Filomat 37:7 (2023), 2013–2026
https://doi.org/10.2298/FIL2307013D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a class of unitary operators on weighted Bergman spaces

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Abstract. In this paper we consider a class of weighted composition operators defined on the weighted Bergman spaces $L^2_a(dA_\alpha)$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),\ \alpha > -1$ and $dA(z)$ is the area measure on $\mathbb{D}$. These operators are also self-adjoint and unitary. We establish here that a bounded linear operator S from $L^2_a(dA_\alpha)$ into itself commutes with all the composition operators $C_a^{(\alpha)},\ a \in \mathbb{D}$, if and only if $B_\alpha S$ satisfies certain averaging condition. Here $B_\alpha S$ denotes the generalized Berezin transform of the bounded linear operator S from $L^2_a(dA_\alpha)$ into itself, $C_a^{(\alpha)} f = (f \circ \phi_a)$, $f \in L^2_a(dA_\alpha)$ and $\phi \in Aut(\mathbb{D})$. Applications of the result are also discussed. Further, we have shown that if $\mathcal{M}$ is a subspace of $L^\infty(\mathbb{D})$ and if for $\phi \in \mathcal{M}$, the Toeplitz operator $T_\phi^{(\alpha)}$ represents a multiplication operator on a closed subspace $\mathcal{S} \subset L^2_a(dA_\alpha)$, then ϕ is bounded analytic on $\mathbb{D}$. Similarly if $q \in L^\infty(\mathbb{D})$ and $\mathcal{B}_n$ is a finite Blaschke product and $M_q^{(\alpha)}\left(Range\ C_{\mathcal{B}_n}^{(\alpha)}\right) \subset L^2_a(dA_\alpha)$, then $q \in H^\infty(\mathbb{D})$. Further, we have shown that if $\psi \in Aut(\mathbb{D})$, then $\mathcal{N} = \left\{q \in L^2_a(dA_\alpha) : M_q^{(\alpha)}\left(Range\ C_\psi^{(\alpha)}\right) \subset L^2_a(dA_\alpha)\right\} = H^\infty(\mathbb{D})$ if and only if ψ is a finite Blaschke product. Here $M_\phi^{(\alpha)}, T_\phi^{(\alpha)}, C_\phi^{(\alpha)}$ denote the multiplication operator, the Toeplitz operator and the composition operator defined on $L^2_a(dA_\alpha)$ with symbol ϕ respectively.

1. Introduction

Let $H(\mathbb{D})$ denote the collection of all holomorphic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $H^2(\mathbb{D})$ be the Hardy space of $\mathbb{D}$ consisting of those functions in $H(\mathbb{D})$ whose Maclaurin coefficients are square summable. The space $H^2(\mathbb{D})$ is a Hilbert space [12], [24]. Let ϕ denotes an analytic self-map of $\mathbb{D}$. Then ϕ induces a bounded [24] composition operator on $H^2(\mathbb{D})$ defined by $C_\phi f = f \circ \phi$. Bourdon and Narayan [5] studied the algebraic properties of the weighted composition operator (induced by ϕ with weight function ψ) $W_{\phi,\psi}$ on $H^2(\mathbb{D})$ defined by $W_{\phi,\psi} f = (f \circ \phi)\psi$ which result from composition with ϕ and then multiplying by a weight function $\psi \in H(\mathbb{D})$. Such weighted composition operators are bounded on $H^2(\mathbb{D})$ when ψ is bounded on $\mathbb{D}$. But the boundedness of ψ on $\mathbb{D}$ is not necessary for $W_{\phi,\psi}$ to be bounded [5]. In this work, we consider a class of weighted composition operator $U_a^\alpha, a \in \mathbb{D}$ defined on the weighted Bergman space $L^2_a(dA_\alpha)$ as $U_a^\alpha f = (f \circ \phi_a) k_a^{1+\frac{\alpha}{2}},\ \alpha > -1$. These operators are self-adjoint, involutive unitary operators. We look at the action of these unitary weighted composition operators $U_a^\alpha, a \in \mathbb{D}$ on some bounded linear operator S defined on $L^2_a(dA_\alpha)$. Such studies on the Segal-Bergman space

2020 *Mathematics Subject Classification*. Primary 47B38; Secondary 47B32
Keywords. Weighted Bergman spaces, reproducing kernel, Toeplitz operator, little Hankel operator, weighted composition operator
Received: 21 February 2022; Revised: 23 July 2022; Accepted: 05 September 2022
Communicated by Dragan S. Djordjević
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(Fock space), Bergman space of the disk, on the Bergman space of the right half plane were carried out in [4], [13], [11], [23] and [18]. Applications of these results can be found in [4]. We have extended the results to weighted Bergman spaces $L^2_a(dA_\alpha), \alpha > -1$. We then considered the weighted composition operator $W_{\psi,q} = M^{(\alpha)}_q C^{(\alpha)}_\psi$ on $L^2_a(dA_\alpha)$ where $\psi \in Aut(\mathbb{D})$ and $q \in L^2_a(dA_\alpha)$. We showed that if $W_{\psi,q}L^2_a(dA_\alpha) \subset L^2_a(dA_\alpha)$ then $q \in H^\infty(\mathbb{D})$ if and only if ψ is a finite Blaschke product.

Let $dA(z) = \frac{1}{\pi}dxdy$ denotes the normalized area measure defined on $\mathbb{D}$. Let the Hilbert space $L^2(\mathbb{D}, dA_\alpha),\ \alpha > -1$ be the space of all Lebesgue measurable functions on $\mathbb{D}$ that are absolutely square-integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),\ z \in \mathbb{D}$. The weighted Bergman space $L^2_a(dA_\alpha)$ is the subspace of all analytic functions of $L^2(\mathbb{D}, dA_\alpha)$. The spaces $L^2_a(dA_\alpha)$ are closed subspaces of the corresponding spaces $L^2(\mathbb{D}, dA_\alpha),\ \alpha > -1$ and these are all reproducing kernel Hilbert spaces. For $\alpha = 0$, we shall denote $L^2_a(dA_0) = L^2_a(\mathbb{D})$ as the unweighted Bergman space of $\mathbb{D}$ whose reproducing kernel is given by $K(z,w) = \frac{1}{(1-z\bar{w})^2},\ z,w \in \mathbb{D}$ and the normalized reproducing kernel of $L^2_a(\mathbb{D})$ is given by $k_z(w) = \frac{1-|z|^2}{(1-w\bar{z})^2}$. Assume $K_z(w) = \overline{K(z,w)}$. The reproducing kernel of $L^2_a(dA_\alpha)$ is given by $K^{(\alpha)}(z,w) = [K(z,w)]^{1+\frac{\alpha}{2}} = \frac{1}{(1-z\bar{w})^{\alpha+2}}$ for $z,w \in \mathbb{D}$. Let $K^{(\alpha)}_z(w) = [K_z(w)]^{1+\frac{\alpha}{2}} = \overline{K^{(\alpha)}(z,w)}$. If $\langle\cdot,\cdot\rangle_\alpha$ denotes the inner product in $L^2(dA_\alpha) = L^2(\mathbb{D}, dA_\alpha)$, then $\langle h, K^{(\alpha)}_z\rangle_\alpha = h(z)$, for every $h \in L^2_a(dA_\alpha)$ and $z \in \mathbb{D}$. The orthogonal projection P_α from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto the closed subspace $L^2_a(dA_\alpha)$ is given by $(P_\alpha f)(z) = \left\langle f, K^{(\alpha)}_z \right\rangle_\alpha = \int_{\mathbb{D}} f(w)\frac{1}{(1-\bar{z}w)^{\alpha+2}} dA_\alpha(z)$ for $f \in L^2(\mathbb{D}, dA_\alpha)$ and $z \in \mathbb{D}$. The normalized reproducing kernels of $L^2_a(dA_\alpha)$ are the functions $k_z^{1+\frac{\alpha}{2}}(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}$. The sequence of functions $\{e^{(\alpha)}_n\} = \left\{\frac{z^n}{\gamma_{n,\alpha}}\right\}$ form as an orthonormal basis [24] for $L^2_a(dA_\alpha)$ where

$$\gamma^2_{n,\alpha} = \|z^n\|^2 = (\alpha+1)\int_{\mathbb{D}} |z|^{2n}(1-|z|^2)^\alpha dA(z) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}.$$

Henceforth we shall suppress the subscript α while writing the inner product and assume $\langle\cdot,\cdot\rangle_\alpha = \langle\cdot,\cdot\rangle$ for simplicity of notations. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$. The space $L^\infty(\mathbb{D})$ is a Banach space with the norm given by $\|f\|_\infty = ess \sup_{z\in\mathbb{D}} \{|f(z)|\},\ f \in L^\infty(\mathbb{D})$. Let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$ and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. A finite Blaschke product $\mathcal{B}_n$ is a function of the form

$$\mathcal{B}_n(z) = z^m \prod_{k=1}^{n} \frac{\overline{\alpha_k}}{\alpha_k}\frac{\alpha_k - z}{1-\overline{\alpha_k}z} \tag{1}$$

where $\alpha_k \neq 0$ and $|\alpha_k| < 1,\ k = 1,2,...,n$.

