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Gradient Ricci-Yamabe solitons on warped product manifolds

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Abstract. We give the necessary and sufficient conditions for a gradient Ricci-Yamabe soliton with warped product metric. As physical applications, we consider gradient Ricci-Yamabe solitons on generalized Robertson-Walker space-times and standard static space-times.

1. Introduction

The Ricci-Yamabe flow is a scalar combination of the Ricci flow and the Yamabe flow [15]. In 1982 and 1989, Hamilton introduced the Ricci flow and the Yamabe flow, respectively [17], [18]. Benefitting from these flows, Güler and Crasmareanu defined the Ricci-Yamabe flow in 2019 [15]. The Ricci-Yamabe flow can be useful in differential geometry and physics, especially in general relativity (i.e. a recent bimetric approach of space-time geometry) [15]. Finally, using [15], Dey introduced the Ricci-Yamabe soliton in 2020 [10].

Definition 1.1. *A Riemannian manifold (M^n, g), $n > 2$ is called a gradient Ricci-Yamabe soliton (briefly GRYS) $((M, g), h, \lambda, \alpha, \beta)$ if there exists a differentiable function $h : M \to \mathbb{R}$ such that*

$$Hess_g h + \alpha Ric_g = \left(\lambda - \frac{1}{2}\beta scal\right) g, \qquad (1)$$

where Ric_g is the Ricci curvature of (M, g), scal is scalar curvature of (M, g), $Hess_g h$ is the Hessian of h and $\lambda, \alpha, \beta \in \mathbb{R}$ [10].

The equation (1) is called gradient Ricci-Yamabe soliton of (α, β)-type, which is a generalization of Ricci and Yamabe solitons. We note that gradient Ricci-Yamabe solitons of type $(\alpha, 0)$, $(0, \beta)$-type are α-Ricci soliton and β-Yamabe soliton, respectively [10]. Specifically, we have:

The equation (1) defines

1) *gradient Ricci soliton [16] when $\alpha = 1, \beta = 0$.*
2) *gradient Yamabe soliton [17] when $\alpha = 0, \beta = 1$.*
3) *gradient Einstein soliton [7] when $\alpha = 1, \beta = -1$.*
4) *gradient ρ-Einstein soliton [8] when $\alpha = 1, \beta = -2\rho$.*

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The gradient Ricci-Yamabe soliton $((M, g), h, \lambda, \alpha, \beta)$ is called shrinking, steady or expanding depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. Gradient Ricci-Yamabe soliton is called proper if $\alpha \neq 0, 1$.

Let us remark that an interpolation soliton between Ricci and Yamabe solitons is considered in [8] where the name Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow but it depends on a single scalar.

Bishop and O'Neill defined the warped product in [6] to construct Riemannian manifolds with negative sectional curvature. The warped product plays an important role in differential geometry and physics.

For semi-Riemannian manifolds (B^r, g_B), (F^m, g_F) and a smooth function $f : B \to (0, \infty)$, the warped product $M^n = B \times_f F$ is the product manifold $M = B \times F$ endowed with the metric tensor

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F), \tag{2}$$

where π and σ are the natural projections on B and F, respectively and the function $f : B \to (0, \infty)$ is the warping function [6].

The Ricci-Yamabe flow was introduced in 2019 [15]. In [10], Dey defined the Ricci-Yamabe soliton. In [28], Shivaprasanna et al. studied Ricci-Yamabe soliton on submanifolds of indefinite Sasakian, Kenmotsu and trans-Sasakian manifolds concerning Riemannian connection and quarter symmetric metric connection. In [27], Siddiqi and Akyol defined and studied η-Ricci-Yamabe soliton soliton on Riemannian submersions from Riemannian manifolds. In [11], Dey and Majhi introduced the notion of generalized Ricci-Yamabe soliton. In [25], Roy et al. studied the conformal Ricci-Yamabe soliton. In [31], Yoldaş studied η-Ricci-Yamabe solitons on Kenmotsu manifolds. In [14], Feitosa et al. obtained a necessary and sufficient condition for constructing a gradient Ricci soliton warped product. In [29], Sousa and Pina studied semi-Riemannian warped product gradient Ricci solitons. In [30], Tokura et al. studied gradient Yamabe soliton on warped product manifolds and obtained nontrivial examples. For recent studies about solitons on warped product manifolds; see [9], [20], [21], [22] and [23]. By a motivation from the above studies, in the present paper, with warped product manifolds, we consider gradient Ricci-Yamabe solitons. We obtain some characteriztions for this kind of solitons. We also give physical applications.

