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The intersection problem for kite-GDDs

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Abstract. In this paper the intersection problem for a pair of kite-GDDs of type 4^u is investigated. The intersection problem for kite-GDDs is the determination of all pairs (T, s) such that there exists a pair of kite-GDDs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ of the same type T and $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. Let $J(u) = \{s : \exists$ a pair of kite-GDDs of type 4^u intersecting in s blocks$\}$; $I(u) = \{0, 1, \ldots, b_u - 2, b_u\}$, where $b_u = 2u(u-1)$ is the number of blocks of a kite-GDD of type 4^u. We show that for any positive integer $u \geq 3$, $J(u) = I(u)$.

1. Introduction

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a partition of a finite set X into subsets (called *holes*), where $|H_i| = n_i$ for $1 \leq i \leq m$. Let $K_{n_1,n_2,\ldots,n_m}$ be the complete multipartite graph on X with the i-th part on H_i, and G be a subgraph of $K_{n_1,n_2,\ldots,n_m}$. A *holey G-design* is a triple $(X, \mathcal{H}, \mathcal{B})$ such that $(X, \mathcal{B})$ is a $(K_{n_1,n_2,\ldots,n_m}, G)$-design. The *hole type* (or *type*) of the holey G-design is $\{n_1, n_2, \ldots, n_m\}$. We use an "exponential" notation to describe hole types: the hole type $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ denotes u_i occurrences of g_i for $1 \leq i \leq r$. Obviously if G is the complete graph K_k, a holey K_k-design is just a k-GDD. A holey K_k-design with the hole type 1^v is called a *Steiner system* $S(2, k, v)$. If G is the graph with vertices a, b, c, d and edges ab, ac, bc, cd (such a graph is called a kite) a holey G-design is said to be a *kite-GDD*.

A pair of holey G-designs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ of the same type is said to *intersect in s blocks* if $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. The intersection problem for $S(2, k, v)$'s was first introduced by Kramer and Mesner in [12]. The intersection problem for $S(2, 4, v)$'s was dealt with by Colbourn et al. [10], apart from three undecided values for $v = 25$, 28 and 37. Chang et al. has completely solved the triangle intersection problem for $S(2, 4, v)$ designs and a pair of disjoint $S(2, 4, v)$s [7, 8]. Butler and Hoffman [2] completely solved the intersection problem for 3-GDDs of type g^u. Zhang, Chang and Feng solved the intersection problem for 4-GDDs of type 3^u [16] and the intersection problem for 4-GDDs of type 4^u [17]. The intersection problem is also considered for many other types of combinatorial structures. The interested reader may refer to [1, 3–6, 9, 13–15]

In this paper we focus on the intersection problem for kite-GDDs. Let $J(u) = \{s : \exists$ a pair of kite-GDD of type 4^u intersecting in s blocks$\}$. Throughout this paper we always assume that $I(u) = \{0, 1, \ldots, b_u - 2, b_u\}$ for $u \geq 3$, where $b_u = 2u(u-1)$ is the number of blocks of a kite-GDD of type 4^u.

As the main result of the present paper, we are to prove the following theorem.

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Theorem 1.1. *$J(u) = I(u)$ for any integer $u \geq 3$.*

Obviously $J(u) \subseteq I(u)$. We need to show that $I(u) \subseteq J(u)$.

2. Basic design constructions

Construction 2.1. ([4])(Weighting Construction) *Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, and let $\omega : X \longmapsto Z^+ \cup \{0\}$ be a weight function. For every block $A \in \mathcal{A}$, suppose that there is a pair of holey G-designs of type $\{\omega(x) : x \in A\}$, which intersect in b_A blocks. Then there exists a pair of holey G-designs of type $\{\sum_{x \in H} \omega(x) : H \in \mathcal{G}\}$, which intersect in $\sum_{A \in \mathcal{A}} b_A$ blocks.*

Construction 2.2. (Filling Construction) *Let m be nonnegative integers and $g_i, a \equiv 0 \pmod m$ for $1 \leq i \leq s$. Suppose that there exists a pair of holey G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in b blocks. If there is a pair of holey G-designs of type $m^{g_i/m}a^1$, which intersect in b_i blocks for $1 \leq i \leq s-1$, and there is a pair of holey G-designs of type $m^{(g_s+a)/m}$ which intersect in b_s blocks, then there exists a pair of holey G-designs of type $m^{(\sum_{i=1}^{s} g_i + a)/m}$ intersecting in $b + \sum_{i=1}^{s} b_i$ blocks.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be two holey G-designs of type $\{g_1, g_2, \ldots, g_s\}$ satisfying $|\mathcal{A} \cap \mathcal{B}| = b$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ with $|G_i| = g_i, 1 \leq i \leq s$ and Y be any given set of length a such that $X \cap Y = \emptyset$. For $1 \leq i \leq s-1$, construct a pair of holey G-designs $(G_i \cup Y, \mathcal{G}_i \cup \{Y\}, \mathcal{C}_i)$ and $(G_i \cup Y, \mathcal{G}_i \cup \{Y\}, \mathcal{D}_i)$ of type $m^{g_i/m}a^1$ satisfying $|\mathcal{C}_i \cap \mathcal{D}_i| = b_i$ and construct a pair of holey G-designs $(G_s \cup Y, \mathcal{G}_s, \mathcal{C}_s)$ and $(G_s \cup Y, \mathcal{G}_s, \mathcal{D}_s)$ of type $m^{(g_s+a)/m}$ satisfying $|\mathcal{C}_s \cap \mathcal{D}_s| = b_s$. Then $(X \cup Y, (\bigcup_{i=1}^{s} \mathcal{G}_i) \cup \{Y\}, \mathcal{A} \cup (\bigcup_{i=1}^{s} \mathcal{C}_i))$ and $(X \cup Y, (\bigcup_{i=1}^{s} \mathcal{G}_i) \cup \{Y\}, \mathcal{B} \cup (\bigcup_{i=1}^{s} \mathcal{D}_i))$ are two holey G-designs of type $m^{(\sum_{i=1}^{s} g_i + a)/m}$. Obviously, the two holey G-designs have $b + \sum_{i=1}^{s} b_i$ common blocks.
□

We quote the following result for later use.

