are smooth functions on $\mathcal{M}^n$ and

$$P_k \circ Hessian\psi(U,W) = g(P_k\nabla_U\nabla_\psi, W), \tag{9}$$

$U, W \in \mathcal{X}(\mathcal{M})$. Placing $k = 0$, (8) gives a gradient almost Ricci soliton. P_k means k-th Newton transformation such that $P_0 = I$ (identity operator).

According to the Gauss equation,

$$(\bar{\mathcal{R}}(U,W)Z)^T = g(BW,Z)BU + \mathcal{R}(U,W)Z - g(BU,Z)BW \tag{10}$$

$\forall U, W, Z \in \mathcal{X}(\mathcal{M}^n)$. In this situation, $()^T$ is used to indicate the tangential components of some vector field of $\mathcal{X}(\mathcal{M}^n)$ along $\mathcal{M}^n$. Moreover, the shape operator B satisfies

$$g(h(U,W),\alpha) = g(B_\alpha U, W), \tag{11}$$

here α means the normal vector field on $\mathcal{M}^n$. We also fix $\bar{\mathcal{R}}$ (resp. $\mathcal{R}$) to denote Riemannian curvature tensor of $\overline{\mathcal{M}}^{n+1}$ (resp. $\mathcal{M}^n$). Further, ρ of $\mathcal{M}^n$ is

$$\sum_{i,j}^{n} g(\bar{R}(E_i,E_j)E_j,E_i) = \rho - n^2H^2 + \|B\|^2, \tag{12}$$

$||B||$ means the Hilbert-Schmidt norm and $\{E_1, \ldots, E_n\}$ indicates orthonormal frame on $T(M)$. Thus, for locally Riemannian product manifold $\overline{\mathcal{M}}^{n+1}$, we have the identity

$$\begin{aligned} \rho &= \frac{1}{8}(c_1 + c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2 + 4q \\ &+ \frac{2}{n(n-1)}[tr^2\varphi - ||\varphi||^2] - \frac{4p}{n}tr\varphi\Big\} \\ &+ \frac{1}{8}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1 - c_2)(4tr\varphi - 2np) \\ &+ \frac{n^2}{2}H^2 - \frac{||B||^2}{2}. \end{aligned} \tag{13}$$

There exist n algebraic invariants corresponding to B of $\mathcal{M}^n$, that are the elementary symmetric functions ρ_k of its principal curvatures $r_1, \ldots, r_m$, and are given by

$$\rho_0 = 1, \quad \rho_k = \sum_{i_1 < \ldots < i_k} r_1 \ldots r_n. \tag{14}$$

Denote with H_k, the k-th mean curvature of the immersion and define it as $\binom{n}{k}H_k = \rho_k$. If $k = 0$, we have $H_1 = \frac{1}{n}tr(A) = H$, tr stands for trace. The Newton transformation $P_k : \mathcal{X}(\mathcal{M}^n) \longrightarrow \mathcal{X}(\mathcal{M}^n)$ of $\mathcal{M}^n$ is defined by putting $P_0 = I$, $0 \leq k \leq m$, by

$$P_k = \sum_{j=0}^{k}(-1)^{k-j}\binom{m}{j}H_j A^{k-j}, \tag{15}$$

B^j represents j times composition of B with itself ($B^0 = I$). Take $\mathcal{L}_k : C^\infty(\mathcal{M}^n) \longrightarrow C^\infty(\mathcal{M}^n)$ described with

$$\mathcal{L}_k u = tr(P_k \circ Hessian\ u). \tag{16}$$

If we take $k = 0$, then there is the Laplacian operator. $\mathcal{L}_0$. Also, we turn up

$$\begin{aligned} div_{\mathcal{M}}(P_k \nabla u) &= \sum_{i=1}^{m} g((\nabla_{E_i}P_k)\nabla_u, E_i) + \sum_{i=1}^{m} g(P_k(\nabla_{E_i}\nabla_u), E_i) \\ &= g(div_{\mathcal{M}}P_k, \nabla_u) + \mathcal{L}_k u, \end{aligned} \tag{17}$$

where

$$div_{\mathcal{M}}P_k = tr(\nabla P_k) = \sum_{i=1}^{m}(\nabla_{E_i}P_k)E_i. \tag{18}$$

If $\overline{\mathcal{M}}^{n+1}$ has constant sectional curvatures, (17) has the following shape

$$\mathcal{L}_k u = div_{\mathcal{M}}(P_k \nabla u), \tag{19}$$

because $div_{\mathcal{M}}P_k = 0$ (also refer [39]).