For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator on the weighted Bergman space $L^2_a(dA_\alpha)$ with symbol ϕ by $T^{(\alpha)}_\phi f = P_\alpha(\phi f),\ f \in L^2_a(dA_\alpha)$. We have $\|T^{(\alpha)}_\phi\| \leq \|\phi\|_\infty$ since the projection P_α has [24] norm 1. In fact, $\left(T^{(\alpha)}_\phi f\right)(w) = \int_{\mathbb{D}} \frac{\phi(z)f(z)}{(1-\bar{z}w)^{\alpha+2}} dA_\alpha(z)$ for $f \in L^2_a(dA_\alpha)$ and $w \in \mathbb{D}$. A Toeplitz operator $T^{(\alpha)}_\phi$ is an analytic (co-analytic) Toeplitz operator if the symbol ϕ belongs to $H^\infty(\mathbb{D})$ $\left(\overline{H^\infty(\mathbb{D})}\right)$.

For $\phi \in L^\infty(\mathbb{D})$, the generalized Berezin transform of ϕ is defined by $(B_\alpha\phi)(z) = \left\langle T^{(\alpha)}_\phi k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle = \int_{\mathbb{D}} \phi(w)|k_z(w)|^{2+\alpha} dA_\alpha(w),\ z \in \mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, we define the big Hankel operator with symbol ϕ from the space $L^2_a(dA_\alpha)$ onto its orthogonal complement $\left(L^2_a(dA_\alpha)\right)^\perp$ by $H^{(\alpha)}_\phi f = (I-P_\alpha)(\phi f),\ f \in L^2_a(dA_\alpha)$. We have $\|H^{(\alpha)}_\phi\| \leq \|\phi\|_\infty$. Let $\overline{L^2_a(dA_\alpha)} = \left\{\overline{f} : f \in L^2_a(dA_\alpha)\right\}$. The space $\overline{L^2_a(dA_\alpha)}$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$. The little Hankel operator $h^{(\alpha)}_\phi$ with symbol ϕ is defined by $h^{(\alpha)}_\phi f = \overline{P_\alpha}(\phi f), f \in L^2_a(dA_\alpha)$ where $\overline{P_\alpha}$ is the orthogonal projection from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto $\overline{L^2_a(dA_\alpha)}$. Clearly, $\|h^{(\alpha)}_\phi\| \leq \|\phi\|_\infty$ as $\|\overline{P_\alpha}\| \leq 1$.

Define J_α from $L^2(\mathbb{D}, dA_\alpha)$ into itself by $(J_\alpha f)(z) = f(\overline{z}),\ z \in \mathbb{D}$. The operator J_α is a unitary operator. For $\phi \in L^\infty(\mathbb{D})$, define $S_\phi^{(\alpha)}$ from $L_a^2(dA_\alpha)$ into itself by $S_\phi^{(\alpha)} f = P_\alpha J_\alpha(\phi f)$. The operator $S_\phi^{(\alpha)}$ is a linear operator and $\|S_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. It is not difficult to verify that $h_\phi^{(\alpha)} = J_\alpha S_\phi^{(\alpha)}$. Thus we shall refer in the sequel, both the operators $h_\phi^{(\alpha)}$ and $S_\phi^{(\alpha)}$ as little Hankel operators on $L_a^2(dA_\alpha)$.

Suppose ϕ is an analytic function from $\mathbb{D}$ into itself. If $\phi \in H^\infty(\mathbb{D}), f \in L_a^2(dA_\alpha)$, the composition operator $C_\phi^{(\alpha)}$ on $L_a^2(dA_\alpha)$ is defined by $(C_\phi^{(\alpha)} f)(z) = f(\phi(z))$ for all $z \in \mathbb{D}$. For a bounded analytic function ϕ on $\mathbb{D}$, the multiplication operator $M_\phi^{(\alpha)}$ on the space $L^2(\mathbb{D}, dA_\alpha)$ is defined by $M_\phi^{(\alpha)} f = \phi f$. Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the Hilbert space H into itself. For $T \in \mathcal{L}(L_a^2(dA_\alpha))$, we define $(B_\alpha T)(z) = \left\langle T k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle,\ z \in \mathbb{D}$. Notice that $|(B_\alpha T)(z)| \leq \|T\|$ as $\left\| k_z^{1+\frac{\alpha}{2}} \right\| = 1$ for all $z \in \mathbb{D}$. The function $B_\alpha T$ is called the generalized transform of T and denote $B_\alpha T_\phi = B_\alpha \phi$. In particular, we shall refer $B_0 T$ as the Berezin transform of T and $B_0 T_\phi = B_0 \phi$, the Berezin transform of the function ϕ. For more details about Berezin transform see [13].

The organization of the paper is as follows: In section 2, we consider a class of weighted composition operators U_z^α defined on the weighted Bergman spaces $L_a^2(dA_\alpha)$. We have shown that these operators are involutions and unitary. Some elementary properties of these operators are also derived. In section 3, we prove that a bounded linear operator S from $L_a^2(dA_\alpha)$ into itself commutes with all the composition operators $C_a^{(\alpha)},\ a \in \mathbb{D}$, if and only if $B_\alpha S$ satisfies certain averaging condition. In section 4, we show that if $\mathcal{M}$ is a subspace of $L^\infty(\mathbb{D})$ and if for $\phi \in \mathcal{M}$, the Toeplitz operator $T_\phi^{(\alpha)}$ represents a multiplication operator on a closed subspace $\mathcal{S} \subset L_a^2(dA_\alpha)$, then ϕ is bounded analytic on $\mathbb{D}$. Similarly if $q \in L^\infty(\mathbb{D})$ and $\mathcal{B}_n$ is a finite Blaschke product and $M_q^{(\alpha)}\left(Range\ C_{\mathcal{B}_n}^{(\alpha)}\right) \subset L_a^2(dA_\alpha)$, then $q \in H^\infty(\mathbb{D})$. Further, we have shown that if $\psi \in Aut(\mathbb{D})$, then $\mathcal{N} = \left\{q \in L_a^2(dA_\alpha) : M_q^{(\alpha)}\left(Range\ C_\psi^{(\alpha)}\right) \subset L_a^2(dA_\alpha)\right\} = H^\infty(\mathbb{D})$ if and only if ψ is a finite Blaschke product. In section 5, we discuss the future scope of the work.

2. Preliminaries

In this section we considered a class of weighted composition operators U_z^α defined on the weighted Bergman spaces $L_a^2(dA_\alpha)$. We showed that these operators are involutions and unitary. We discussed many elementary properties of these operators which will be used in establishing the main result of the paper.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that,
(i) $(\phi_a \circ \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a,\ \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$. In fact, $\phi_a(w) = \frac{a-w}{1-\bar{a}w}$, for all $a, w \in \mathbb{D}$. Given $z \in \mathbb{D}$, and h any measurable function on $\mathbb{D}$, we define

$$U_z^\alpha h = (h \circ \phi_z) k_z^{1+\frac{\alpha}{2}}.$$

Using the identity $1 - \overline{\phi_z(w)} z = \frac{1-|z|^2}{1-\overline{w}z}$, we have $k_z^{1+\frac{\alpha}{2}}(\phi_z(w)) = \frac{1}{k_z^{1+\frac{\alpha}{2}}}$. Since $\phi_z \circ \phi_z(w) \equiv w$, we see that $(U_z^\alpha(U_z^\alpha h))(z) = h(z)$ for all $z \in \mathbb{D}$ and $h \in L_a^2(dA_\alpha)$. For $a \in \mathbb{D}$, define $C_a^{(\alpha)} : L_a^2(dA_\alpha) \to L_a^2(dA_\alpha)$ as $C_a^{(\alpha)} f = f \circ \phi_a$.

Lemma 2.1. *The following hold:*
(i) The operator U_w^α is unitary and is an involution.
(ii) For $z, w \in \mathbb{D}$, $U_z^\alpha k_w^{1+\frac{\alpha}{2}} = \lambda k_{\phi_z(w)}^{1+\frac{\alpha}{2}}$ for some constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.
(iii) For all $w \in \mathbb{D}$, $U_w^\alpha k_w^{1+\frac{\alpha}{2}} = 1$.
(iv) For any $z, w \in \mathbb{D}$, there exists a unitary map $U \in G_0 = \{\psi \in Aut(\mathbb{D}) : \psi(0) = 0\}$ such that $\phi_w \circ \phi_z = U \phi_{\phi_z(w)}$.
(v) If $S \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is invertible and is an involution with polar decomposition $S = \mathcal{V}|S|$, then $\mathcal{V}$ is an involution which is also self-adjoint.