Lemma 2.3. [11]

(1) *A* 4-GDD of type g^u exists if and only if $u \geq 4$, $(u-1)g \equiv 0 \pmod 3$, and $u(u-1)g^2 \equiv 0 \pmod{12}$, with the exception of $(g, u) \in \{(2, 4), (6, 4)\}$.

(2) A 3-GDD of type g^u exists if and only if $u \geq 3$, $(u-1)g \equiv 0 \pmod 2$, and $u(u-1)g^2 \equiv 0 \pmod 6$.

Lemma 2.4. [18] *There is a pair of kite-GDD of type 2^4 intersecting in s blocks, then $s \in \{0, \ldots, 4, 6\}$.*

3. Ingredients

Lemma 3.1. *$J(3) = I(3)$.*

Proof. Take the vertex set $X = \{0, 1, \ldots, 11\}$ and $\mathcal{G} = \{\{0, 1, 2, 3\}, \{4, 5, 10, 11\}, \{6, 7, 8, 9\}\}$. Let $\mathcal{B}_1 = \{[9, 3, 10-7], [8, 2, 10-6], [2, 4, 6-3], [6, 5, 1-10], [11, 7, 1-8], [0, 6, 11-8], [4, 8, 3-11], [5, 8, 0-10], [1, 4, 9-5], [7, 4, 0-9], [3, 7, 5-2], [9, 11, 2-7]\}$. $\mathcal{B}_2 = (\mathcal{B}_1 \setminus \{[9, 3, 10-7], [8, 2, 10-6]\}) \cup \{[9, 3, 10-6], [8, 2, 10-7]\}$, $\mathcal{B}_3 = (\mathcal{B}_1 \setminus \{[9, 3, 10-7], [8, 2, 10-6], [2, 4, 6-3]\}) \cup \{[9, 10, 3-6], [8, 2, 10-7], [2, 4, 6-10]\}$, $\mathcal{B}_4 = (\mathcal{B}_2 \setminus \{[6, 5, 1-10], [11, 7, 1-8]\}) \cup \{[6, 5, 1-8], [11, 7, 1-10]\}$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus \{[6, 5, 1-10], [11, 7, 1-8]\}) \cup \{[6, 5, 1-8], [11, 7, 1-10]\}$. Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 4^3 for $i = 1, 2, 3, 4, 5$. Consider the following permutations on X.

$\pi_0 = (2\ 3)(4\ 11\ 5)(6\ 8\ 9\ 7)$, $\pi_1 = (0\ 1\ 2\ 3)(4\ 11)(6\ 7)(8\ 9)$, $\pi_2 = (0\ 3)(1\ 2)(4\ 5)(6\ 9\ 7)(10\ 11)$,
$\pi_3 = (6\ 8)(10\ 11)$, $\pi_4 = (0\ 2)(1\ 3)(4\ 5)(6\ 8)(10\ 11)$, $\pi_5 = (4\ 5)$,
$\pi_6 = (5\ 10)$, $\pi_7 = \pi_8 = \pi_9 = \pi_{10} = \pi_{12} = (1)$.

We have that for each $s \in I(3) \setminus \{7, 8, 9, 10\}$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. For each $s \in \{7, 8, 9, 10\}$, $|\pi_s \mathcal{B}_{12-s} \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. □

Lemma 3.2. *$J(4) = I(4)$.*

Proof. Take the vertex set $X = \{0, 1, \ldots, 15\}$ and $\mathcal{G} = \{\{0, 1, 2, 15\}, \{3, 4, 13, 14\}, \{5, 6, 11, 12\}, \{7, 8, 9, 10\}\}$. Let $\mathcal{B}_1 = [14, 15, 7-3], [6, 0, 7-2], [5, 13, 7-11], [4, 1, 7-12], [10, 4, 11-13], [2, 3, 11-14], [9, 1, 11-0], [4, 5, 15-3], [13, 6, 15-8], [12, 14, 8-11], [6, 1, 8-4], [12, 3, 10-0], [2, 12, 13-8], [0, 5, 14-1], [4, 2, 6-10], [5, 2, 10-14], [3, 6, 9-4], [9, 12, 15-11], [13, 0, 9-5], [1, 13, 10-15], [9, 2, 14-6], [0, 3, 8-2], [1, 3, 5-8], [0, 4, 12-1]$.

Table 1. The blocks of kite-GDD of type 4^4

i	A_i	C_i
1	[14,15,7-3],[6,0,7-2]	[14,15,7-2],[6,0,7-3]
2	[14,15,7-3],[6,0,7-2],[5,13,7-11]	[14,15,7-11],[6,0,7-3],[5,13,7-2]
3	[10,4,11-13],[2,3,11-14]	[10,4,11-14],[2,3,11-13]
4	[4,5,15-3],[13,6,15-8]	[4,5,15-8],[13,6,15-3]
5	[12,14,8-11],[6,1,8-4]	[12,14,8-4],[6,1,8-11]

Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 4^4 for $i = 1, 2, \ldots, 8$, where $\mathcal{B}_2 = (\mathcal{B}_1 \setminus A_1) \cup C_1$, $\mathcal{B}_3 = (\mathcal{B}_1 \setminus A_2) \cup C_2$, $\mathcal{B}_4 = (\mathcal{B}_2 \setminus A_3) \cup C_3$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus A_3) \cup C_3$, $\mathcal{B}_6 = (\mathcal{B}_4 \setminus A_4) \cup C_4$, $\mathcal{B}_7 = (\mathcal{B}_5 \setminus A_4) \cup C_4$, $\mathcal{B}_8 = (\mathcal{B}_6 \setminus A_5) \cup C_5$. Consider the following permutations on X.