Since the s.f.f. of $\mathcal{M}^n$ is trace-less, which is produced as

$$\Phi = BHI, \qquad tr(\Phi) = 0, \tag{20}$$

$$|\Phi|^2 = tr(\Phi^2) = ||B||^2 - mH^2 \geq 0. \tag{21}$$

$|\Phi|^2 = 0 \Leftrightarrow \mathcal{M}^n$ is totally umbilical.

Let us use the maximal principle to obtain our results (for more information, check [24]). As a result, for every $s \geq 1$, we use the expression

$$\mathcal{L}^s(L) = \left\{u : \mathcal{M}^n \longrightarrow \mathcal{R}; \int_{\mathcal{M}} |u|^s\, dL < +\infty\right\}. \tag{22}$$

Lemma 3.1. *Assume $\mathcal{M}^n$ denotes non-compact, complete, oriented Riemannian manifold and for smooth vector field U, $div_{\mathcal{M}}U$ keeps sign on $\mathcal{M}^n$ unchanged. Then $|U| \in \mathcal{L}^1(\mathcal{M})$ implies $div_{\mathcal{M}}U = 0$.*

Theorem 1.2 [10] is extended in the following manner.

Theorem 3.2. *Consider (g, ψ, λ, k) denotes a complete k-ANRS on hypersurface $\mathcal{M}^n$ of metallic Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2 and p.f. $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ s.t. $\left|\nabla\psi\right| \in \mathcal{L}^1(\mathcal{M})$. When*

1. *$\lambda > 0, c_1 + c_2 \leq 0, c_1 - c_2 \leq 0 \implies \mathcal{M}^n$ can not be minimal,*
2. *$c_1 - c_2 < 0, \lambda \geq 0$, $c_1 + c_2 < 0 \implies \mathcal{M}^n$ can not be minimal,*
3. *$\mathcal{M}^n$ is minimal, $c_1 - c_2 = 0$, $\lambda \geq 0, c_1 + c_2 = 0 \implies \mathcal{M}^n$ will be isometric to $\mathbb{R}^n$.*

Proof. Since c_1 and c_2 are the constant sectional curvatures of the ambient space, then from (19) we can see that the operator $\mathcal{L}_k$ is of the divergent kind. Furthermore, B is bounded on $\mathcal{M}^n$, so (15) indicates P_k has a bounded norm implying

$$\left|P_k\nabla\psi\right| \leq |P_k|\left|\nabla\psi\right| \in \mathcal{L}^1(\mathcal{M}). \tag{23}$$

To prove (1) and (2), consider on contrary that $\mathcal{M}^n$ is minimal. In that situation, (13) together with $c_1 - c_2 \leq 0$, $c_1 + c_2 \leq 0$ and $c_1 - c_2 < 0$, $c_1 + c_2 < 0$ shows that $\rho \leq 0$ ($\rho < 0$). Thus, contraction on (8) produces $\mathcal{L}_r\psi = n\lambda - \rho > 0$ in both cases, and that contradicts Lemma 3.1 establishing assertions (1) and (2).