Proof. (i) Since $\phi_w \circ \phi_w(z) \equiv z$, we see that for $h \in L^2_a(dA_\alpha)$,
$U^\alpha_w U^\alpha_w h = U^\alpha_w (h \circ \phi_w) k_w^{1+\frac{\alpha}{2}} = \left(h \circ \phi_w \circ \phi_w\right)\left(k_w^{1+\frac{\alpha}{2}} \circ \phi^\alpha_w\right) k_w^{1+\frac{\alpha}{2}} = h$. Thus $(U^\alpha_w)^2 = I$ for all $w \in \mathbb{D}$ and therefore $(U^\alpha_w)^{-1} = U^\alpha_w$ and U^α_w is unitary on $L^2_a(dA_\alpha)$.
(ii) Let $z, w \in \mathbb{D}$ and $f \in L^2_a(dA_\alpha)$. Then

$$\left\langle f, U^\alpha_z K^{(\alpha)}_w \right\rangle = \left\langle U^\alpha_z f, K^{(\alpha)}_w \right\rangle = (U^\alpha_z f)(w) = (f \circ \phi_z)(w) k_z^{1+\frac{\alpha}{2}}(w) = \left\langle f, \overline{k_z^{1+\frac{\alpha}{2}}(w)} K^{(\alpha)}_{\phi_z(w)} \right\rangle.$$

Thus $U^\alpha_z K^{(\alpha)}_w = \overline{k_z^{1+\frac{\alpha}{2}}(w)} K^{(\alpha)}_{\phi_z(w)}$. This implies

$$\begin{aligned} U^\alpha_z k_w^{1+\frac{\alpha}{2}} &= \frac{\overline{k_z^{1+\frac{\alpha}{2}}(w)}}{\|K^{(\alpha)}_w\|} \frac{K^{(\alpha)}_{\phi_z(w)}}{\left\|K^{(\alpha)}_{\phi_z(w)}\right\|} . \left\|K^{(\alpha)}_{\phi_z(w)}\right\| = \frac{\overline{k_z^{1+\frac{\alpha}{2}}(w)}}{\|K^{(\alpha)}_w\|} k_{\phi_z(w)}^{1+\frac{\alpha}{2}} \left\|K^{(\alpha)}_{\phi_z(w)}\right\| \\ &= \frac{\overline{k_z^{1+\frac{\alpha}{2}}(w)}}{\|K^{(\alpha)}_w\|} \left\|U^\alpha_z K^{(\alpha)}_w\right\| k_{\phi_z(w)}^{1+\frac{\alpha}{2}} = \lambda k_{\phi_z(w)}^{1+\frac{\alpha}{2}} \end{aligned}$$

for some constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This is so, since U^α_z is unitary and $\left\|k_w^{1+\frac{\alpha}{2}}\right\|_2 = \left\|k_{\phi_z(w)}^{1+\frac{\alpha}{2}}\right\|_2 = 1$.
(iii) Notice that $1 - \overline{\phi_w(z)} w = \frac{1-|w|^2}{1-\bar{z}w}$. Hence $k_w^{1+\frac{\alpha}{2}}\left(\phi_w(z)\right) = \frac{1}{k_w^{1+\frac{\alpha}{2}}(z)}$ for all $w \in \mathbb{D}$ and $z \in \mathbb{D}$.
(iv) Let $U = \phi_w \circ \phi_z \circ \phi_{\phi_z(w)}$, then $U(0) = \phi_w \circ \phi_z\left(\phi_z(w)\right) = \phi_w(w) = 0$; thus $U \in G_0$ is unitary.
(v) We know that $R, T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ and $RT = TR$ then $\sqrt{R}\sqrt{T} = \sqrt{T}\sqrt{R}$. Hence $(S^*S)(SS^*) = (SS^*)(S^*S)$ implies that $|S||S^*| = |S^*||S|$. Thus, it follows that

$$(|S^*||S|)^2 = |S^*|^2|S|^2 = (SS^*)(S^*S) = I.$$

Now since the product of two commuting positive operators will be positive, we obtain from the [12] uniqueness of the square root of a positive operator that $|S||S^*| = |S^*||S| = I$. Further, $S^*(S^*S) = (SS^*)S^*$ implies $S^*|S| = |S^*|S^*$. Now since $\mathcal{V} = S^*|S|$, we obtain $\mathcal{V}^2 = (|S^*|S^*)(S^*|S|) = |S^*||S| = I$. Since $\mathcal{V}$ is unitary and $\mathcal{V}^2 = I$, we have $\mathcal{V}^* = \mathcal{V}$ and $\mathcal{V}$ is self-adjoint. □

The operators U^α_w satisfy the following intertwining properties with Toeplitz, multiplication, Hankel and little Hankel operators defined on $L^2_a(dA_\alpha)$.

Lemma 2.2. *The following is valid for* $\phi \in L^\infty(\mathbb{D})$*:*
(i) $U^\alpha_w T^{(\alpha)}_\phi U^\alpha_w = T^{(\alpha)}_{\phi\circ\phi_w}$.
(ii) $U^\alpha_w H^{(\alpha)}_\phi U^\alpha_w = H^{(\alpha)}_{\phi\circ\phi_w}$.
(iii) $U^\alpha_w M^{(\alpha)}_\phi U^\alpha_w = M^{(\alpha)}_{\phi\circ\phi_w}$.
(iv) $U^\alpha_w h^{(\alpha)}_\phi U^\alpha_w = h^{(\alpha)}_{\phi\circ\phi_w}$.

Proof. Notice that $U^\alpha_w\left(L^2_a(dA_\alpha)\right) \subset L^2_a(dA_\alpha)$ and $U^\alpha_w\left(\left(L^2_a(dA_\alpha)\right)^\perp\right) \subset \left(L^2_a(dA_\alpha)\right)^\perp$. Hence $P_\alpha U^\alpha_w = U^\alpha_w P_\alpha$. Now let $f \in L^2_a(dA_\alpha)$. Then from Lemma 2.1, it follows that

$$\begin{aligned} U^\alpha_w T^{(\alpha)}_\phi U^\alpha_w f &= U^\alpha_w T^{(\alpha)}_\phi \left((f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right) = U^\alpha_w P_\alpha \left(\phi (f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right) = P_\alpha U^\alpha_w \left(\phi (f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right) \\ &= P_\alpha \left((\phi \circ \phi_w)(f \circ \phi_w \circ \phi_w)\left(k_w^{1+\frac{\alpha}{2}} \circ \phi_w\right) k_w^{1+\frac{\alpha}{2}}\right) = P_\alpha\left((\phi \circ \phi_w) f\right) = T^{(\alpha)}_{\phi\circ\phi_w} f. \end{aligned}$$

Hence (i) follows. Again let $f \in L^2_a(dA_\alpha)$. Then from Lemma 2.1, it follows that

$$\begin{aligned} U^\alpha_w H^{(\alpha)}_\phi U^\alpha_w f &= U^\alpha_w H^{(\alpha)}_\phi \left[(f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right] = U^\alpha_w \left[(I - P_\alpha)\left(\phi (f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right)\right] \\ &= (I - P_\alpha) U^\alpha_w \left[\phi (f \circ \phi_w) k_w^{1+\frac{\alpha}{2}}\right] = (I - P_\alpha)\left[(\phi \circ \phi_w)(f \circ \phi_w \circ \phi_w)\left(k_w^{1+\frac{\alpha}{2}} \circ \phi_w\right) k_w^{1+\frac{\alpha}{2}}\right] \\ &= (I - P_\alpha)\left[(\phi \circ \phi_w) f\right] = H^{(\alpha)}_{\phi\circ\phi_w} f. \end{aligned}$$

Thus (ii) follows. The proof of (iii) and (iv) are similar. □

Lemma 2.3. *Fix $\alpha > -1$. If $S, T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ and $(B_\alpha S)(z) = (B_\alpha T)(z)$ for all $z \in \mathbb{D}$, then $S = T$.*

Proof. Assume $\left\langle (S - T) k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = 0$ for all $z \in \mathbb{D}$. Then $\left\langle (S - T) K^{(\alpha)}_z, K^{(\alpha)}_z \right\rangle = K^{(\alpha)}(z, z) \left\langle (S - T) k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = K^{(\alpha)}(z, z) \cdot 0 = 0$. Let $A = S - T$ and define $G(x, y) = \left\langle A K^{(\alpha)}_{\overline{x}}, K^{(\alpha)}_y \right\rangle$. The function G is holomorphic in x and y and $G(x, y) = 0$ if $x = \overline{y}$. It can now be verified that such functions must vanish identically. Let $x = u + iv,\ y = u - iv$. Let $F(u, v) = G(x, y)$. The function F is holomorphic and vanishes if u and v are real. Hence $G(x, y) = F(u, v) \equiv 0$. Thus even $\left\langle A K^{(\alpha)}_x, K^{(\alpha)}_y \right\rangle = 0$ for any $x, y \in \mathbb{D}$. Since the linear combinations of $K^{(\alpha)}_x, x \in \mathbb{D}$, are dense in $L^2_a(dA_\alpha)$, it follows that $A = 0$. That is, $S = T$. □

Lemma 2.4. *If $f \in L^1_a(\mathbb{D}, dA_\alpha)$, then $f(z) = \int_{\mathbb{D}} f(w) K^{(\alpha)}(z, w) dA_\alpha(w)$ for all $z \in \mathbb{D}$ and*

$$\|K^{(\alpha)}(\cdot, w)\|_2 \approx \frac{1}{(1 - |w|^2)^{1+\frac{\alpha}{2}}}.$$

Proof. It follows from [24] that

$$\begin{aligned} \|K^{(\alpha)}(\cdot, w)\|_2 &= \left(\int_{\mathbb{D}} \left|K^{(\alpha)}(z, w)\right|^2 dA_\alpha(z)\right)^{\frac{1}{2}} = \left(\int_{\mathbb{D}} \frac{\left(1 - |z|^2\right)^\alpha}{\left|1 - \overline{w} z\right|^{2(\alpha+2)}} dA(z)\right)^{\frac{1}{2}} (\alpha + 1)^{\frac{1}{2}} \\ &\approx \left(\frac{1}{(1 - |w|^2)^{\alpha+2}}\right)^{\frac{1}{2}} = \frac{1}{(1 - |w|^2)^{1+\frac{\alpha}{2}}} . \end{aligned}$$

□

For any $f \in L^2(\mathbb{D}, dA_\alpha)$, we define a function $B_\alpha f$ on $\mathbb{D}$ by

$$(B_\alpha f)(z) = \int_{\mathbb{D}} f(\phi_z(w)) dA_\alpha(w) = \int_{\mathbb{D}} f(w) \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w).$$

From [1],[24] it follows that there exists a constant C such that $\frac{|K^{(\alpha)}(z,w)|}{K^{(\alpha)}(z,z)} = \frac{1}{|K^{(\alpha)}(z,\phi_z(w))|} \leq C$, for all z and w in $\mathbb{D}$. It thus follows that $|B_\alpha f(z)| \leq C \int_{\mathbb{D}} |f(w)| \left|K^{(\alpha)}(z, w)\right| dA_\alpha(w)$. This implies that the transform B_α is a bounded linear operator on $L^2(\mathbb{D}, dA_\alpha)$.