$\pi_0 = (2\,15)(3\,14\,4)(5\,11\,12\,6)(7\,8\,10)$, $\pi_1 = (0\,15)(1\,2)(3\,13\,14\,4)(5\,12\,6)(7\,10\,8\,9)$,
$\pi_3 = (1\,15)(4\,14)(5\,11\,6)$, $\pi_2 = (3\,14\,4)(5\,11\,12)(8\,9)$,
$\pi_4 = (2\,15)(6\,12)(8\,10)$, $\pi_5 = (3\,13)(5\,12)$,
$\pi_8 = (1\,15)(7\,9)$, $\pi_6 = (3\,14)(11\,12)$,
$\pi_7 = (1\,15)(8\,9)$, $\pi_{12} = (2\,15)$,
$\pi_{14} = (7\,8)$, $\pi_{13} = (8\,10)$,
$\pi_{11} = (3\,13)$, $\pi_9 = (5\,11\,12)$,
$\pi_{10} = (1\,2)$, $\pi_{15} = (7\,9)$,
$\pi_{16} = \pi_{17} = \pi_{18} = \pi_{19} = (1)$ $\pi_{20} = \pi_{21} = \pi_{22} = \pi_{24} = (1)$.

We have that for each $s \in I(4) \setminus \{16, \ldots, 22\}$, $|\pi_s\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. For each $s \in \{16, \ldots, 22\}$, $|\pi_s\mathcal{B}_{24-s} \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. □

Lemma 3.3. $J(5) = I(5)$.

Proof. Take the vertex set $X = \{0, 1, \ldots, 19\}$ and $\mathcal{G} = \{\{0, 1, 2, 3\}, \{4, 5, 18, 19\}, \{6, 7, 16, 17\}, \{8, 9, 14, 15\}, \{10, 11, 12, 13\}\}$. Let

$\mathcal{B}_1$:	$[0, 19, 10-6]$,	$[9, 1, 10-5]$,	$[2, 4, 10-14]$,	$[16, 18, 10-17]$,	$[5, 0, 7-1]$,
	$[14, 11, 7-10]$,	$[9, 4, 7-2]$,	$[17, 18, 8-2]$,	$[6, 1, 8-5]$,	$[11, 16, 8-13]$,
	$[0, 4, 8-12]$,	$[5, 16, 15-1]$,	$[17, 4, 15-2]$,	$[13, 4, 14-18]$,	$[8, 19, 7-3]$,
	$[5, 6, 14-19]$,	$[3, 4, 12-6]$,	$[0, 18, 9-6]$,	$[17, 19, 9-3]$,	$[16, 3, 14-1]$,
	$[13, 15, 6-2]$,	$[14, 12, 2-16]$,	$[9, 2, 11-5]$,	$[8, 3, 10-15]$,	$[6, 4, 11-0]$,
	$[3, 5, 13-18]$,	$[1, 19, 11-18]$,	$[19, 13, 2-18]$,	$[1, 12, 18-6]$,	$[16, 19, 12-7]$,
	$[16, 0, 13-9]$,	$[12, 0, 15-19]$,	$[12, 5, 9-16]$,	$[3, 19, 6-0]$,	$[2, 5, 17-3]$,
	$[0, 14, 17-12]$,	$[15, 7, 18-3]$,	$[1, 17, 13-7]$,	$[15, 3, 11-17]$,	$[4, 16, 1-5]$.

Table 1. The blocks of kite-GDD of type 4^5

i	A_i	C_i
1	[0,19,10-6],[9,1,10-5]	[0,19,10-5],[9,1,10-6]
2	[5,0,7-1],[14,11,7-10],[9,4,7-2]	[5,0,7-2],[14,11,7-1],[9,4,7-10]
3	[2,4,10-14],[16,18,10-17]	[2,4,10-17],[16,18,10-14]
4	[17,18,8-2],[6,1,8-5]	[17,18,8-5],[6,1,8-2]
5	[11,16,8-13],[0,4,8-12]	[11,16,8-12],[0,4,8-13]
6	[5,16,15-1],[17,4,15-2]	[5,16,15-2],[17,4,15-1]

Then $(X, \mathcal{G}, \mathcal{B}_i)$ is a kite-GDD of type 4^5 for $i = 1, 2, \ldots, 10$, where $\mathcal{B}_2 = (\mathcal{B}_1 \setminus A_1) \cup C_1$, $\mathcal{B}_3 = (\mathcal{B}_1 \setminus A_2) \cup C_2$, $\mathcal{B}_4 = (\mathcal{B}_2 \setminus A_3) \cup C_3$, $\mathcal{B}_5 = (\mathcal{B}_3 \setminus A_3) \cup C_3$, $\mathcal{B}_6 = (\mathcal{B}_4 \setminus A_4) \cup C_4$, $\mathcal{B}_7 = (\mathcal{B}_5 \setminus A_4) \cup C_4$, $\mathcal{B}_8 = (\mathcal{B}_6 \setminus A_5) \cup C_5$, $\mathcal{B}_9 = (\mathcal{B}_7 \setminus A_5) \cup C_5$, $\mathcal{B}_{10} = (\mathcal{B}_8 \setminus A_6) \cup C_6$ Consider the following permutations on X.