Next, since c_1 and c_2 are the constant sectional curvatures of the ambient space and $\mathcal{M}^n$ is minimal, then equation (13) becomes

$$\rho = -\frac{||B||^2}{2} \leq 0. \tag{24}$$

Next, $\lambda \geq 0 \implies \mathcal{L}_r(\psi) = n\lambda - \rho \geq 0$. Taking $\mathcal{L}_r u = div_{\mathcal{M}}(P_k\nabla u)$ and $\left|P_k\nabla\psi\right| \in \mathcal{L}^1(\mathcal{M})$, Lemma (3.1) contributed once more $\mathcal{L}_r\psi = 0$ on $\mathcal{M}^n$. Therefore $0 \geq \rho = n\lambda \geq 0$, $\implies \rho = \lambda = 0$ establishing $||B||^2 = 0$. Hence k-ANRS $\mathcal{M}^n$ is geodesic and flat. □

Recall the following result corresponding to Theorem 3 of [48].

Lemma 3.3. *Assume M^n stands for complete Riemannian manifold with non-negative smooth subharmonic function u. Assume $u \in \mathcal{L}^s(\mathcal{M})$, then u is constant $\forall s > 1$.*

Thus, one writes:

Theorem 3.4. *When (g, ψ, λ, k) denotes complete k-ANRS on hypersurface $\mathcal{M}^n$ of $\overline{\mathcal{M}}^{n+1}$, P_k is bounded from above (in the sense of quadratic forms) and $\psi \in \mathcal{L}^s(\mathcal{M})$, $\forall s > 1$. Then*

1. *$K_{\mathcal{M}} \leq 0, \lambda > 0 \implies \mathcal{M}^n$ is not minimal,*
2. *$\lambda \geq 0, K_{\mathcal{M}} < 0 \implies \mathcal{M}^n$ will not be minimal,*
3. *$\mathcal{M}^n$ is minimal, $\lambda \geq 0$, $K_{\mathcal{M}} \leq 0 \implies \mathcal{M}^n$ will be flat and totally geodesic.*

Proof. Let $\mathcal{M}^n$ is minimal. Then, (12) with given hypothesis results $\rho \leq 0$ and contraction of (8) implies

$$\mathcal{L}_k\psi = n\lambda - \rho > 0. \tag{25}$$

Since P_k has been considered bounded from above, therefore

$$\omega\Delta\psi \geq \mathcal{L}_k\psi > 0, \tag{26}$$

in above case, ω indicates any positive constant. As a result of Lemma 3.3, ψ is constant, which is not appropriate, establishing (1). (2) and (3) are simply found in light of the evidence of Theorem 3.2. □

The next result extends Theorem 1.5 [9] for $U = \nabla\psi$. We also provide the terms for an k-ANRS on hypersurface of metallic Riemannian manifold to be totally umbilical, provided s.f.f. of $\mathcal{M}^n$ is bounded. Thus, one has

Theorem 3.5. *If (g, ψ, λ, k) be a complete k-ANRS on hypersurface $\mathcal{M}^n$ of $\overline{\mathcal{M}}^{n+1}$ of sectional curvatures c_1 and c_2, with bounded s.f.f. and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ s.t. $\left|\nabla\psi\right| \in \mathcal{L}^1(\mathcal{M})$. Therefore, for*

1. $\lambda \geq \frac{(n-1)(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)}{8n(p^2+q^2)\sqrt{p^2+q^2}}$, *$\mathcal{M}^n$ is totally geodesic with* $\lambda = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$, *and* $\rho = \frac{n(n-1)(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)}{8n(p^2+q^2)\sqrt{p^2+q^2}}$,
2. *$\mathcal{M}^n$ is compact and* $\lambda \geq \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}}{8n(p^2+q^2)\sqrt{p^2+q^2}}$, *$\mathcal{M}^n$ is isometric to a Euclidean sphere,*
3. $\lambda \geq \frac{[(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}]}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$, *$\mathcal{M}^n$ is totally umbilical. Particularly, $\rho = n(n-1)K_{\mathcal{M}}$ is constant,* $K_{\mathcal{M}} = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}}{8n(p^2+q^2)\sqrt{p^2+q^2}}$ *is the sectional curvature of $\mathcal{M}^n$.*

Proof. Using (8) and (13), one derives

$$\begin{aligned}\mathcal{L}_r\psi &= n\lambda - \left[\frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\right]\left[\frac{(n-1)}{n}\right] \\ &\quad -\frac{n^2}{2}H^2 + \frac{\|B\|^2}{2}, \end{aligned} \tag{27}$$

where $D_1 = (p^2+4q)[2p^2+4q\frac{2}{n(n-1)}(tr^2\varphi - \|\varphi\|^2) - \frac{4p}{n}tr\varphi]$ and
$D_2 = (4tr\varphi - 2np)$.