2. Gradient Ricci-Yamabe Solitons

Assume that $M = B \times_f F$ is a warped product manifold endowed with the metric tensor $g = g_B \oplus f^2 g_F$, where $f : B \to (0, \infty)$.

Now, we can state:

Proposition 2.1. *If $(M = B \times_f F, g, h, \lambda, \alpha, \beta)$ is a GRYS with $h : M \to \mathbb{R}$, then $h : B \to \mathbb{R}$ i.e., h depends only on the base.*

Proof. Let $(M = B \times_f F, g, h, \lambda, \alpha, \beta)$ be a GRYS with $h : M \to \mathbb{R}$. From $scal = scal_B + \frac{scal_F}{f^2} - 2m\frac{\Delta_B f}{f} - m(m-1)\frac{\|grad_B f\|^2}{f^2}$ ([6]) and the equation (1), we have

$$\begin{aligned} &Hess_g h(X, V) + \alpha Ric_g(X, V) \\ &= \left[\lambda - \frac{1}{2}\beta\left(scal_B + \frac{scal_F}{f^2} - 2m\frac{\Delta_B f}{f} - m(m-1)\frac{\|grad_B f\|^2}{f^2}\right)\right] g(X, V) \end{aligned} \tag{3}$$

for $X \in \chi(B)$ and $V \in \chi(F)$.

As $Ric_g(X, V) = 0$ ([6], [24]), we find

$$\frac{X(f)}{f} g(grad_g h, V) = 0, \tag{4}$$

for $X \in \chi(B)$ and $V \in \chi(F)$. Then, we can write

$$g(grad_g h, V) = g(\upsilon\left(grad_g h\right), V) + g(\hat{H}\left(grad_g h\right), V), \tag{5}$$

where $\hat{H}\left(grad_g h\right)$ and $v\left(grad_g h\right)$ are the horizontal part and vertical part of $grad_g h$, respectively.
Using (4) and (5), we obtain $h = h_B \circ \pi$. This proves the proposition. □

Using Proposition 2.1, we can state:

Theorem 2.2. *($M = B \times_f F, g, h, \lambda, \alpha, \beta$) is a GRYS with $scal_F = c$ and $m > 1$ if and only if $f, h, , \lambda, \alpha, \beta$ satisfy:*

$$\alpha Ric_B - \alpha \frac{m}{f} Hess_B(f) + Hess_B h_B$$

$$= \left[\lambda - \frac{1}{2}\beta\left(scal_B + \frac{c}{f^2} - 2m\frac{\Delta_B f}{f} - m(m-1)\frac{\left\|grad_B f\right\|^2}{f^2}\right)\right] g_B. \tag{6}$$

F is an Einstein manifold with $Ric_F = \frac{\mu}{\alpha} g_F$, where

$$\mu = \lambda f^2 - \frac{1}{2}\beta\left(f^2 scal_B + c - 2mf\Delta_B f - m(m-1)\left\|grad_B f\right\|^2\right)$$

$$-\alpha\left((m-1)\left\|grad_B f\right\|^2 - f\Delta_B f\right) - f grad_B h_B\left(f\right). \tag{7}$$

Proof. (1) Assume that ($M = B \times_f F, g, h, \lambda, \alpha, \beta$) is a GRYS and the fiber F is with constant scalar curvature $scal_F = c$, $m > 1$. From $Ric_g(X, Y) = Ric_B(X, Y) - \frac{m}{f} Hess f(X, Y)$ ([6], [24]) and the equation (1), we obtain (6) for $X, Y \in \chi(B)$. Similarly, using $Ric_g(V, W) = Ric_F(V, W) - \left[\frac{-\Delta_B f}{f} + (m-1)\frac{\left\|grad_B f\right\|^2}{f^2}\right] g(V, W)$ ([6], [24]) and the equation (1), we obtain

$$Hess h(V, W) + \alpha Ric_F(V, W) - \alpha\left[\frac{-\Delta_B f}{f} + (m-1)\frac{\left\|grad_B f\right\|^2}{f^2}\right] f^2 g_F(V, W)$$

$$= \left[\lambda - \frac{1}{2}\beta\left(scal_B + \frac{c}{f^2} - 2m\frac{\Delta_B f}{f} - m(m-1)\frac{\left\|grad_B f\right\|^2}{f^2}\right)\right] f^2 g_F(V, W) \tag{8}$$