$\pi_0 = (2\,3)(4\,19\,5)(6\,16\,17\,7)(8\,9\,15)(10\,12)(11\,13)$, $\pi_1 = (0\,1\,3\,2)(4\,18\,19\,5)(7\,17)(8\,14)(10\,12\,11\,13)$,
$\pi_2 = (0\,2\,1)(4\,5)(6\,17\,16\,7)(9\,14)(11\,13)(18\,19)$, $\pi_3 = (4\,18\,5)(6\,16\,17)(11\,13)$,
$\pi_4 = (0\,1\,3)(7\,16)(8\,15)(11\,12)$, $\pi_5 = (5\,18)(7\,16)(8\,9\,14)(10\,12)$,
$\pi_6 = (0\,3)(6\,17)(8\,15)(10\,12)$, $\pi_7 = (4\,18\,19)(8\,9)(10\,13)$,
$\pi_8 = (7\,16\,17)(9\,15)(11\,13)$, $\pi_9 = (8\,15)(10\,12\,11\,13)$,
$\pi_{10} = (8\,15\,14)(10\,12\,13)$, $\pi_{11} = (0\,2)(7\,16)$,
$\pi_{12} = (4\,19)(7\,16)$, $\pi_{13} = (9\,14)(10\,12\,13)$,
$\pi_{14} = (7\,17)(8\,15\,9)$, $\pi_{15} = (4\,5)(9\,15)$,
$\pi_{16} = (4\,19)(10\,11)$, $\pi_{17} = (7\,16)(10\,11)$,
$\pi_{18} = (9\,14)(10\,13)$, $\pi_{19} = (8\,14)(10\,12)$,
$\pi_{20} = (11\,13\,12)$, $\pi_{21} = (10\,12\,11)$,
$\pi_{22} = (0\,2)$, $\pi_{23} = (4\,19)$,
$\pi_{24} = (18\,19)$, $\pi_{25} = (7\,16)$,
$\pi_{26} = (7\,17)$, $\pi_{27} = (8\,14)$,
$\pi_{28} = (8\,15)$, $\pi_{29} = (10\,11)$,
$\pi_{30} = \pi_{31} = \pi_{32} = \pi_{33} = \pi_{34} = (1)$, $\pi_{35} = \pi_{36} = \pi_{37} = \pi_{38} = \pi_{40} = (1)$.

We have that for each $s \in I(5) \setminus \{30, \ldots, 38\}$, $|\pi_s\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. For each $s \in \{30, \ldots, 38\}$, $|\pi_s\mathcal{B}_{40-s} \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. $\square$

4. Input designs

For counting $J(u)$ for $6 \le u \le 14$, we may search for a large number of instances of kite-GDDs. However, to reduce the computation, when $6 \le u \le 14$, we shall first determine the intersection numbers of a pair of kite-GDDs of type $a^m b^1$ with the same group set.

Lemma 4.1. *Let $M_1 = \{0, 1, \ldots, 26, 36\}$ and $s \in M_1$. Then there is a pair of kite-GDDs of type $4^3 8^1$ with the same group set, which intersect in s blocks.*

Proof. Take the vertex set $X = \{0, 1, \ldots, 19\}$ and the group set $\mathcal{G} = \{\{8, 9, 18, 19\}, \{10, 11, 16, 17\}, \{12, 13, 14, 15\}, \{0, 1, \ldots, 7\}\}$. Let

$\mathcal{B}_1$:, $[19, 10, 0-16]$, $[17, 18, 7-19]$, $[15, 16, 6-19]$, $[1, 11, 12-7]$, $[9, 10, 2-17]$,
$[8, 16, 7-14]$, $[0, 18, 11-7]$, $[17, 19, 5-16]$, $[16, 4, 18-3]$, $[15, 17, 3-19]$,
$[16, 2, 14-0]$, $[11, 2, 13-17]$, $[12, 10, 3-8]$, $[9, 4, 11-6]$, $[10, 14, 8-0]$,
$[9, 15, 7-13]$, $[8, 6, 13-18]$, $[1, 19, 13-0]$, $[1, 18, 10-7]$, $[17, 9, 0-15]$,
$[18, 5, 15-1]$, $[17, 4, 14-5]$, $[16, 3, 13-4]$, $[14, 19, 11-5]$, $[10, 5, 13-9]$,
$[12, 9, 6-17]$, $[8, 15, 11-3]$, $[19, 16, 12-5]$, $[1, 17, 8-5]$, $[2, 18, 12-0]$,
$[19, 15, 2-8]$, $[4, 8, 12-17]$, $[15, 10, 4-19]$, $[18, 14, 6-10]$, $[1, 16, 9-5]$,
$[3, 9, 14-1]$.

Then $(X, \mathcal{G}, \mathcal{B}_1)$ is a kite-GDD of type $4^3 8^1$. Consider the following permutations on X.

$\pi_0 = (0\ 3)(1\ 2)(8\ 9)(10\ 17\ 11)(18\ 19)$, $\pi_1 = (0\ 1\ 2\ 3)(8\ 18\ 19)(10\ 16\ 11\ 17)$,
$\pi_2 = (2\ 3)(8\ 19\ 9)(10\ 16\ 17\ 11)$, $\pi_3 = (1\ 2)(8\ 9\ 18)(10\ 17\ 11)$,
$\pi_4 = (0\ 1)(2\ 3)(8\ 9)(11\ 17)$, $\pi_5 = (0\ 3)(1\ 2)(10\ 16)(18\ 19)$,
$\pi_6 = (0\ 3)(8\ 9\ 19)$, $\pi_7 = (8\ 19\ 18)(11\ 16)$,
$\pi_8 = (0\ 2)(9\ 18\ 19)$, $\pi_9 = (2\ 3)(8\ 18\ 9)$,
$\pi_{10} = (0\ 3\ 1\ 2)(10\ 11)$, $\pi_{11} = (8\ 19)(11\ 16)$,
$\pi_{12} = (0\ 3\ 1)(10\ 11)$, $\pi_{13} = (0\ 3\ 2)(10\ 11)$,
$\pi_{14} = (0\ 1\ 2)(10\ 11)$, $\pi_{15} = (0\ 1\ 2\ 3)$,
$\pi_{16} = (0\ 1)(10\ 11)$, $\pi_{17} = (0\ 2)(10\ 11)$,
$\pi_{18} = (16\ 17)$, $\pi_{19} = (1\ 3\ 2)$,
$\pi_{20} = (0\ 3\ 2)$, $\pi_{21} = (11\ 16)$,
$\pi_{22} = (10\ 11)$, $\pi_{23} = (0\ 3)$,
$\pi_{24} = (1\ 2)$, $\pi_{25} = (2\ 3)$,
$\pi_{26} = (0\ 2)$, $\pi_{36} = (1)$.