On λ, simply obtain that $\mathcal{L}_r\psi$ is non-negative function on $\mathcal{M}^n$. Lemma 3.1 implies $\mathcal{L}_k\psi$ vanishes identically. Thus, (27) makes $\mathcal{M}^n$ totally geodesic and we turn up to

$$\lambda = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]. \tag{28}$$

Additionally, (13) implies

$$\rho = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right],$$

completing proof of (1). If $\mathcal{M}^n$ is compact, being totally geodesic results ambient space must be $\mathcal{S}^{n+1}$ isometric to $\overline{\mathcal{M}}^{n+1}$, completing (2). From equation (27), we have

$$\begin{aligned}\mathcal{L}_k\psi &= n\left[\lambda - (n-1)\frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)}{8n(p^2+q^2)\sqrt{p^2+q^2}}\right] \\ &\quad + |\Phi|^2. \end{aligned} \tag{29}$$

Now, Theorem (3.5) (1) entails.

Corollary 3.6. *When (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of $\overline{\mathcal{M}}^{n+1}$, then $\mathcal{M}^n$ admits the steady k-ANRS.*

As a result of our assumption on λ, we get $\mathcal{L}_k\psi \geq 0$. We get $\mathcal{L}_k\psi = 0$ from Lemma (3.1). This demonstrates that $\mathcal{M}^n$ is completely umbilical. As a result, it implies that κ be constant, so $\mathcal{M}^n$ has a constant sectional curvature.

$$K_{\mathcal{M}} = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}}{8n(p^2+q^2)\sqrt{p^2+q^2}}.$$

This relation together with (29) gives

$$\begin{aligned}\lambda &= \frac{[(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}]}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right] \\ &= (n-1)K_{\mathcal{M}}, \end{aligned} \tag{30}$$

establishing $\rho = n(n-1)K_{\mathcal{M}}$. □

Theorem 1.6 [9] states that if a minimal immersed nontrivial almost Ricci soliton $\mathcal{M}^n$ in S^{n+1} satisfies $\rho \geq n(n \geq 2)$ and $\|B\|$ obtains its maximum, then S^n will be isometric. Now, with help of Theorem 3.5, one obtains

Corollary 3.7. *Assume that the data (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of metallic Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2. Then*

1. $\lambda \geq \frac{(n-1)H^2}{2} \implies L^m$ *will be isometric to* $\mathcal{S}^m$.
2. $\|B\|$ *obtains maximum,* $\lambda \geq \frac{(n-1)H^2}{2}$, $\rho \geq n(n-2) \implies \mathcal{M}^n$ *will be isometric to* $\mathcal{S}^n$.

Proof. (2) We turn up through Simon's formula [44].

$$\Delta\|B\|^2 - \|\nabla B\|^2 = (2n - \|B\|^2)\|B\|^2 \geq 0. \tag{31}$$

In addition, for $\rho \geq m(m-2)$, the immersion is minimal, therefore (13) turns to be

$$\frac{\|B\|^2}{2} = n(n-1) - \rho \leq n.$$

We can deduce from Hopf's strong maximum principle and equation (31) that $\nabla B = 0$ on $\overline{\mathcal{M}}^{n+1}$. Thus, Proposition 1 [37] deduces that $\mathcal{M}^n$ is compact, and the result follows from Theorem 3.5. □

Theorem 3.8. *Let (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of metallic Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2 and $\psi \in \mathcal{L}^s(\mathcal{M})$, $\forall s > 1$. For*