for $V, W \in \chi(F)$. From the definition of Hessian of a function, we obtain

$$Hess h(V, W) = f grad_B h_B(f)\, g_F(V, W). \tag{9}$$

Substituting the equation (9) in (8), we find

$$Ric_F(V, W) = \frac{1}{\alpha}\left[\lambda f^2 - \frac{1}{2}\beta\left(f^2 scal_B + c - 2mf\Delta_B f - m(m-1)\left\|grad_B f\right\|^2\right)\right.$$

$$\left.\alpha\left(-f\Delta_B f + (m-1)\left\|grad_B f\right\|^2\right) - f grad_B h_B(f)\right] g_F(V, W).$$

Therefore, F is an Einstein manifold. This proves the theorem. □

Let $\left(M = \left(\mathbb{R}^r, \varphi^{-2} g_{\mathbb{R}}\right) \times_f F, g = \varphi^{-2} g_{\mathbb{R}} + f^2 g_F\right)$ be a warped product manifold where $\left(\mathbb{R}^r, \varphi^{-2} g_{\mathbb{R}}\right)$ is conformal to r-dimensional semi-Euclidean space, $(g_{\mathbb{R}})_{i,j} = \epsilon_i \delta_{i,j}$ is the canonical semi-Riemannian metric and φ is the conformal factor. We define the function $\xi(x_1, x_2, ..., x_r) = \sum_{i=1}^{r} \theta_i x_i$, $\theta_i \in \mathbb{R}$ where $x = (x_1, x_2, ..., x_r) \in \mathbb{R}^r$.

Now, we give the following theorem:

Theorem 2.3. *($M = \mathbb{R}^r \times_f F, g = \varphi^{-2} g_{\mathbb{R}} + f^2 g_F, h, \lambda, \alpha, \beta$) is a GRYS with $scal_F = c$ and $f = f \circ \xi$, $h = h \circ \xi$, $\varphi = \varphi \circ \xi$ defined in $\left(\mathbb{R}^r, \varphi^{-2} g_{\mathbb{R}}\right)$ if and only if the functions f, h, φ satisfy:*

$$\alpha(r-2)\frac{\varphi''}{\varphi} - \alpha m \frac{f''}{f} - 2\alpha m \frac{f'}{f}\frac{\varphi'}{\varphi} + h'' + 2\frac{\varphi'}{\varphi}h' = 0, \tag{10}$$

$$\begin{aligned} &\left\{ -\beta m \frac{f''}{f} - (\alpha m + \beta m(r-2))\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{1}{2}\beta m(m-1)\left(\frac{f'}{f}\right)^2 \right. \\ &\left. + (\alpha + \beta(r-1))\frac{\varphi''}{\varphi} - \left[\alpha(r-1) + \frac{1}{2}\beta r(r-1)\right]\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{\varphi'}{\varphi}h' \right\} \|\theta\|^2 \\ &= \frac{1}{\varphi^2}\left(\lambda - \frac{1}{2}\beta\frac{c}{f^2}\right), \end{aligned} \tag{11}$$

and

$$\begin{aligned} &f^2\varphi^2 \left\{ -\frac{1}{2}\beta\left[(r-1)\left(2\frac{\varphi''}{\varphi} - r\left(\frac{\varphi'}{\varphi}\right)^2\right) - 2m\left(\frac{f''}{f} - (r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right) - m(m-1)\left(\frac{f'}{f}\right)^2\right] \right. \\ &\left. +\alpha\left(\frac{f''}{f} - (r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right) + \alpha(m-1)\left(\frac{f'}{f}\right)^2 - \frac{f'}{f}h' \right\} \|\theta\|^2 \\ &= \mu - \lambda f^2 + \frac{1}{2}\beta c. \end{aligned} \tag{12}$$

Proof. Let $h(\xi)$, $f(\xi)$ and $\varphi(\xi)$ be functions of ξ, where $\xi : \mathbb{R}^r \to \mathbb{R}$ given by $\xi(x_1, x_2, ..., x_r) = \sum_{i=1}^{r} \theta_i x_i$, $\theta_i \in \mathbb{R}$. Hence, we have

$$\begin{array}{lll} h_{x_i} = h'\theta_i, & f_{x_i} = (b_s)'\,\theta_i, & \varphi_{x_i} = \varphi'\theta_i \\ h_{x_i x_j} = h''\theta_i\theta_j, & f_{x_i x_j} = (b_s)''\,\theta_i\theta_j, & \varphi_{x_i x_j} = \varphi''\theta_i\theta_j. \end{array} \tag{13}$$