We have that for each $s \in M_1$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. □

Lemma 4.2. *Let $M_2 = \{0, 1, \ldots, 35, 48\}$ and $s \in M_2$. Then there is a pair of kite-GDDs of type $4^3 12^1$ with the same group set, which intersect in s blocks.*

Proof. Take the vertex set $X = \{0, 1, \ldots, 23\}$ and the group set $\mathcal{G} = \{\{12, 13, 22, 23\}, \{14, 15, 20, 21\}, \{16, 17, 18, 19\}, \{0, 1, \ldots, 11\}\}$. Let

$\mathcal{B}_1$:

[0, 14, 23 − 5],	[22, 11, 21 − 8],	[20, 10, 19 − 0],	[1, 16, 15 − 11],	[13, 2, 14 − 8],
[11, 20, 12 − 8],	[23, 9, 21 − 6],	[22, 8, 20 − 0],	[19, 7, 21 − 3],	[18, 6, 20 − 1],
[17, 3, 15 − 10],	[16, 5, 14 − 6],	[13, 0, 15 − 9],	[12, 1, 14 − 4],	[11, 19, 13 − 10],
[10, 21, 12 − 7],	[1, 23, 17 − 5],	[0, 16, 22 − 7],	[0, 21, 17 − 6],	[23, 2, 20 − 5],
[22, 1, 19 − 6],	[2, 18, 21 − 5],	[20, 4, 17 − 9],	[18, 4, 15 − 8],	[16, 7, 13 − 6],
[15, 6, 12 − 5],	[14, 11, 18 − 8],	[12, 9, 16 − 8],	[14, 7, 17 − 11],	[15, 5, 19 − 8],
[20, 3, 16 − 2],	[17, 10, 22 − 5],	[22, 9, 18 − 7],	[19, 3, 23 − 11],	[1, 21, 13 − 4],
[2, 15, 22 − 6],	[3, 14, 22 − 4],	[4, 21, 16 − 11],	[5, 18, 13 − 3],	[6, 23, 16 − 10],
[7, 15, 23 − 4],	[17, 13, 8 − 23],	[9, 19, 14 − 10],	[9, 13, 20 − 7],	[12, 0, 18 − 3],
[4, 19, 12 − 3],	[23, 10, 18 − 1],	[17, 12, 2 − 19].		

Then $(X, \mathcal{G}, \mathcal{B}_1)$ is a kite-GDD of type $4^3 12^1$. Consider the following permutations on X.

$\pi_0 = (12\,13\,22\,23)(14\,20\,21)(16\,18\,17\,19)$, $\pi_1 = (13\,22)(14\,20)(15\,21)(16\,18\,19)$,
$\pi_2 = (14\,21\,15)(16\,18\,19\,17)(22\,23)$, $\pi_3 = (12\,22\,23)(15\,21\,20)(17\,18\,19)$,
$\pi_4 = (12\,23)(13\,22)(16\,18)(20\,21)$, $\pi_5 = (12\,23\,22)(17\,19\,18)(20\,21)$,
$\pi_6 = (14\,20)(16\,19\,18)(22\,23)$, $\pi_7 = (13\,23)(15\,21\,20)(18\,19)$,
$\pi_8 = (2\,7\,9\,5\,10)(8\,11)$, $\pi_9 = (0\,10\,11\,2\,5)(8\,9)$,
$\pi_{10} = (0\,7\,11\,10\,8\,9\,2)$, $\pi_{11} = (0\,8)(5\,10\,11\,7)$,
$\pi_{12} = (2\,8)(5\,11)(9\,10)$, $\pi_{13} = (2\,10\,8\,11)(7\,9)$,
$\pi_{14} = (0\,7\,8\,2\,9\,11)$, $\pi_{15} = (0\,9\,7)(8\,10\,11)$,
$\pi_{16} = (0\,8\,11\,9\,5)$, $\pi_{17} = (2\,10\,11)(5\,8)$,
$\pi_{18} = (0\,5\,2\,8\,11)$, $\pi_{19} = (0\,11)(5\,9)$,
$\pi_{20} = (0\,8\,5\,10)$, $\pi_{21} = (2\,11\,5\,10)$,
$\pi_{22} = (0\,2\,5\,7)$, $\pi_{23} = (0\,2)(8\,9)$,
$\pi_{24} = (0\,9\,7\,2)$, $\pi_{25} = (5\,9\,8)$,
$\pi_{26} = (0\,8\,10)$, $\pi_{27} = (0\,10\,5)$,
$\pi_{28} = (2\,9\,10)$, $\pi_{29} = (0\,2\,5)$,
$\pi_{30} = (0\,9\,7)$, $\pi_{31} = (5\,11)$,
$\pi_{32} = (5\,10)$, $\pi_{33} = (9\,11)$,
$\pi_{34} = (2\,7)$, $\pi_{35} = (2\,9)$,
$\pi_{48} = (1)$.

We have that for each $s \in M_2$, $|\pi_s \mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s \mathcal{G} = \mathcal{G}$. □

Lemma 4.3. *Let $M_3 = \{0, 1, \ldots, 53, 64\}$ and $s \in M_3$. Then there is a pair of kite-GDDs of type $8^2 12^1$ with the same group set, which intersect in s blocks.*

Proof. Take the vertex set $X = \{0, 1, \ldots, 27\}$ and the group set $\mathcal{G} = \{\{0, \ldots, 11\}, \{12, \ldots, 19\}, \{20, \ldots, 27\}\}$. Let