1. $\lambda \geq \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$, *$\mathcal{M}^n$ is totally geodesic with* $\lambda = \frac{(n-1)(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)}{8n(p^2+q^2)\sqrt{p^2+q^2}}$, *and the scalar curvature* $\rho = \frac{n(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$,
2. $\lambda \geq (n-1)\frac{[(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}]}{8n(p^2+q^2)\sqrt{p^2+q^2}}$, *$\mathcal{M}^n$ is totally umbilical. Particularly, $\rho = n(n-1)K_{\mathcal{M}}$ is constant, where* $K_{\mathcal{M}} = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(n-1)(c_1-c_2)D_2(p^2+q^2)+\frac{H^2}{2}}{8n(p^2+q^2)\sqrt{p^2+q^2}}$ *is the sectional curvature of $\mathcal{M}^n$.*

Proof. The hypothesis on λ and equation (27) give

$$\begin{aligned}\mathcal{L}_r\psi &= n\lambda - \left[\frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\right]\frac{(n-1)}{n} \\ &\quad -\frac{n^2}{2}H^2 + \frac{\|B\|^2}{2} \\ &\geq 0. \end{aligned} \tag{32}$$

As P_k is bounded from above, therefore $\omega\Delta\psi \geq \mathcal{L}_k\psi \geq 0$ in the case of a positive constant ω We conclude that ψ is constant using Lemma 3.3. As a result, $\mathcal{L}_n\psi = 0$, and equation (32) proves that $\mathcal{M}^n$ is totally geodesic.

$$\lambda = \frac{(c_1+c_2)D_1\sqrt{p^2+q^2}-(c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$$

and the scalar curvature

$$\rho = \frac{n(c_1+c_2)D_1\sqrt{p^2+q^2} - (c_1-c_2)D_2(p^2+q^2)}{8(p^2+q^2)\sqrt{p^2+q^2}}\left[\frac{(n-1)}{n}\right]$$

establishing (1). Assertion (2) follows through process of Theorem 3.5. □

4. Some Applications

As an application, we obtain the following results for golden Riemannian manifold $\overline{\mathcal{M}}$ ($p = q = 1, \sigma = \frac{1+\sqrt{5}}{2}$ ([20],[35])).

Theorem 4.1. *Consider complete k-ANRS (g, ψ, λ, k) on hypersurface $\mathcal{M}^n$ of golden Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2 with bounded B and p.f. $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ s.t. that $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. We have*

1. $\lambda > 0, c_1 + c_2 \leq 0, c_1 - c_2 \leq 0 \implies \mathcal{M}^n$ *is not minimal,*
2. $c_1 - c_2 < 0, \lambda \geq 0, c_1 + c_2 < 0 \implies \mathcal{M}^n$ *will not be minimal.*
3. $\mathcal{M}^n$ *be minimal,* $\lambda \geq 0, c_1 - c_2 = 0, c_1 + c_2 = 0 \implies \mathcal{M}^n$ *will be isometric to* $\mathbb{R}^n$.

This generalizes Theorem 1.2 of [10] to golden Riemannian manifold. Next, we have:

Theorem 4.2. *Assume (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of golden Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ and $\psi \in \mathcal{L}^s(\mathcal{M}), \forall s > 1$. If*

1. $\lambda > 0, K_{\mathcal{M}} \leq 0, \implies \mathcal{M}^n$ *is not minimal,*
2. $K_{\mathcal{M}} < 0, \lambda \geq 0 \implies \mathcal{M}^n$ *will not be minimal,*
3. $\mathcal{M}^n$ *be minimal,* $\lambda \geq 0, K_{\mathcal{M}} \leq 0 \implies \mathcal{M}^n$ *will be flat and totally geodesic.*

Next result generalizes the Theorem 1.5 of [9] for $U = \nabla\psi$.