It is well-known that $h = h_B \circ \pi$, $\varphi = \varphi_B \circ \pi$, $f = f_B \circ \pi$ where $(B, g_B) = \left(\mathbb{R}^r, \varphi^{-2} g_{\mathbb{R}}\right)$. In [5], the Ricci curvature with $g_B = \varphi^{-2} g_{\mathbb{R}}$ is given by

$$Ric_B = \frac{1}{\varphi^2}\left\{(r-2)\varphi Hess_{g_{\mathbb{R}}}(\varphi) + \left[\varphi\Delta_{g_{\mathbb{R}}}\varphi - (r-1)\left\|grad_{g_{\mathbb{R}}}\varphi\right\|^2\right] g_{\mathbb{R}}\right\}. \tag{14}$$

From (13) and (14), we easily see that the scalar curvature with $g_B = \varphi^{-2} g_{\mathbb{R}}$ is obtained

$$scal_B = (r-1)\left[2\varphi\Delta_{g_{\mathbb{R}}}\varphi - r\left\|grad_{g_{\mathbb{R}}}\varphi\right\|^2\right]. \tag{15}$$

Using $\left(Hess_{g_{\mathbb{R}}}(\varphi)\right)_{i,j} = \varphi''\theta_i\theta_j$, $\Delta_{g_{\mathbb{R}}}\varphi = \varphi''\|\theta\|^2$ and $\left\|grad_{g_{\mathbb{R}}}\varphi\right\|^2 = (\varphi')^2\|\theta\|^2$, we find

$$(Ric_B)(X_i, X_j) = \frac{1}{\varphi}(r-2)\varphi''\left(\theta_i\theta_j\right), \tag{16}$$

for $\forall i \neq j = 1, 2, ..., r$ and

$$(Ric_B)(X_i, X_i) = \frac{1}{\varphi^2}\left\{(r-2)\varphi\varphi''(\theta_i)^2 + \epsilon_i\left[\varphi\varphi'' - (r-1)(\varphi')^2\right]\|\theta\|^2\right\}, \tag{17}$$

for $\forall i = 1, 2, ..., r$. Using (15), we obtain

$$(scal_B)_{i,j} = 0, \tag{18}$$

for $\forall i \neq j = 1, 2, ..., r$ and

$$(scal_B)_{i,i} = (r-1)\left[2\varphi\varphi'' - r(\varphi')^2\right]\|\theta\|^2, \tag{19}$$

for $\forall i = 1, 2, ..., r$. Then, the Christoffel symbols Γ_{ij}^k for distinct i, j, k are given by

$$\Gamma_{ij}^k = 0,\ \Gamma_{ij}^i = -\frac{\varphi_{x_j}}{\varphi},\ \Gamma_{ii}^k = \epsilon_i\epsilon_k\frac{\varphi_{x_k}}{\varphi} \text{ and } \Gamma_{ii}^i = -\frac{\varphi_{x_i}}{\varphi}. \tag{20}$$

By the use of the equation (20) and the definition of Hessian function, we find

$$(Hess_B(h))_{ij} = h_{x_ix_j} - \sum_{k=1}^{r}\Gamma_{ij}^k h_{x_k}$$

$$= h''\theta_i\theta_j + \left(2\theta_i\theta_j - \delta_{ij}\epsilon_i\|\theta\|^2\right)\varphi^{-1}\varphi'h'. \tag{21}$$

Then, the Laplacian of f with $g_B = \varphi^{-2}g_{\mathbb{R}}$ is

$$\Delta_B f = \sum_k \varphi^2\epsilon_k(Hess_B(f))_{kk}$$

$$= \varphi^2\|\theta\|^2\left[f'' - (r-2)\varphi^{-1}\varphi'f'\right]. \tag{22}$$

Moreover, we obtain

$$\begin{cases} grad_B f(h) = \varphi^2\|\theta\|^2 f'h', \\ \left\|grad_B f\right\|^2 = \varphi^2\|\theta\|^2(f')^2. \end{cases} \tag{23}$$