$\mathcal{B}_1$:	$[0,27,14-5]$,	$[0,26,13-4]$,	$[0,25,12-6]$,	$[24,12,1-26]$,	$[27,15,1-25]$
	$[2,14,23-11]$,	$[19,22,3-26]$,	$[3,21,18-5]$,	$[2,22,18-1]$,	$[23,19,4-27]$,
	$[26,17,4-25]$,	$[25,16,5-26]$,	$[17,24,5-23]$,	$[6,23,15-5]$,	$[6,22,16-1]$,
	$[12,27,7-21]$,	$[13,25,7-15]$,	$[12,21,8-23]$,	$[13,24,8-22]$,	$[16,21,9-17]$,
	$[15,22,9-12]$,	$[14,24,10-23]$,	$[13,20,11-21]$,	$[14,26,11-22]$,	$[27,19,5-21]$,
	$[26,6,18-0]$,	$[25,17,8-16]$,	$[24,16,2-17]$,	$[25,15,10-21]$,	$[23,18,7-24]$,
	$[22,17,10-19]$,	$[21,19,2-13]$,	$[20,18,4-24]$,	$[20,17,6-27]$,	$[21,17,0-15]$,
	$[24,0,19-6]$,	$[16,7,20-0]$,	$[20,15,2-25]$,	$[20,14,3-25]$,	$[23,17,1-22]$,
	$[24,18,9-27]$,	$[23,16,0-22]$,	$[21,15,4-16]$,	$[25,19,9-26]$,	$[25,18,11-17]$,
	$[22,14,7-17]$,	$[26,7,19-1]$,	$[21,14,1-20]$,	$[22,13,5-20]$,	$[27,18,8-14]$,
	$[24,15,11-19]$,	$[16,27,11-12]$,	$[19,8,20-9]$,	$[13,23,9-14]$,	$[10,27,13-3]$,
	$[26,16,10-18]$,	$[21,6,13-1]$,	$[26,12,2-27]$,	$[8,26,15-3]$,	$[10,20,12-5]$,
	$[25,14,6-24]$,	$[12,22,4-14]$,	$[27,17,3-24]$,	$[12,23,3-16]$.	

Then $(X,\mathcal{G},\mathcal{B}_1)$ is a kite-GDD of type $8^2 12^1$. Consider the following permutations on X.

$$\pi_0 = (0\ 10\ 8\ 4)(2\ 9\ 6\ 7\ 3)(5\ 11)(13\ 16\ 14)(18\ 19)(20\ 22\ 25)(24\ 27)$$
$$\pi_1 = (0\ 2\ 3)(1\ 6\ 10\ 9\ 5)(4\ 11)(12\ 15\ 14\ 19\ 13\ 18\ 16\ 17)(20\ 22\ 24\ 26\ 23\ 21\ 27)$$
$$\pi_2 = (0\ 11\ 5\ 3\ 1\ 7\ 2)(4\ 10)(12\ 17\ 13\ 14\ 19\ 16)(15\ 18)(20\ 25\ 22\ 21)(23\ 26\ 27)$$
$$\pi_3 = (0\ 10\ 11\ 9\ 7\ 5\ 3\ 8)(1\ 4)(2\ 6)(12\ 14\ 13\ 19\ 17)(15\ 16\ 18)(20\ 22\ 23\ 27\ 24\ 21\ 26\ 25)$$
$$\pi_4 = (0\ 7\ 10\ 3\ 11\ 6)(1\ 5\ 8\ 2\ 4)(12\ 13\ 15\ 16\ 14\ 17\ 18\ 19)(20\ 22\ 27\ 26)(21\ 23\ 24)$$
$$\pi_5 = (1\ 11\ 4\ 3\ 7\ 2\ 9)(14\ 16\ 17)(20\ 21)$$
$$\pi_6 = (0\ 2\ 9\ 10)(1\ 3\ 6)(8\ 11)(12\ 16\ 14)(13\ 19\ 18\ 15\ 17)(20\ 21\ 23)(22\ 27\ 24)$$

$\pi_7 = (2\ 4\ 3\ 5)(12\ 15\ 14\ 13)(24\ 25\ 27\ 26)$	$\pi_8 = (4\ 5)(12\ 13\ 14\ 15)(24\ 26)(25\ 27)$
$\pi_9 = (2\ 3\ 4\ 5)(12\ 14\ 15)(24\ 26\ 25\ 27)$,	$\pi_{10} = (2\ 5)(3\ 4)(12\ 13)(14\ 15)(24\ 27\ 25)$
$\pi_{11} = (4\ 5)(12\ 15\ 13)(24\ 26\ 27\ 25)$	$\pi_{12} = (2\ 5)(13\ 15)(24\ 25\ 26\ 27)$
$\pi_{13} = (2\ 5\ 3\ 4)(13\ 14\ 15)(26\ 27)$	$\pi_{14} = (2\ 5\ 4)(12\ 15\ 13\ 14)(24\ 27)$
$\pi_{15} = (2\ 3\ 5)(12\ 13)(24\ 25\ 27)$	$\pi_{16} = (2\ 5)(3\ 4)(14\ 15)(24\ 26)$
$\pi_{17} = (2\ 3\ 4\ 5)(12\ 14\ 13\ 15)$	$\pi_{18} = (3\ 5\ 4)(12\ 15\ 13)(24\ 27)$
$\pi_{19} = (2\ 5\ 4\ 3)(24\ 25\ 26)$	$\pi_{20} = (3\ 5)(24\ 26\ 27\ 25)$
$\pi_{21} = (2\ 5)(14\ 15)(24\ 25)$	$\pi_{22} = (2\ 4)(12\ 15)(24\ 26)$
$\pi_{23} = (2\ 5)(12\ 14)(24\ 27)$	$\pi_{24} = (3\ 5\ 4)(12\ 15)(25\ 27)$
$\pi_{25} = (12\ 15\ 14)(25\ 26)$	$\pi_{26} = (2\ 3\ 5\ 4)(26\ 27)$
$\pi_{27} = (2\ 5\ 3)(24\ 25)$	$\pi_{28} = (12\ 15)(13\ 14)$
$\pi_{29} = (2\ 5)(12\ 13\ 15)$	$\pi_{30} = (2\ 4)(13\ 14\ 15)$
$\pi_{31} = (3\ 4\ 5)(13\ 14)$	$\pi_{32} = (12\ 15)(25\ 26)$
$\pi_{33} = (4\ 5)(14\ 15)$	$\pi_{34} = (2\ 5)(12\ 14)$
$\pi_{35} = (3\ 5)(12\ 15)$	$\pi_{36} = (2\ 3)(12\ 13)$
$\pi_{37} = (3\ 4)(13\ 15)$	$\pi_{38} = (2\ 3\ 4\ 5)$
$\pi_{39} = (3\ 4\ 9\ 7)$	$\pi_{40} = (22\ 25)$
$\pi_{41} = (26\ 27)$	$\pi_{42} = (24\ 27)$
$\pi_{43} = (25\ 26)$	$\pi_{44} = (2\ 5\ 4)$
$\pi_{45} = (2\ 4\ 3)$	$\pi_{46} = (13\ 15)$
$\pi_{47} = (7\ 9\ 8)$	$\pi_{48} = (0\ 1)$
$\pi_{49} = (0\ 3)$	$\pi_{50} = (0\ 2)$
$\pi_{51} = (1\ 2)$	$\pi_{52} = (4\ 7)$
$\pi_{53} = (8\ 9)$	$\pi_{64} = (1)$.