3. Main results

In this section, we proved that a bounded linear operator S from $L^2_a(dA_\alpha)$ into itself commutes with all the composition operators $C^{(\alpha)}_a,\ a \in \mathbb{D}$, if and only if $B_\alpha S$ satisfies certain averaging condition. That is, if and only if $\widehat{S} = S$ where $\widehat{S} = \int_{\mathbb{D}} U^\alpha_a S U^\alpha_a dA_\alpha(a)$. Since the mapping $a \mapsto U^{(\alpha)}_a$ is strong operator continuous, we can define for each bounded linear operator S on $L_a^2(dA_\alpha)$, a bounded linear operator $\widehat{S}$ (an averaging operation) on the space by $\widehat{S} = \int_{\mathbb{D}} U_a^\alpha S U_a^\alpha dA_\alpha(a)$ where the integral is taken in the sense that $\left\langle \left(\int_{\mathbb{D}} U_a^\alpha S U_a^\alpha dA_\alpha(a) \right) f, g \right\rangle = \int_{\mathbb{D}} \langle U_a^\alpha S U_a^\alpha f, g \rangle \, dA_\alpha(a)$. Notice that the integrand of $\int_{\mathbb{D}} U_a^\alpha S U_a^\alpha dA_\alpha(a)$ is strongly continuous in a and uniformly bounded for each fixed S. For a discussion of such integrals see [6] and [7]. The idea of averaging an operator against some unitary operators were considered by many authors [4], [14]. We will also present some applications of Theorem 3.1 in form of corollaries at the end of this section.

Theorem 3.1. *A bounded linear operator $S \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ commutes with all the composition operators $C_a^{(\alpha)}, a \in \mathbb{D}$, if and only if*

$$(B_\alpha S)(z) = \int_{\mathbb{D}} (B_\alpha S)(\phi_a(z)) dA_\alpha(a)$$

for all $z \in \mathbb{D}$.

Proof. Suppose $(B_\alpha S)(z) = \int_{\mathbb{D}} (B_\alpha S)(\phi_a(z)) dA_\alpha(a)$ for all $z \in \mathbb{D}$. Then by Lemma 2.1, there exists a constant λ with $|\lambda| = 1$ such that for all $z \in \mathbb{D}$,

$$\begin{aligned}
(B_\alpha S)(z) &= \left\langle S k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = \int_{\mathbb{D}} (B_\alpha S)(\phi_a(z)) dA_\alpha(a) = \int_{\mathbb{D}} \left\langle S k_{\phi_a(z)}^{1+\frac{\alpha}{2}}, k_{\phi_a(z)}^{1+\frac{\alpha}{2}} \right\rangle dA_\alpha(a) \\
&= \int_{\mathbb{D}} \left\langle \lambda S U_a^\alpha k_z^{1+\frac{\alpha}{2}}, \lambda U_a^\alpha k_z^{1+\frac{\alpha}{2}} \right\rangle dA_\alpha(a) = \int_{\mathbb{D}} \left\langle U_a^\alpha S U_a^\alpha k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle dA_\alpha(a) \\
&= \left\langle \left(\int_{\mathbb{D}} U_a^\alpha S U_a^\alpha dA_\alpha(a) \right) k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = \left\langle \widehat{S} k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \\
&= \left(B_\alpha \widehat{S} \right)(z)
\end{aligned}$$

where $\widehat{S} = \int_{\mathbb{D}} U_a^\alpha S U_a^\alpha dA_\alpha(a)$. Thus by Lemma 2.3, $S = \widehat{S}$. Hence for all $f, g \in L_a^2(dA_\alpha), \langle Sf, g \rangle = \langle \widehat{S} f, g \rangle$. That is,

$$\int_{\mathbb{D}} (Sf)(z) \overline{g(z)} dA_\alpha(z) = \int_{\mathbb{D}} \langle S U_a^\alpha f, U_a^\alpha g \rangle dA_\alpha(a). \tag{2}$$

The boundedness of S and the antianalyticity of $K^{(\alpha)}(z,a)$ in a imply that for each $z \in \mathbb{D}$, the function, $S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z) K^{(\alpha)}(z,a)$ is antianalytic in a. Therefore, by the mean value property of harmonic functions, we have [19]

$$\int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z) K^{(\alpha)}(z,a) dA_\alpha(a) = S\left(\frac{f}{K^{(\alpha)}(\cdot,0)}\right) K^{(\alpha)}(z,0) = Sf(z). \tag{3}$$

Thus, from (3), it follows that

$$\langle Sf, g \rangle = \int_{\mathbb{D}} \overline{g(z)} \int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z) K^{(\alpha)}(z,a) dA_\alpha(a) dA_\alpha(z).$$

Using Fubini's theorem [22] , we get $\langle Sf, g \rangle = \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z) \overline{g(z)} K^{(\alpha)}(z,a) dA_\alpha(z) dA_\alpha(a)$. Now since $k_a^{1+\frac{\alpha}{2}}(z) = \frac{K^{(\alpha)}(z,a)}{\sqrt{K^{(\alpha)}(a,a)}}$ and $\left(k_a^{1+\frac{\alpha}{2}} \circ \phi_a\right)(z) k_a^{1+\frac{\alpha}{2}}(z) = 1$ for all $a, z \in \mathbb{D}$, we obtain

$$\begin{aligned}
\langle Sf, g \rangle &= \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(z) \overline{g(z)} k_a^{1+\frac{\alpha}{2}}(z) dA_\alpha(z) dA_\alpha(a) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(z) \overline{g(z)} \overline{k_a^{1+\frac{\alpha}{2}}(\phi_a(z))} \left| k_a^{1+\frac{\alpha}{2}}(z) \right|^2 dA_\alpha(z) dA_\alpha(a).
\end{aligned}$$

Finally, as $(\phi_a \circ \phi_a)(z) \equiv z$ and $J_{\phi_a(z)} = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^4}$, we obtain using Lemma 2.1 that

$$\langle Sf, g\rangle = \int_{\mathbb{D}}\int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(\phi_a(z))\overline{k_a^{1+\frac{\alpha}{2}}(z)}\,\overline{g(\phi_a(z))}dA_\alpha(z)dA_\alpha(a).$$

By our hypothesis, and using (3) we have $\langle Sf, g\rangle = \int_{\mathbb{D}}\langle SU_a^\alpha f, U_a^\alpha g\rangle dA_\alpha(a)$. Using Lemma 2.1, we obtain

$$\begin{aligned}
\langle SU_a^\alpha f, U_a^\alpha g\rangle &= \left\langle S\left(\frac{f\circ\phi_a}{k_a^{1+\frac{\alpha}{2}}\circ\phi_a}\right), (g\circ\phi_a)k_a^{1+\frac{\alpha}{2}}\right\rangle \\
&= \left\langle S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\circ\phi_a\right), (g\circ\phi_a)k_a^{1+\frac{\alpha}{2}}\right\rangle \\
&= \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\circ\phi_a\right)(z)\overline{g(\phi_a(z))}\,\overline{k_a^{1+\frac{\alpha}{2}}(z)}dA_\alpha(z).
\end{aligned}$$

Thus we obtain for all $f, g \in L_a^2(dA_\alpha)$,

$$\int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\circ\phi_a\right)(z)\overline{g(\phi_a(z))}k_a^{1+\frac{\alpha}{2}}(z)dA_\alpha(z) = \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(\phi_a(z))\overline{k_a^{1+\frac{\alpha}{2}}(z)}\,\overline{g(\phi_a(z))}dA_\alpha(z).$$

Hence for all $f, g \in L_a^2(dA_\alpha)$, $a \in \mathbb{D}$, we have

$$\left\langle S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\circ\phi_a\right), U_a^\alpha g\right\rangle = \left\langle S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)\circ\phi_a, U_a^\alpha g\right\rangle.$$

Since $U_a^\alpha \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is unitary, we obtain $S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\circ\phi_a\right) = S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)\circ\phi_a$ for all $f \in L_a^2(dA_\alpha)$ and $a \in \mathbb{D}$. Thus $SC_a^{(\alpha)}\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right) = C_a^{(\alpha)}S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)$. Since $\left(k_a^{1+\frac{\alpha}{2}}\right)^{-1} \in H^\infty(\mathbb{D})$, hence $SC_a^{(\alpha)} = C_a^{(\alpha)}S$ for all $a \in \mathbb{D}$. Now to prove the converse, assume that $C_a^{(\alpha)}S = SC_a^{(\alpha)}$ for all $a \in \mathbb{D}$. That is, for all $f \in L_a^2(dA_\alpha)$, $a \in \mathbb{D}$, we have $(Sf)\circ\phi_a = S(f\circ\phi_a)$. Hence by Lemma 2.1, we obtain for all $f \in L_a^2(dA_\alpha)$,

$$SU_a^\alpha f = S\left((f\circ\phi_a)k_a^{1+\frac{\alpha}{2}}\right) = S\left(\frac{f\circ\phi_a}{k_a^{1+\frac{\alpha}{2}}\circ\phi_a}\right) = S\left(\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)\circ\phi_a\right) = S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)\circ\phi_a.$$