Replacing the equations (17), (19), (21), (22) and (23) for $i = j$ in (6), we find (11). Similarly, using (16), (18), (21), (22) and (23) for $i \neq j$ in (6), we have

$$\left[\alpha(r-2)\frac{\varphi''}{\varphi} - \alpha m\frac{f''}{f} - 2\alpha m\frac{f'}{f}\frac{\varphi'}{\varphi} + h'' + 2\frac{\varphi'}{\varphi}h'\right]\theta_i\theta_j = 0. \tag{24}$$

From (24), if there exist i, j for $i \neq j$ such that $\theta_i\theta_j \neq 0$, then we find (10). Finally, using the equations (19) ,(22) and (23) in (7), we find (12). Hence, we obtain the desired result. □

Remark 2.4.
- *If we take* $\alpha = 1, \beta = 0$ *in Theorem 2.3, then we obtain Theorem 1.3 in [29]. Thus, the gradient Ricci-Yamabe soliton turns into the gradient Ricci soliton.*
- *If we take* $\alpha = 0, \beta = 1$ *in Theorem 2.3, then we obtain Theorem 1.6 in [30]. Thus, the gradient Ricci-Yamabe soliton turns into the gradient Yamabe soliton.*

Let $\left(M = \left(\mathbb{R}^r, \varphi^{-2}g_{\mathbb{R}}\right)\times_f\left(\mathbb{R}^m, \tau^{-2}g_{\mathbb{R}}\right), g = \varphi^{-2}g_{\mathbb{R}} + f^2\tau^{-2}g_{\mathbb{R}}\right)$ be a warped product manifold where $\left(\mathbb{R}^r, \varphi^{-2}g_{\mathbb{R}}\right)$ and $\left(\mathbb{R}^m, \tau^{-2}g_{\mathbb{R}}\right)$ are conformal to r-dimensional and m-dimensional semi-Euclidean spaces, φ and τ are the conformal factors of base and fiber, respectively. Similarly, we define the function $\zeta(x_{r+1}, x_{r+2}, ..., x_{r+m}) = a_{r+1}x_{r+1} + ... + a_{r+m}x_{r+m}$, with an arbitrary choice of non-zero vectors $a = (a_{r+1}, ..., a_{r+m})$ and $y = (x_{r+1}, ..., x_{r+m}) \in \mathbb{R}^m$.

Now, we can state:

Theorem 2.5. *$(M = \mathbb{R}^r \times_f \mathbb{R}^m, g = \varphi^{-2} g_{\mathbb{R}} + f^2 \tau^{-2} g_{\mathbb{R}}, h, \lambda, \alpha, \beta)$ is a GRYS with $scal_F = c$ and $f = f \circ \xi$, $h = h \circ \xi$, $\varphi = \varphi \circ \xi$, $\tau = \tau \circ \zeta$ defined in $\left(\mathbb{R}^r, \varphi^{-2} g_{\mathbb{R}}\right)$ and $\left(\mathbb{R}^m, \tau_s^{-2} g_{\mathbb{R}}\right)$ if and only if the functions f, h, φ, τ satisfy:*

$$\alpha(r-2)\frac{\varphi''}{\varphi} - \alpha m \frac{f''}{f} - 2\alpha m \frac{f'}{f}\frac{\varphi'}{\varphi} + h'' + 2\frac{\varphi'}{\varphi}h' = 0, \tag{25}$$

$$\begin{aligned}&\left\{-\beta m \frac{f''}{f} - (\alpha m + \beta m(r-2))\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{1}{2}\beta m(m-1)\left(\frac{f'}{f}\right)^2\right.\\ &\left. + (\alpha + \beta(r-1))\frac{\varphi''}{\varphi} - \left[\alpha(r-1) + \frac{1}{2}\beta r(r-1)\right]\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{\varphi'}{\varphi}h'\right\}\|\theta\|^2\\ &= \frac{1}{\varphi^2}\left(\lambda - \frac{1}{2}\beta\frac{c}{f^2}\right),\end{aligned} \tag{26}$$

$$\begin{aligned}&f^2\varphi^2\left\{-\frac{1}{2}\frac{\beta}{\alpha}\left[(r-1)\left(\frac{\varphi''}{\varphi} - r\left(\frac{\varphi'}{\varphi}\right)^2\right) - 2m\left(\frac{f''}{f} - (r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right) - m(m-1)\left(\frac{f'}{f}\right)^2\right]\right.\\ &\left.\left(\frac{f''}{f} - (r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right) + (m-1)\left(\frac{f'}{f}\right)^2 - \frac{1}{\alpha}\frac{f'}{f}h'\right\}\|\theta\|^2 + \frac{\lambda}{\alpha}f^2 + \frac{1}{2}\frac{\beta}{\alpha}c\\ &= \left[\tau\tau'' - (m-1)(\tau')^2\right]\|a\|^2,\end{aligned} \tag{27}$$