We have that for each $s \in M_3$, $|\pi_s\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $\pi_s\mathcal{G} = \mathcal{G}$. □

5. For $6 \leq u \leq 14$

Lemma 5.1. *$J(6) = I(6)$.*

Proof. Take the same set M_2 as in Lemma 4.2. Let $\alpha \in M_2$. Then there is a pair of kite-GDDs of type $4^3 12^1$ $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ with the same group set, which intersect in α blocks. Here the subgraph K_{12} is constructed on $Y \subset X$. Let $\beta \in I(3)$, By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 $(Y, \mathcal{B}'_1)$ and $(Y, \mathcal{B}'_2)$ intersecting in β common blocks. Then $(X, \mathcal{B}_1 \cup \mathcal{B}'_1)$ and $(X, \mathcal{B}_2 \cup \mathcal{B}'_2)$ are a pair of kite-GDDs of type 4^6 with $\alpha + \beta$ common blocks. Thus we have

$$J(6) \supseteq \{\alpha + \beta : \alpha \in M_2, \beta \in I(3)\} = M_2 + I(3) = I(6).$$

□

Lemma 5.2. $J(8) = I(8)$.

Proof. Take the same set M_3 as in Lemma 4.3. Let $\alpha \in M_3$. Then there is a pair of kite-GDDs of type $8^2 12^1$ with the same group set, which intersect in α blocks. Let $\gamma_1, \gamma_2 \in I(3)$. By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 intersecting in γ_i common blocks for each $i = 1, 2$. Let $\gamma_3 \in I(4)$. By Lemma 3.2, there is a pair of kite-GDDs of type 4^4 with γ_3 common blocks. Now applying Construction 2.2, we obtain a pair of kite-GDDs of type 4^8 with $\alpha + \sum_{i=1}^{3} \gamma_i$ common blocks. Thus we have

$$J(8) \supseteq \{\alpha + \sum_{i=1}^{3} \gamma_i : \alpha \in M_3, \gamma_1, \gamma_2 \in I(3), \gamma_3 \in I(4)\} = I(8).$$

□

Lemma 5.3. $J(u) = I(u)$ *for* $u = 7, 10, 13$.

Proof. Start from a 4-GDD of type 2^u, $u = 7, 10, 13$, by Lemma 2.3. Give each point of the GDD weight 2. By Lemma 2.4, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in \{0, \ldots, 4, 6\}$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 4^u with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = u(u-1)/3$ is the number of blocks of the 4-GDD of type 2^u and $\alpha_i \in \{0, \ldots, 4, 6\}$ for $1 \le i \le b$. Which implies, for $u = 7, 10, 13$

$$J(u) \supseteq \{\sum_{i=1}^{b} \alpha_i : \alpha_i \in \{0, \ldots, 4, 6\}, 1 \le i \le b\} = b * \{0, \ldots, 4, 6\} = I(u).$$

□

Lemma 5.4. $J(u) = I(u)$ *for* $u = 9, 11$.

Proof. Start from a 3-GDD of type 3^3 by Lemma 2.3. Give each point of the GDD weight 4. By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 with α common blocks, $\alpha \in I(3)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^3 with $\sum_{i=1}^{9} \alpha_i$ common blocks, where $b = 9$ is the number of blocks of the 3-GDD of type 3^3 and $\alpha_i \in I(3)$ for $1 \le i \le 9$.

Let $u = 9$. By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 with β_j common blocks, where $\beta_j \in I(3)$, $1 \le j \le 3$. By Construction 2.2, we have a pair of kite-GDDs of type 4^9 with $\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{3} \beta_j$ common blocks, which implies

$$\begin{aligned} J(9) &\supseteq \{\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{3} \beta_j : \alpha_i \in I(3), \beta_j \in I(3), 1 \le i \le 9, 1 \le j \le 3\} \\ &= 9 * \{0, \ldots, 10, 12\} + 3 * \{0, \ldots, 10, 12\} = I(9). \end{aligned}$$

Let $u = 11$. By Lemma 4.1, there is a pair of kite-GDDs of type $4^3 8^1$ with β_j common blocks, where $\beta_j \in M_1$, $1 \le j \le 2$. By Lemma 3.3, there is a pair of kite-GDDs of type 4^5 with γ common blocks. By

Construction 2.2, we have a pair of kite-GDDs of type 4^{11} with $\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{2}\beta_j+\gamma$ common blocks, which implies

$$J(11)\supseteq\{\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{2}\beta_j+\gamma:\alpha_i\in I(3),\beta_j\in M_1,\gamma\in I(5),1\le i\le 9,1\le j\le 2\}$$
$$=9*\{0,\dots,10,12\}+2*\{0,\dots,24,36\}+\{0,\dots,38,40\}=I(11).$$

□

Lemma 5.5. *$J(u)=I(u)$ for $u=12,14$.*

Proof. Start from a 4-GDD of type 3^4 by Lemma 2.3. Give each point of the GDD weight 4. By Lemma 3.2, there is a pair of kite-GDDs of type 4^4 with α common blocks, $\alpha\in I(4)$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^4 with $\sum_{i=1}^{9}\alpha_i$ common blocks, where $b=9$ is the number of blocks of the 4-GDD of type 3^4 and $\alpha_i\in I(4)$ for $1\le i\le 9$.