Now since $k_a^{1+\frac{\alpha}{2}}(z) = \frac{K^{(\alpha)}(z,a)}{\sqrt{K^{(\alpha)}(a,a)}}$ for all $a, z \in \mathbb{D}$ and by using Lemma 2.1, we get for all $f, g \in L_a^2(dA_\alpha)$,

$$\begin{aligned}
\langle SU_a^\alpha f, U_a^\alpha g\rangle &= \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(\phi_a(z))\overline{(g\circ\phi_a)(z)}\,\overline{k_a^{1+\frac{\alpha}{2}}(z)}dA_\alpha(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(z)\overline{g(z)}\,\overline{\left(k_a^{1+\frac{\alpha}{2}}\circ\phi_a\right)(z)}\left|k_a^{1+\frac{\alpha}{2}}(z)\right|^2 dA_\alpha(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{k_a^{1+\frac{\alpha}{2}}}\right)(z)\overline{g(z)}k_a^{1+\frac{\alpha}{2}}dA_\alpha(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot, a)}\right)(z)\overline{g(z)}K^{(\alpha)}(z,a)dA_\alpha(z).
\end{aligned}$$

By using Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \langle SU_a^\alpha f, U_a^\alpha g\rangle dA_\alpha(a) &= \int_{\mathbb{D}}\int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z)\overline{g(z)}K^{(\alpha)}(z,a)dA_\alpha(z)dA_\alpha(a)\\ &= \int_{\mathbb{D}} \overline{g(z)}dA_\alpha(z)\int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z)K^{(\alpha)}(z,a)dA_\alpha(a).\end{aligned}$$

In the first part of the proof, we have already checked that for all $z \in \mathbb{D}$, $\int_{\mathbb{D}} S\left(\frac{f}{K^{(\alpha)}(\cdot,a)}\right)(z)K^{(\alpha)}(z,a)dA_\alpha(a) = S\left(\frac{f}{K^{(\alpha)}(\cdot,0)}\right)(z)K^{(\alpha)}(z,0) = Sf(z)$. Thus $\int_{\mathbb{D}} \langle SU_a^\alpha f, U_a^\alpha g\rangle dA_\alpha(a) = \int_{\mathbb{D}} Sf(z)\overline{g(z)}dA_\alpha(z) = \langle Sf, g\rangle$. Taking $f = g = k_z^{1+\frac{\alpha}{2}}, z \in \mathbb{D}$, we obtain by Lemma 2.1 that

$$\begin{aligned}(B_\alpha S)(z) &= \left\langle Sk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle = \int_{\mathbb{D}} \left\langle SU_a^\alpha k_z^{1+\frac{\alpha}{2}}, U_a^\alpha k_z^{1+\frac{\alpha}{2}}\right\rangle dA_\alpha(a)\\ &= \int_{\mathbb{D}} \left\langle Sk_{\phi_a(z)}^{1+\frac{\alpha}{2}}, k_{\phi_a(z)}^{1+\frac{\alpha}{2}}\right\rangle dA_\alpha(a) = \int_{\mathbb{D}} (B_\alpha S)(\phi_a(z))dA_\alpha(a). \qquad (4)\end{aligned}$$

This completes the proof. □

Example 3.2. *The operator B_α defined on $L^2(\mathbb{D}, dA_\alpha)$ commutes with the composition operators $C_a^{(\alpha)}, a \in \mathbb{D}$. To verify this, let $f \in L^2(\mathbb{D}, dA_\alpha)$. By a change of variable,*

$$\begin{aligned}(B_\alpha f)(\phi_a(z)) &= \int_{\mathbb{D}} f(w)\left|k_{\phi_a(z)}^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w)\\ &= \int_{\mathbb{D}} f(\phi_a(w))\left|k_{\phi_a(z)}^{1+\frac{\alpha}{2}} \circ \phi_a(w)\right|^2 \left|k_a^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w).\end{aligned}$$

Applying Lemma 2.1, we obtain an unitary U with $\phi_{\phi_a(z)} \circ \phi_a = U\phi_{\phi_a\circ\phi_a(z)} = U\phi_z$. Taking the real Jacobian determinants of the above equation, we obtain $\left|k_{\phi_a(z)}^{1+\frac{\alpha}{2}} \circ \phi_a(w)\right|^2 \left|k_a^{1+\frac{\alpha}{2}}(w)\right|^2 = \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2$ for all a, z and w in $\mathbb{D}$. Therefore,

$$(B_\alpha f)(\phi_a(z)) = \int_{\mathbb{D}} f(\phi_a(w))\left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w) = B_\alpha(f \circ \phi_a)(z).$$

This implies that $B_\alpha C_a^{(\alpha)} = C_a^{(\alpha)} B_\alpha$ on $L^{2,\alpha}(\mathbb{D})$ and hence $\widehat{B_\alpha} = B_\alpha$.

For $\phi \in L^\infty(\mathbb{D})$, define the functions

$$(D_\alpha\phi)(z) = \int_{\mathbb{D}} \phi(\phi_a(z))dA_\alpha(a),$$

and

$$(B_\alpha\phi)(z) = \int_{\mathbb{D}} \phi(\phi_z(w))dA_\alpha(w).$$

Now we present some applications of our main result Theorem 3.1.

Corollary 3.3. *If $\phi \in L^\infty(\mathbb{D})$, then there exists a constant δ of modulus 1 such that*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \phi\left(\phi_{\phi_a(z)}(w)\right)dA_\alpha(w)dA_\alpha(a) = \int_{\mathbb{D}}\int_{\mathbb{D}} \phi\left(\delta\phi_{\phi_z(a)}(w)\right)dA_\alpha(a)dA_\alpha(w).$$

Proof. From (4) it follows that

$$\begin{aligned}\int_{\mathbb{D}}\left(B_\alpha T_\phi^{(\alpha)}\right)(\phi_a(z))dA_\alpha(a) &= \int_{\mathbb{D}}\left\langle T_\phi^{(\alpha)}k_{\phi_a(z)}^{1+\frac{\alpha}{2}}, k_{\phi_a(z)}^{1+\frac{\alpha}{2}}\right\rangle dA_\alpha(a)\\ &= \int_{\mathbb{D}}\left\langle \phi k_{\phi_a(z)}^{1+\frac{\alpha}{2}}, k_{\phi_a(z)}^{1+\frac{\alpha}{2}}\right\rangle dA_\alpha(a)\\ &= \int_{\mathbb{D}}(B_\alpha\phi)(\phi_a(z))dA_\alpha(a)\\ &= \int_{\mathbb{D}}\int_{\mathbb{D}}\phi\left(\phi_{\phi_a(z)}(w)\right)dA_\alpha(w)dA_\alpha(a).\end{aligned}$$

Let $f, g \in L_a^2(dA_\alpha)$. Then by Lemma 2.1 and Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{D}}\left\langle U_a^\alpha T_\phi^{(\alpha)}U_a^\alpha f, g\right\rangle dA_\alpha(a) &= \int_{\mathbb{D}}dA_\alpha(a)\int_{\mathbb{D}}\phi(z)(f\circ\phi_a)(z)k_a^{1+\frac{\alpha}{2}}(z)\overline{(g\circ\phi_a)(z)}\overline{k_a^{1+\frac{\alpha}{2}}}(z)dA_\alpha(z)\\ &= \int_{\mathbb{D}}dA_\alpha(a)\int_{\mathbb{D}}\phi(\phi_a(w))f(w)\overline{g(w)}\left|\left(k_a^{1+\frac{\alpha}{2}}\circ\phi_a\right)(w)\right|^2\left|k_a^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w)\\ &= \int_{\mathbb{D}}dA_\alpha(a)\int_{\mathbb{D}}\phi(\phi_a(w))f(w)\overline{g(w)}dA_\alpha(w)\\ &= \int_{\mathbb{D}}f(w)\overline{g(w)}dA_\alpha(w)\int_{\mathbb{D}}\phi(\phi_a(w))dA_\alpha(a)\\ &= \int_{\mathbb{D}}(D_\alpha\phi)(w)f(w)\overline{g(w)}dA_\alpha(w).\end{aligned}$$

Thus
$$\begin{aligned}\int_{\mathbb{D}}\left(B_\alpha T_\phi^{(\alpha)}\right)(\phi_a(z))dA_\alpha(a) &= \int_{\mathbb{D}}\left\langle U_a^\alpha T_\phi^{(\alpha)}U_a^\alpha k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle dA_\alpha(a)\\ &= \int_{\mathbb{D}}(D_\alpha\phi)(w)\left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w) = \int_{\mathbb{D}}(D_\alpha\phi)(\phi_z(w))dA_\alpha(w)\\ &= \int_{\mathbb{D}}\int_{\mathbb{D}}(\phi\circ\phi_a\circ\phi_z)(w)dA_\alpha(a)dA_\alpha(w).\end{aligned}$$

Hence by Theorem 3.1, we obtain

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\phi\left(\phi_{\phi_a(z)}(w)\right)dA_\alpha(w)dA_\alpha(a) = \int_{\mathbb{D}}\int_{\mathbb{D}}\phi(\phi_a\circ\phi_z)(w)dA_\alpha(a)dA_\alpha(w).$$