and

$$(m-2)\frac{\tau''}{\tau} = 0. \tag{28}$$

Proof. Let $h(\xi)$, $f(\xi)$, $\varphi(\xi)$ and $\tau(\zeta)$ be functions of ξ and ζ, where $\xi : \mathbb{R}^r \to \mathbb{R}$ and $\zeta : \mathbb{R}^m \to \mathbb{R}$. In [5], the Ricci curvature with $g_F = \tau^{-2} g_{\mathbb{R}}$ is given by

$$Ric_F = \frac{1}{\tau^2}\left\{(m-2)\tau Hess_{g_{\mathbb{R}}}(\tau) + \left[\tau\Delta_{g_{\mathbb{R}}}\tau - (m-1)\left\|grad_{g_{\mathbb{R}}}\tau\right\|^2\right]g_{\mathbb{R}}\right\}. \tag{29}$$

Using $\left(Hess_{g_{\mathbb{R}}}(\tau)\right)_{i,j} = \tau_i'' a_i a_j$, $\Delta_{g_{\mathbb{R}}}\tau = \tau''\|a\|^2$ and $\left\|grad_{g_{\mathbb{R}}}\tau\right\|^2 = (\tau')^2\|a\|^2$, we obtain

$$(Ric_F)(X_i, X_j) = \frac{1}{\tau}(m-2)\tau''\left(a_i a_j\right), \tag{30}$$

for $\forall i \neq j = 1, 2, ..., m$ and

$$(Ric_F)(X_i, X_i) = \frac{1}{\tau^2}\left\{(m-2)\tau\tau''(a_i)^2 + \epsilon_i\left[\tau\tau'' - (m-1)(\tau')^2\right]\|a\|^2\right\}, \tag{31}$$

for $\forall i = 1, 2, ..., m$. Firstly, substituting the equations (16), (18), (21), (22) and (23) in (6) for $i \neq j$ and using the same method in the proof of Theorem 2.3, we find (25). Then, substituting the equations (17), (19), (21), (22) and (23) in (6), for $i = j$, we obtain (26).

From Theorem 2.2, F is an Einstein manifold with $Ric_F = \frac{\mu}{\alpha}g_F$, we have

$$Ric_F = \rho g_F, \tag{32}$$

where

$$\rho = \frac{\lambda}{\alpha}f^2 - \frac{1}{2}\frac{\beta}{\alpha}\left(f^2 scal_B + c - 2mf\Delta_B f - m(m-1)\left\|grad_B f\right\|^2\right)$$

$$-\left((m-1)\left\|grad_B f\right\|^2 - f\Delta_B f\right) - \frac{1}{\alpha} f grad_B h_B(f). \tag{33}$$

Using (19), (22) and (23) in (33), we find

$$f^2\varphi^2\left\{-\frac{1}{2}\frac{\beta}{\alpha}\left[(r-1)\left(\frac{\varphi''}{\varphi}-r\left(\frac{\varphi'}{\varphi}\right)^2\right)-2m\left(\frac{f''}{f}-(r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right)-m(m-1)\left(\frac{f'}{f}\right)^2\right]\right.$$

$$\left.\left(\frac{f''}{f}-(r-2)\frac{\varphi'}{\varphi}\frac{f'}{f}\right)+(m-1)\left(\frac{f'}{f}\right)^2-\frac{1}{\alpha}\frac{f'}{f}h'\right\}\|\theta\|^2+\frac{\lambda}{\alpha}f^2+\frac{1}{2}\frac{\beta}{\alpha}c=\rho \tag{34}$$

Substituting the equations (31) and (34) in (32) for $i = j$, we obtain (27). Then, using the equations (30) and (34) in (32) for $i \neq j$, we have

$$\left[(m-2)\frac{\tau''}{\tau}\right]a_i a_j = 0. \tag{35}$$

From the equation (35), if there exist i, j for $i \neq j$ such that $a_i a_j \neq 0$, then we have (28). This proves the theorem. □

3. Applications

Applications of warped products have been increased in recent years, especially in differential geometry and physics [4]. There are two well-known examples of warped products, namely generalized Robertson-Walker space-times and standard static space-times. Generalized Robertson-Walker space-times are clearly a generalization of Robertson-Walker space-times and standard static space-times are a generalization of the Einstein static universe [12].