Theorem 4.3. *If (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of golden Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2, with bounded s.f.f. and $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ satisfies $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. When*

1. $\lambda \geq \frac{(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$, $\mathcal{M}^n$ *is totally geodesic with* $\lambda = \frac{(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$, *and* $\rho = \frac{n(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$,
2. $\mathcal{M}^n$ *is compact and* $\lambda \geq \frac{(c_1+c_2)D_1\sqrt{2}-2(n-1)(c_1-c_2)D_2+\frac{H^2}{2}}{16n\sqrt{2}}$, $\mathcal{M}^n$ *will be isometric to a Euclidean sphere,*
3. $\lambda \geq \frac{[(c_1+c_2)D_1\sqrt{2}-2(n-1)(c_1-c_2)D_2+\frac{H^2}{2}]}{16n\sqrt{2}}(n-1)$, $\mathcal{M}^n$ *is totally umbilical. Particularly,* $\rho = n(n-1)K_{\mathcal{M}}$ *is constant, in this case,* $K_{\mathcal{M}} = \frac{(c_1+c_2)D_1\sqrt{2}-2(n-1)(c_1-c_2)D_2+\frac{H^2}{2}}{16n\sqrt{2}}$ *is the sectional curvature of* $\mathcal{M}^n$.

Theorem (4.3) (1) produces the following.

Corollary 4.4. *If (g, ψ, λ, k) be complete k-ANRS on hypersurface $\mathcal{M}^n$ of golden Riemannian manifold $\overline{\mathcal{M}}^{n+1}$, then $\mathcal{M}^n$ admits the steady k-ANRS.*

Corollary 4.5. *Let (g, ψ, λ, k) denotes complete k-ANRS on hypersurface $\mathcal{M}^n$ of golden Riemannian manifold $\overline{\mathcal{M}}^{n+1}$ of constant sectional curvatures c_1 and c_2. We notice*

1. $\lambda \geq \frac{(n-1)H^2}{2} \implies L^m$ *will be isometric to* $\mathcal{S}^m$,

2. $\|B\|$ *obtains its maximum*, $\rho \geq n(n-2)$, $\lambda \geq \frac{(n-1)H^2}{2} \Longrightarrow \mathcal{M}^n$ *is isometric to* $\mathcal{S}^n$.

Theorem 4.6. *Assume* (g, ψ, λ, k) *be complete k-ANRS on hypersurface* $\mathcal{M}^n$ *of* $\overline{\mathcal{M}}^{n+1}$ *of constant sectional curvatures* c_1 *and* c_2 *and* $\psi \in \mathcal{L}^s(\mathcal{M})$, $\forall s > 1$. *For*

1. $\lambda \geq \frac{(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$, $\mathcal{M}^n$ *is totally geodesic with*
   $\lambda = \frac{(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$,
   and $\rho = \frac{n(c_1+c_2)D_1\sqrt{2}-2(c_1-c_2)D_2}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$,
2. $\lambda \geq \frac{[(c_1+c_2)D_1\sqrt{2}-2(n-1)(c_1-c_2)D_2+\frac{H^2}{2}]}{16\sqrt{2}}\left[\frac{(n-1)}{n}\right]$, $\mathcal{M}^n$ *is totally umbilical. Particularly,* $\rho = n(n-1)K_{\mathcal{M}}$ *will be constant, where* $K_{\mathcal{M}} = \frac{(c_1+c_2)D_1\sqrt{2}-2(n-1)(c_1-c_2)D_2+\frac{H^2}{2}}{16n\sqrt{2}}$ *is the sectional curvature of* $\mathcal{M}^n$.

Remark. The following cases can also be analyzed in the same manner [35]:

- the *silver ratio* ($p = 2, q = 1$, $\sigma_{2,1} = 1 + \sqrt{2}$)
- the *bronze ratio* ($p = 3, q = 1, \sigma_{3,1} = \frac{3+\sqrt{13}}{2}$)
- the *copper ratio* ($p = 1, q = 2, \sigma_{1,2} = 2$)
- the *nickel ratio* ($p = 1, q = 3, \sigma_{1,3} = \frac{1+\sqrt{13}}{2}$)
- the *subtle mean* ($p = 4, q = 1, \sigma_{4,1} = 2 + \sqrt{5}$) etc.

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