Let $U = \phi_a\circ\phi_z\circ\phi_{\phi_z(a)}$. Then $U \in Aut(\mathbb{D})$ and $U(0) = \phi_a\circ\phi_z(\phi_z(a)) = \phi_a(a) = 0$ and $U\phi_{\phi_z(a)} = \phi_a\circ\phi_z$. It is well known [9] that if $\phi \in Aut(\mathbb{D})$, then $\phi(z) = e^{i\theta}\frac{z-p}{1-\bar{p}z}$ for some $\theta \in \mathbb{R}$ and $p \in \mathbb{D}$. Furthermore, $\phi(0) = 0$ if and only if $\phi(z) = e^{i\theta}z$. Thus $Uz = e^{i\theta}z$ and $\phi_a\circ\phi_z = U\phi_{\phi_z(a)} = e^{i\theta}\phi_{\phi_z(a)} = \delta\phi_{\phi_z(a)}$, where $\delta = e^{i\theta}, \theta \in \mathbb{R}$. Hence it follows that $\int_{\mathbb{D}}\int_{\mathbb{D}}\phi\left(\phi_{\phi_a(z)}(w)\right)dA_\alpha(w)dA_\alpha(a) = \int_{\mathbb{D}}\int_{\mathbb{D}}\phi\left(\delta\phi_{\phi_z(a)}(w)\right)dA_\alpha(a)dA_\alpha(w)$. □

Notice that one can define U_a^α on $L^2(\mathbb{D}, dA_\alpha)$ also. Suppose $\phi \in L^\infty(\mathbb{D})$, $f, g \in L^2(\mathbb{D}, dA_\alpha)$. Then by using

Fubini's theorem and making a change of variable, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, U_a^\alpha g \right\rangle dA_\alpha(a) &= \int_{\mathbb{D}} dA_\alpha(a) \int_{\mathbb{D}} \phi(z)(f\circ\phi_a)(z)k_a^{1+\frac{\alpha}{2}}\overline{(g\circ\phi_a)(z)k_a^{1+\frac{\alpha}{2}}(z)}dA_\alpha(z)\\ &= \int_{\mathbb{D}} dA_\alpha(a)\int_{\mathbb{D}} \phi(\phi_a(w))f(w)\overline{g(w)}dA_\alpha(w)\\ &= \int_{\mathbb{D}} f(w)\overline{g(w)}dA_\alpha(w)\int_{\mathbb{D}}\phi(\phi_a(w))dA_\alpha(a)\\ &= \int_{\mathbb{D}} (D_\alpha\phi)(w)f(w)\overline{g(w)}dA_\alpha(w) = \left\langle (D_\alpha\phi)f, g\right\rangle. \end{aligned} \tag{5}$$

Define $J_\alpha : L^2(\mathbb{D}, dA_\alpha) \to L^2(\mathbb{D}, dA_\alpha)$ as $J_\alpha f(z) = f(\overline{z})$. The map J_α is an unitary operator and $J_\alpha^* = J_\alpha$. Let $\overline{L_a^2(dA_\alpha)} = \{\overline{f} : f \in L_a^2(dA_\alpha)\}$. Define $h_\phi^{(\alpha)} : L_a^2(dA_\alpha) \to \overline{L_a^2(dA_\alpha)}$ such that $h_\phi^{(\alpha)} f = \overline{P_\alpha}(\phi f)$, where $\overline{P_\alpha}$ is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $\overline{L_a^2(dA_\alpha)}$. The operator $h_\phi^{(\alpha)}$ is called the little Hankel operator on $L_a^2(dA_\alpha)$.

In Corollary 3.4, we show that $\widehat{H}_\phi^{(\alpha)} = H_{D_\alpha\phi}^{(\alpha)}$, $\widehat{h}_\phi^{(\alpha)} = h_{D_\alpha\phi}^{(\alpha)}$, $\widehat{T}_\phi^{(\alpha)} = T_{D_\alpha\phi}^{(\alpha)}$. Thus $T_\phi^{(\alpha)}$, $H_\phi^{(\alpha)}$, $h_\phi^{(\alpha)}$ commutes with all $C_\alpha^{(\alpha)}, a \in \mathbb{D}$ if and only if $D_\alpha\phi = \phi$.

Corollary 3.4. *If $\phi \in L^\infty(\mathbb{D})$, $f \in L_a^2(dA_\alpha)$, then*

(i) $\int_{\mathbb{D}} \left\langle U_a^\alpha H_\phi^{(\alpha)} U_a^\alpha f, g\right\rangle dA_\alpha(a) = \left\langle H_{(D_\alpha\phi)}^{(\alpha)} f, g\right\rangle$ *for all* $g \in \left(L_a^2(dA_\alpha)\right)^\perp$.

(ii) $\int_{\mathbb{D}} \left\langle U_a^\alpha h_\phi^{(\alpha)} U_a^\alpha f, g\right\rangle dA_\alpha(a) = \left\langle h_{(D_\alpha\phi)}^{(\alpha)} f, g\right\rangle$ *for all* $g \in \overline{L_a^2(dA_\alpha)}$.

(iii) $\int_{\mathbb{D}} \left\langle U_a^\alpha T_\phi^{(\alpha)} U_a^\alpha f, g\right\rangle dA_\alpha(a) = \left\langle T_{(D_\alpha\phi)}^{(\alpha)} f, g\right\rangle$ *for all* $g \in L_a^2(dA_\alpha)$.

Proof. (i) If $f \in L_a^2(dA_\alpha)$, $g \in \left(L_a^2(dA_\alpha)\right)^\perp$, then from (5), it follows that

$$\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, U_a^\alpha g\right\rangle dA_\alpha(a) = \left\langle (D_\alpha\phi)f, g\right\rangle.$$

This implies that $\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, U_a^\alpha (I - P_\alpha)g\right\rangle dA_\alpha(a) = \left\langle (D_\alpha\phi)f, (I-P_\alpha)g\right\rangle$. Hence since $U_a^\alpha P_\alpha = P_\alpha U_a^\alpha$, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \left\langle U_a^\alpha (I-P_\alpha)(\phi U_a^\alpha f), g\right\rangle dA_\alpha(a) &= \int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, (I-P_\alpha)U_a^\alpha g\right\rangle dA_\alpha(a)\\ &= \left\langle (I-P_\alpha)((D_\alpha\phi)f), g\right\rangle.\end{aligned}$$

Therefore, we get $\int_{\mathbb{D}} \left\langle U_a^\alpha H_\phi^{(\alpha)} U_a^\alpha f, g\right\rangle dA_\alpha(a) = \left\langle H_{(D_\alpha\phi)}^{(\alpha)} f, g\right\rangle$.

(ii) If $f \in L_a^2(dA_\alpha)$, $g \in \overline{L_a^2(dA_\alpha)}$, then from the above discussion it follows that $\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, U_a^\alpha g\right\rangle dA_\alpha(a) = \langle (D_\alpha\phi)f, g\rangle$. This implies

$$\int_{\mathbb{D}} \left\langle \phi U_a^\alpha P_\alpha f, U_a^\alpha \overline{P_\alpha} g\right\rangle dA_\alpha(a) = \langle (D_\alpha\phi)P_\alpha f, \overline{P_\alpha} g\rangle.$$

Since $\overline{P_\alpha} = J_\alpha P_\alpha J_\alpha$, hence we obtain $\int_{\mathbb{D}} \left\langle \phi U_a^\alpha P_\alpha f, U_a^\alpha J_\alpha P_\alpha J_\alpha g\right\rangle dA_\alpha(a) = \langle (D_\alpha\phi)P_\alpha f, J_\alpha P_\alpha J_\alpha g\rangle$. Now $U_a^\alpha P_\alpha = P_\alpha U_a^\alpha$. Thus we obtain

$$\begin{aligned}\int_{\mathbb{D}} \left\langle U_a^\alpha J_\alpha P_\alpha J_\alpha \phi P_\alpha U_a^\alpha f, g\right\rangle dA_\alpha(a) &= \int_{\mathbb{D}} \left\langle \phi U_a^\alpha P_\alpha f, J_\alpha P_\alpha J_\alpha U_a^\alpha g\right\rangle dA_\alpha(a)\\ &= \left\langle J_\alpha P_\alpha J_\alpha (D_\alpha\phi)P_\alpha f, g\right\rangle.\end{aligned}$$

Thus $\int_{\mathbb{D}} \left\langle U_a^\alpha h_\phi^{(\alpha)} U_a^\alpha f, g \right\rangle dA_\alpha(a) = \left\langle h_{D_\alpha \phi}^{(\alpha)} f, g \right\rangle$.