Let $u=12$. By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 with β_j common blocks, where $\beta_j\in I(3)$, $1\le j\le 4$. By Construction 2.2, we have a pair of kite-GDDs of type 4^{12} with $\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{4}\beta_j$ common blocks, which implies

$$J(12)\supseteq\{\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{4}\beta_j:\alpha_i\in I(4),\beta_j\in I(3),1\le i\le 9,1\le j\le 4\}$$
$$=9*\{0,\dots,22,24\}+4*\{0,\dots,10,12\}=I(12).$$

Let $u=14$. By Lemma 4.1, there is a pair of kite-GDDs of type $4^3 8^1$ with β_j common blocks, where $\beta_j\in M_1$, $1\le j\le 3$. By Lemma 3.3, there is a pair of kite-GDDs of type 4^5 with γ common blocks, where $\gamma\in I(5)$. By Construction 2.2, we have a pair of kite-GDDs of type 4^{14} with $\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{3}\beta_j+\gamma$ common blocks, which implies

$$J(14)\supseteq\{\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{3}\beta_j+\gamma:\alpha_i\in I(4),\beta_j\in M_1,\gamma\in I(5),1\le i\le 9,1\le j\le 3\}$$
$$=9*\{0,\dots,22,24\}+3*\{0,\dots,24,36\}+\{0,\dots,38,40\}=I(14).$$

□

6. Proof of Theorem 1.1

First we need the following definition. Let s_1 and s_2 be two non-negative integers. If X and Y are two sets of pairs of non-negative integers, then $X+Y$ denotes the set $\{s_1+s_2:s_1\in X,s_2\in Y\}$. If X is a set of pairs of non-negative integers and h is some positive integer, then $h*X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X allowed).

Lemma 6.1. *For any integer $u\equiv 0 \pmod 3$ and $u\ge 15$, $J(u)=I(u)$.*

Proof. Let $u=3t$ and $t\ge 5$. Start from a 4-GDD of type 6^t by Lemma 2.3. Give each point of the GDD weight 2. By Lemma 2.4, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha\in\{0,\dots,4,6\}$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^t with $\sum_{i=1}^{b}\alpha_i$ common blocks, where $b=3t(t-1)$ is the number of blocks of the 4-GDD of type 6^t and $\alpha_i\in\{0,\dots,4,6\}$ for $1\le i\le b$.

By Lemma 3.1, there is a pair of kite-GDDs of type 4^3 with β_j common blocks, where $\beta_j \in I(3)$, $1 \le j \le t$. By Construction 2.2, we have a pair of kite-GDDs of type 4^{3t} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t) &\supseteq \{\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j : \alpha_i \in \{0, \ldots, 4, 6\}, \beta_j \in I(3), 1 \le i \le b, 1 \le j \le t\} \\ &= b * \{0, \ldots, 4, 6\} + t * \{0, \ldots, 10, 12\} \\ &= I(3t) = I(u). \end{aligned}$$

□

Lemma 6.2. *For any integer $u \equiv 1 \pmod 3$ and $u \ge 16$, $J(u) = I(u)$.*

Proof. Let $u = 3t + 1$ and $t \ge 5$. Start from a 4-GDD of type 6^t by Lemma 2.3. Give each point of the GDD weight 2. By Lemma 2.4, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in \{0, \ldots, 4, 6\}$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^t with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = 3t(t-1)$ is the number of blocks of the 4-GDD of type 6^t and $\alpha_i \in \{0, \ldots, 4, 6\}$ for $1 \le i \le b$.

By Lemma 3.2, there is a pair of kite-GDDs of type 4^4 with β_j common blocks, where $\beta_j \in I(4)$, $1 \le j \le t$. By Construction 2.2, we have a pair of kite-GDDs of type 4^{3t+1} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t+1) &\supseteq \{\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t} \beta_j : \alpha_i \in \{0, \ldots, 4, 6\}, \beta_j \in I(4), 1 \le i \le b, 1 \le j \le t\} \\ &= b * \{0, \ldots, 4, 6\} + t * \{0, \ldots, 22, 24\} \\ &= I(3t+1) = I(u). \end{aligned}$$

□

Lemma 6.3. *For any integer $u \equiv 2 \pmod 3$ and $u \ge 17$, $J(u) = I(u)$.*

Proof. Let $u = 3t + 2$ and $t \ge 5$. Start from a 4-GDD of type 6^t by Lemma 2.3. Give each point of the GDD weight 2. By Lemma 2.4, there is a pair of kite-GDDs of type 2^4 with α common blocks, $\alpha \in \{0, \ldots, 4, 6\}$. Then apply Construction 2.1 to obtain a pair of kite-GDDs of type 12^t with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = 3t(t-1)$ is the number of blocks of the 4-GDD of type 6^t and $\alpha_i \in \{0, \ldots, 4, 6\}$ for $1 \le i \le b$.

By Lemma 4.1, there is a pair of kite-GDDs of type $4^3 8^1$ with β_j common blocks, where $\beta_j \in M_1$, $1 \le j \le t-1$. By Lemma 3.3, there is a pair of kite-GDDs of type 4^5 with γ common blocks, where $\gamma \in I_5$. By Construction 2.2, we have a pair of kite-GDDs of type 4^{3t+2} with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-1} \beta_j + \gamma$ common blocks, which implies

$$\begin{aligned} J(u) = J(3t+2) &\supseteq \{\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{t-1} \beta_j + \gamma : \alpha_i \in \{0, \ldots, 4, 6\}, \beta_j \in M_1, \gamma \in I_5, 1 \le i \le b, 1 \le j \le t\} \\ &= b * \{0, \ldots, 4, 6\} + (t-1) * \{0, \ldots, 24, 36\} + \{0, \ldots, 38, 40\} \\ &= I(3t+2) = I(u). \end{aligned}$$

Proof of Theorem 1.1: When $u \in \{3, 4, \ldots, 14\}$, the conclusion follows from Lemmas 3.1-3.3, and Lemmas 5.1-5.5. When $u \ge 15$, combining the results of Lemmas 6.1-6.3, we complete the proof. □

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