Let (F, g_F) be m-dimensional Riemannian manifold and I be an open, connected interval endowed with the negative definite metric $\left(-dt^2\right)$. Let $f : I \to (0,\infty)$ be a positive smooth function. Generalized Robertson-Walker space-time $M = I \times_f F$ is the product manifold $I \times F$ endowed with the metric tensor

$$g = (-dt^2) \oplus f^2 g_F, \tag{36}$$

[13], [26].

Let (B, g_B) be r-dimensional Riemannian manifold and $f : B \to (0,\infty)$ be positive smooth function. Standart static space-time $M = B \times_f I$ is the product manifold $B \times I$ endowed with the metric tensor

$$g = g_B + f^2(-dt^2), \tag{37}$$

[2], [6], [19].

Let $\left(M = I \times_f F, g = (-dt^2) \oplus f^2 g_F\right)$ be a generalized Robertson-Walker space-time. From Proposition 2.1, we have:

Theorem 3.1. *$(M = I \times_f F, g = (-dt^2) \oplus f^2 g_F, h, \lambda, \alpha, \beta)$ is a GRYS with $scal_F = c$ and $m > 1$ if and only if $f, h, \lambda, \alpha, \beta$ satisfy:*

$$h'' - \alpha m\frac{f''}{f} = \frac{1}{2}\beta\left[\frac{c}{f^2} + 2m\frac{f''}{f}m(m-1)\left(\frac{f'}{f}\right)^2\right] - \lambda, \tag{38}$$

F is an Einstein manifold with $Ric_F = \frac{\mu}{\alpha} g_F$, where

$$\mu = \lambda f^2 - \frac{1}{2}\beta\left(c + 2mff'' + m(m-1)(f')^2\right)$$

$$-\alpha\left(ff'' - (m-1)(f')^2\right) + ff'h'. \tag{39}$$

Proof. By substituting $grad_B f = -f'$, $Hess_B f(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = f''$, $\Delta_B f = -f''$, $g_B(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$, $\left\|grad_B f\right\|^2 = -(f')^2$ in Theorem 2.2, we have the equations (38) and (39). Hence, we obtain the desired result. □

Let $\left(M = B \times_f I, g = g_B + f^2(-dt^2)\right)$ be a standard static space-time. From Proposition 2.1, we can state:

Theorem 3.2. *$(M = B \times_f I, g = g_B + f^2(-dt^2), h, \lambda, \alpha, \beta)$ is a GRYS if and only if $f, h, \lambda, \alpha, \beta$ satisfy:*

$$\alpha Ric_B - \frac{\alpha}{f} Hess_B f + Hess_B h_B = \left[\lambda - \frac{1}{2}\beta\left(scal_B - 2\frac{\Delta_B f}{f}\right)\right] g_B \tag{40}$$

and

$$grad_B h_B (f) + \alpha \Delta_B f = \lambda f - \frac{1}{2}\beta f \left[scal_B - 2\frac{\Delta_B f}{f}\right]. \tag{41}$$

Proof. Using the same method in the proof of Theorem 2.2 for $m = 1$, we find the equation (40). From definition of Hessian, we find

$$Ric_g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -f \Delta_B f \tag{42}$$

and

$$Hess_g h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -f grad_B h_B (f). \tag{43}$$

By the use of (42) and (43) in (1) for $m = 1$, we have the equation (41). This completes the proof. □

4. Conclusion

Ricci-Yamabe solitons are very useful in differential geometry and relativity theory. Recently, a bi-metric approach of the space-time geometry is used in [1] and [3]. The application of Ricci-Yamabe solitons do not only play an important and significant role in differential geometry but also they have a motivational contribution in relativity theory. On the other hand, the warped product is a great importance in relativity theory. So, we considered a GRYS with the structure of warped product manifold. Firstly, we find the main relations for a warped product manifold to be a gradient Ricci-Yamabe soliton in Theorem 2.2. Then, we obtain some characteriztions for this kind of solitons in Theorem 2.3 and Theorem 2.5. Finally, we also obtained the main relations for generalized Robertson-Walker space-times and standard static space-times to be gradient Ricci-Yamabe solitons in Theorem 3.1 and Theorem 3.2.

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