(iii) If $f, g \in L_a^2(dA_\alpha)$, then from equation (5), it follows that

$$\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, U_a^\alpha g \right\rangle dA_\alpha(a) = \left\langle (D_\alpha \phi) f, g \right\rangle.$$

Hence we obtain $\int_{\mathbb{D}} \left\langle \phi U_a^\alpha f, P_\alpha U_a^\alpha g \right\rangle dA_\alpha(a) = \left\langle (D_\alpha \phi) f, P_\alpha g \right\rangle$. Thus

$$\begin{aligned}\int_{\mathbb{D}} \left\langle U_a^\alpha P_\alpha(\phi U_a^\alpha f), g \right\rangle dA_\alpha(a) &= \int_{\mathbb{D}} \left\langle P_\alpha(\phi U_a^\alpha f), U_a^\alpha g \right\rangle dA_\alpha(a) = \left\langle (D_\alpha \phi) f, P_\alpha g \right\rangle \\ &= \left\langle P_\alpha((D_\alpha \phi) f), g \right\rangle.\end{aligned}$$

It follows therefore that $\int_{\mathbb{D}} \left\langle U_a^\alpha T_\phi^{(\alpha)} U_a^\alpha f, g \right\rangle dA_\alpha(a) = \left\langle T_{(D_\alpha \phi)}^{(\alpha)} f, g \right\rangle$. □

Example 3.5. *Let $\alpha = 0$ and consider the Berezin transform B_0. Notice that if g is harmonic on $\mathbb{D}$, then g is the sum of an analytic function and the conjugate of another analytic function. It follows from [1], [10], [13] that $B_0 g = g$ and $D_0 g = g(0) - \frac{1}{2}\frac{\partial g}{\partial z}(0) z - \frac{1}{2}\frac{\partial g}{\partial \bar{z}}(0)\bar{z}$. Let $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H^\infty(\mathbb{D})$. Then from [1], [13], [10] that $B_0 g = g$ and $D_0 g = c_0 - \frac{c_1}{2} z$. Hence if $g(z) = 3 - 2z + 7z^2 - 5z^3, z \in \mathbb{D}$, then $B_0 g = g$ but $D_0 g = 3 - z$. Hence $\widehat{T_g^{(0)}} = T_{D_0 g}^{(0)} \neq T_g^{(0)}$. By Theorem 3.1, $T_g^{(0)}$ does not commute with all $C_a^{(0)}, a \in \mathbb{D}$. Now let $f(z) = -2\bar{z} - 7\,\bar{z}^2$. Then $B_0 f = f$ but $(D_0 f)(z) = \bar{z}$. Thus $\widehat{H_f^{(0)}} = H_{D_0 f}^{(0)} = H_{\bar{z}}^{(0)} \neq H_f^{(0)}$ and similarly $\widehat{h_f^{(0)}} = h_{D_0 f}^{(0)} = h_{\bar{z}}^{(0)} \neq h_f^{(0)}$. By Theorem 3.1, $H_f^{(0)}$ and $h_f^{(0)}$ does not commute with all $C_a^{(0)}, a \in \mathbb{D}$.*

4. Bounded analytic functions and composition operators

It is not difficult to verify that $M_\phi^{(\alpha)} L_a^2(dA_\alpha) \subset L_a^2(dA_\alpha)$ if and only if $\phi \in H^\infty(\mathbb{D})$. In section 3, we considered the weighted composition operator $U_a^\alpha f = (f \circ \phi_a) k_a^{1+\frac{\alpha}{2}}$, $f \in L_a^2(dA_\alpha)$. Here $\phi_a' = -k_a \in H^\infty(\mathbb{D})$ and observe that the inducing function of the weighted composition operator belongs to $Aut(\mathbb{D})$ and the weight function belongs to $H^\infty(\mathbb{D})$. Now consider the weighted composition operator $W_{\psi,q}$ on $L_a^2(dA_\alpha)$ where $q \in L_a^2(dA_\alpha)$ and $\psi \in Aut(\mathbb{D})$. If $W_{\psi,q} L_a^2(dA_\alpha) \subset L_a^2(dA_\alpha)$ then what will be the relation between q and ψ. In this section we have shown that $q \in H^\infty(\mathbb{D})$ if and only if ψ is a finite Blaschke product. More specifically, we established the following. We showed that if $\mathcal{M}$ is a subspace of $L^\infty(\mathbb{D})$ and if for $\phi \in \mathcal{M}$, the Toeplitz operator $T_\phi^{(\alpha)}$ represents a multiplication operator on a closed subspace $\mathcal{S} \subset L_a^2(dA_\alpha)$, then ϕ is bounded analytic on $\mathbb{D}$. Similarly if $q \in L^\infty(\mathbb{D})$ and $\mathcal{B}_n$ is a finite Blaschke product and $M_q^{(\alpha)}\left(Range\ C_{\mathcal{B}_n}^{(\alpha)}\right) \subset L_a^2(dA_\alpha)$, then $q \in H^\infty(\mathbb{D})$. Further, we have shown that if $\psi \in Aut(\mathbb{D})$ and $q \in L_a^2(dA_\alpha)$, then $\mathcal{N} = \left\{q \in L_a^2(dA_\alpha) : M_q^{(\alpha)}\left(Range\ C_\psi^{(\alpha)}\right) \subset L_a^2(dA_\alpha)\right\} = H^\infty(\mathbb{D})$ if and only if ψ is a finite Blaschke product. Akeroyd and Ghatage (2008,[2]) showed that if ϕ is univalent, analytic self-map of the disk, then C_ϕ has closed range on the Bergman space $L_a^2(\mathbb{D})$ if and only if ϕ is a conformal automorphism of the disk.

Theorem 4.1. *(i) Let $\mathcal{M}$ be a subspace of $L^\infty(\mathbb{D})$ such that for $\phi \in \mathcal{M}$, there exists a closed subspace $\mathcal{S}$ of $L_a^2(dA_\alpha)$ for which $T_\phi^{(\alpha)} f = \phi f$, for all $f \in \mathcal{S}$. Then $\mathcal{M} \subset H^\infty(\mathbb{D})$.*

(ii) Let $q \in L^\infty(\mathbb{D})$ and $\mathcal{B}_n$ is a finite Blaschke product as defined in (1). If $M_q^{(\alpha)}\left(Range\ C_{\mathcal{B}_n}^{(\alpha)}\right) \subset L_a^2(dA_\alpha)$, then $q \in H^\infty(\mathbb{D})$.

Proof. (i) Suppose $T_\phi^{(\alpha)} f = \phi f,\ f \in \mathcal{S} \subset L_a^2(dA_\alpha)$. Then $\phi(z) = \frac{T_\phi^{(\alpha)} f(z)}{f(z)}$. Hence ϕ is analytic on $\mathbb{D} \setminus \{zeros\ of\ f\}$. Thus each isolated singularity of ϕ in $\mathbb{D}$ is removable since ϕ is assumed to be bounded. Thus ϕ is analytic on $\mathbb{D}$. Since $\phi \in L^\infty(\mathbb{D})$, hence $\phi \in H^\infty(\mathbb{D})$.

(ii) Since $M_q^{(\alpha)}\left(C_{\mathcal{B}_n}^{(\alpha)} L_a^2(dA_\alpha)\right) \subset L_a^2(dA_\alpha)$, hence $M_q^{(\alpha)} C_{\mathcal{B}_n}^{(\alpha)}$ is bounded (see [3],[24]). Let $f \in L_a^2(dA_\alpha)$. Then

$$\begin{aligned}\left\langle \left(C_{\mathcal{B}_n}^{(\alpha)}\right)^* M_{\overline{q}}^{(\alpha)} K^{(\alpha)}(\cdot, z), f\right\rangle &= \left\langle K^{(\alpha)}(\cdot, z), M_q^{(\alpha)} C_{\mathcal{B}_n}^{(\alpha)} f\right\rangle = \overline{q(z)}\ \overline{f(\mathcal{B}_n(z))} \\ &= \overline{q(z)} \left\langle K^{(\alpha)}(\cdot, \mathcal{B}_n(z)), f\right\rangle.\end{aligned}$$

Hence $\left(C_{\mathcal{B}_n}^{(\alpha)}\right)^* M_{\overline{q}}^{(\alpha)} K^{(\alpha)}(\cdot, z) = \overline{q(z)} K^{(\alpha)}(\cdot, \mathcal{B}_n(z))$. Since $M_q^{(\alpha)} C_{\mathcal{B}_n}^{(\alpha)}$ is bounded, so is $\left(C_{\mathcal{B}_n}^{(\alpha)}\right)^* M_{\overline{q}}^{(\alpha)}$ as $\left(M_q^{(\alpha)}\right)^* = M_{\overline{q}}^{(\alpha)}$ (for details see [24]). Thus there exists $R > 0$ such that $\left\|\left(C_{\mathcal{B}_n}^{(\alpha)}\right)^* M_{\overline{q}}^{(\alpha)} K^{(\alpha)}(\cdot, z)\right\|_2 \leq R\|K^{(\alpha)}(\cdot, z)\|_2$. Hence $|q(z)|\ \|K^{(\alpha)}(\cdot, \mathcal{B}_n(z))\|_2 \leq R\|K^{(\alpha)}(\cdot, z)\|_2$ and we obtain from Lemma 2.4 that

$$|q(z)| \frac{1}{(1 - |\mathcal{B}_n(z)|^2)^{1+\frac{\alpha}{2}}} \leq R \frac{1}{(1 - |z|^2)^{1+\frac{\alpha}{2}}} \quad .$$

That is,

$$|q(z)| \leq R \left(\frac{1 - |\mathcal{B}_n(z)|^2}{1 - |z|^2}\right)^{1+\frac{\alpha}{2}} \quad .$$

Let $l = \max_{1 \leq i \leq n}\{|\alpha_i|\}$ and $p = \min_{1 \leq i \leq n}\{|\alpha_i|\}$. It follows from [8] that for $l < |z| < 1$, we have

$$\frac{1 - |\mathcal{B}_n(z)|^2}{1 - |z|^2} \leq m + 2n \frac{1 + p}{1 - p}.$$

Hence $q \in H^\infty(\mathbb{D})$. □

Theorem 4.2. *Let $\psi \in Aut(\mathbb{D})$ and $\mathcal{N} = \left\{q \in L_a^2(dA_\alpha) : M_q^{(\alpha)}(Range\ C_\psi^{(\alpha)}) \subset L_a^2(dA_\alpha)\right\}$. If $\mathcal{N} = H^\infty(\mathbb{D})$, then there exist constants $L > 0$ and $R > 0$ such that*

$$L\ \|M_q^{(\alpha)} C_\psi^{(\alpha)}\| \leq \|q\|_\infty \leq R\ \|M_q^{(\alpha)} C_\psi^{(\alpha)}\|\ .$$

Proof. The set $\mathcal{N}$ is a vector space. Define for $q \in \mathcal{N}$, the norm $\|q\|_{\mathcal{N}} := \|M_q^{(\alpha)} C_\psi^{(\alpha)}\|$. The space $\mathcal{N}$ is complete with respect to the metric induced from $\|\cdot\|_{\mathcal{N}}$. Let Ξ_n be a sequence in $\mathcal{N}$ which is Cauchy. Then $M_{\Xi_n}^{(\alpha)} C_\psi^{(\alpha)}$ is a Cauchy sequence in $\mathcal{L}(L_a^2(dA_\alpha))$. Since the space $\mathcal{L}(L_a^2(dA_\alpha))$ is complete, hence there exists $S \in \mathcal{L}(L_a^2(dA_\alpha))$ such that $\lim_{n\to\infty} M_{\Xi_n}^{(\alpha)} C_\psi^{(\alpha)} = S$. For $f \in L_a^2(dA_\alpha)$, $\lim_{n\to\infty} M_{\Xi_n}^{(\alpha)} C_\psi^{(\alpha)} f = Sf$. That is, $\lim_{n\to\infty} \Xi_n(f \circ \psi) = Sf$ and for $z \in \mathbb{D}$, $\lim_{n\to\infty} \Xi_n(z) f(\psi(z)) = (Sf)(z)$. For $f = 1$, we obtain $\lim_{n\to\infty} \Xi_n = S1$. Let $q = S1$. Then for $q \in L_a^2(dA_\alpha)$, $z \in \mathbb{D}$, we have $\lim_{n\to\infty} \Xi_n(z) f(\psi(z)) = q(z) f(\psi(z))$. Hence we get $(Sf)(z) = q(z) f(\psi(z))$. It follows therefore that $S = M_q^{(\alpha)} C_\psi^{(\alpha)}$ and $q \in \mathcal{N}$ and $\lim_{n\to\infty} \|\Xi_n - q\|_{\mathcal{N}} = \lim_{n\to\infty} \|M_{\Xi_n}^{(\alpha)} C_\psi^{(\alpha)} - M_q^{(\alpha)} C_\psi^{(\alpha)}\| = 0$ and $\mathcal{N}$ is complete with respect to the metric induced from the norm $\|\cdot\|_{\mathcal{N}}$. Since $\mathcal{N} = H^\infty(\mathbb{D})$, we obtain by inverse mapping theorem [20] that there exist constants $L > 0$ and $R > 0$ such that $L\|q\|_{\mathcal{N}} \leq \|q\|_\infty \leq R\|q\|_{\mathcal{N}}$. Thus $L\|M_q^{(\alpha)} C_\psi^{(\alpha)}\| \leq \|q\|_\infty \leq R\|M_q^{(\alpha)} C_\psi^{(\alpha)}\|$. The theorem follows. □

Theorem 4.3. *Let $\psi \in Aut(\mathbb{D})$ and $q \in L_a^2(dA_\alpha)$. Then*

$$\mathcal{N} = \left\{q \in L_a^2(dA_\alpha) : M_q^{(\alpha)}\left(Range\ C_\psi^{(\alpha)}\right) \subset L_a^2(dA_\alpha)\right\} = H^\infty(\mathbb{D})$$

if and only if ψ is a finite Blaschke product.

Proof. The sufficiency part follows from Theorem 4.1. For the necessary part, define for $z, w \in \mathbb{D}$, the function $K_w^{(\alpha)}(z) = \left(\frac{1}{1-z\overline{w}}\right)^{\alpha+2}$. Then for any $f \in L_a^2(dA_\alpha)$, it follows from Lemma 2.4 that

$$\begin{aligned}
\left\|M_{K_w^{(\alpha)}}^{(\alpha)} C_\psi^{(\alpha)} f\right\|_2^2 &= \int_{\mathbb{D}} |K_w^{(\alpha)}(z)|^2 |f(\psi(z))|^2 dA_\alpha(z) \\
&= \int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^{2(\alpha+2)}} |f(\psi(z))|^2 dA_\alpha(z) \\
&= \frac{1}{(1-|w|^2)^{\alpha+2}} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+2}}{|1-z\overline{w}|^{2(\alpha+2)}} |f(\psi(z))|^2 dA_\alpha(z) \\
&= \frac{1}{(1-|w|^2)^{\alpha+2}} \int_{\mathbb{D}} |f(\psi(z))|^2 |k_z^{1+\frac{\alpha}{2}}|^2 dA_\alpha(z) \\
&= \frac{1}{(1-|w|^2)^{\alpha+2}} \int_{\mathbb{D}} |f((\psi \circ \phi_w)(z))|^2 dA_\alpha(z) \\
&\leq \frac{1}{(1-|w|^2)^{\alpha+2}} \left(\frac{1+|\psi(w)|}{1-|\psi(w)|}\right)^{\alpha+2} \|f\|_2^2.
\end{aligned}$$

The last inequality follows from [16]. So

$$\left\|M_{K_w^{(\alpha)}}^{(\alpha)} C_\psi^{(\alpha)}\right\| \leq \frac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} \left(\frac{1+|\psi(w)|}{1-|\psi(w)|}\right)^{1+\frac{\alpha}{2}}.$$

From Theorem 4.2, it follows that there exists a constant $R' > 0$ such that

$$\|K_w^{(\alpha)}\|_\infty \leq R' \frac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} \left(\frac{1+|\psi(w)|}{1-|\psi(w)|}\right)^{1+\frac{\alpha}{2}}.$$

Since $\|K_w^{(\alpha)}\|_\infty = \left(\frac{1}{1-|w|}\right)^{\alpha+2}$, we obtain $\left(\frac{1}{1-|w|}\right)^{\alpha+2} \leq R' \frac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} \left(\frac{1+|\psi(w)|}{1-|\psi(w)|}\right)^{1+\frac{\alpha}{2}}$. That is,

$$\left(\frac{1+|w|}{1-|w|}\right)^{1+\frac{\alpha}{2}} \leq R' \left(\frac{1+|\psi(w)|}{1-|\psi(w)|}\right)^{1+\frac{\alpha}{2}} \leq R' \left(\frac{2}{1-|\psi(w)|}\right)^{1+\frac{\alpha}{2}}.$$

Thus when $|w| \to 1$, then $|\psi(w)| \to 1$ and the function ψ is a finite Blaschke product. □

5. Conclusion

(i) In this work, we only dealt with the weights $(1-|z|^2)^\alpha dA(z)$, $z \in \mathbb{D}, \alpha > -1$ which is a Mӧbius invariant. Whether such result holds for other weights like (i) $\frac{1}{\Gamma(\alpha+1)}\left(log\frac{1}{|z|^2}\right)^\alpha, \alpha > -1$ (ii) $exp\left(\frac{-c}{(1-|z|)^\alpha}\right)$, $\alpha, c > 0$ (iii) $exp\left(-\gamma\, exp\left(\frac{\beta}{(1-|z|)^\alpha}\right)\right) \alpha, \beta, \gamma > 0$ defined on $\mathbb{D}$ and in the weighted Bergman spaces $L_a^2(\Omega)$ where Ω is any bounded symmetric domain in $\mathbb{C}$?

(ii) De Leeuw [17] showed that the isometries in the Hardy space $H^1(\mathbb{D})$ are weighted composition operators and Forelli [15] obtained the same result for the Hardy spaces $H^p, 1 < p < \infty, p \neq 2$. Further, it is well-known [17] that if T is any Banach space isometry of $H^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{D})$, then T has the form $(Tf)(\lambda) = \alpha f(\tau(\lambda))$, $f \in H^\infty(\mathbb{D})$ and where α is a complex constant of modulus 1 and τ is a conformal map of the open unit disk onto itself. Bourdon and Narayan [5] gave a characterization of the unitary weighted composition operators on $H^2(\mathbb{D})$ in 2010. They showed that if the weighted composition operator $W_{\phi,\psi}$ from $H^2(\mathbb{D})$ into itself is unitary, then $\phi \in Aut(\mathbb{D})$. Further in 2014, Matache [21] proved that if $W_{\phi,\psi}$ is isometric on $H^2(\mathbb{D})$ then ϕ must be an inner function and ψ must belong to $H^2(\mathbb{D})$ and $\|\psi\| = 1$. In this context it is also important to analyse what are all the isometries from $L_a^p(dA_\alpha), 1 \leq p < \infty$ into itself ?

(iii) In section 2, we have seen that the map $U_a^\alpha = (f \circ \phi_a)k_a^{1+\frac{\alpha}{2}},\ a \in \mathbb{D}$ is bounded, unitary and self-adjoint. Notice that, $\phi_a' = -k_a$. That is, if the inducing function of the composition operator is ϕ_a then the weight function is $k_a^{1+\frac{\alpha}{2}}$ and the resulting operator is unitary. In section 4, we have shown that if the inducing function of the composition operator is a finite Blaschke product if and only if the weight function belong to $H^\infty(\mathbb{D})$. Now we ask if the inducing function is an infinite Blaschke product or an inner function then to which class the weight function ψ must belong to, so that $W_{\phi,\psi}$ will be bounded and unitary.

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