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New integral inequalities for Atangana-Baleanu fractional integral operators and various comparisons via simulations

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Abstract. Integral identities created in inequality theory studies help to prove many inequalities. Recently, different fractional integral and derivative operators have been used to achieve these identities. In this article, with the help of Atangana-Baleanu integral operators, an integral identity was first obtained and various integral inequalities for convex functions have been proved using this identity. In the last part of the article, various simulation graphs are given to reveal the consistency of Atangana-Baleanu fractional integral operators and Riemann-Liouville fractional integral operators for different α values. The prominent motivating idea in this work is to obtain new and general form integral inequalities with the help of fractional integral operators with strong kernel structure.

1. Introduction

Convex functions, the definition of which is expressed as an inequality, distinguish it from other function classes with its applications in many areas of inequality theory, convex programming, statistics, numerical analysis and mathematics, as well as providing important features such as continuity and limitation in domain of the function. This interesting class of functions is defined as follow.

Definition 1.1. *The function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Convex functions are a concept widely used in inequality theory. The Hermite-Hadamard inequality, which produces upper and lower bounds based on the averages of the mean value of a convex function, is given as follows.

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Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave. Some recent studies related to convexity and Hadamard type inequalities, see the papers [11–13, 16, 17]. In [20], Sarikaya et. al. gave a different perspective to the inequality (2) by using the Riemann-Liouville fractional integral operators. This study played a key role in generalizing, expanding and obtaining variations of classical integral inequalities with the help of fractional integrals. On the other hand, by defining different versions of Riemann-Liouville fractional integral operators in the last decade, new versions and generalizations of inequalities on both convex functions and differentiable functions have been obtained (see the paper [22]). Studies in the field of fractional analysis have brought a new perspective and orientation to many fields of applied sciences and mathematics in addition to the theory of inequality in recent years. It has shed light on many real world problems with the applications of newly defined fractional integral and derivative operators. In these new operators, several important criteria have differentiated them and have made some advantageous in applications compared to others. Exponentially or power law expressions used in the kernel of fractional operators revealed their features such as locality and singularity, and it became important to obtain the initial conditions for the special versions of the parameters used in the definition. Another important detail is to reveal the spaces where the operators are defined and to show the suitability for the real world problems. For more results related to different kinds of fractional operators, we suggest to the interested readers the papers [1, 3, 4, 6, 8–10, 14, 15, 18, 19, 21, 23, 24]. Also, several new findings for discrete versions, different kinds of convex functions and different kinds of fractional integral operators have been provided in the papers [25–33].
In the sequel of this paper, we will denote the normalization function with $B(\alpha)$. The definition of Atangana-Baleanu fractional operators will be given as follows:

Definition 1.2. *[5] Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1)$ then, the definition of the new fractional derivative is given:*

$${}^{ABC}_{\ \ a}D^{\alpha}_{t}\left[f(t)\right] = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{(1-\alpha)}\right]dx. \tag{3}$$

Definition 1.3. *[5] Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1)$ then, the definition of the new fractional derivative is given:*

$${}^{ABR}_{\ \ a}D^{\alpha}_{t}\left[f(t)\right] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{(1-\alpha)}\right]dx. \tag{4}$$

Equations (3) and (4) have a non-local kernel. Also in equation (3) when the function is constant we get zero.

With the help of Laplace transform and convolution theorem, Atangana-Baleanu described the fractional integral operator as follows.

Definition 1.4. *[5] The fractional integral associate to the new fractional derivative with non-local kernel of a function $f \in H^1(a,b)$ as defined:*

$${}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy$$

where $b > a, \alpha \in (0,1]$.

In [2], Abdeljawad and Baleanu introduced right hand side of integral operator as following. The right fractional new integral with ML kernel of order $\alpha \in (0,1]$ is defined by

$$\left({}^{AB}I_b^{\alpha}\right)\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.$$

The main purpose of this article is to obtain an integral identity that includes the Atangana-Baleanu integral operators and to demonstrate Hermite-Hadamard type integral inequalities for differentiable convex functions with the help of this identity. Several simulations have been considered to show the effectiveness of the fractional integral operators.

2. Main Results

We will start with a new integral identity that will be used the proofs of our main findings as following:

Lemma 2.1. *$f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
& {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \\
= \; & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha} f'(kt+(1-k)a)\,dk - \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha} f'(kb+(1-k)t)\,dk
\end{aligned}$$

where $\alpha \in (0,1]$, $t \in [a,b]$, $k \in [0,1]$ and $\Gamma(.)$ is Gamma function.

Proof. *By using integration by parts for the right hand side of equation, we have*

$$\begin{aligned}
& \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha} f'(kt+(1-k)a)\,dk + \frac{1-\alpha}{B(\alpha)}f(t) + \frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} \\
= \; & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left.\frac{(1-k)^{\alpha}f(kt+(1-k)a)}{(t-a)}\right|_0^1 + \alpha\int_0^1 \frac{(1-k)^{\alpha-1}f(kt+(1-k)a)}{(t-a)}dk\right] \\
& + \frac{1-\alpha}{B(\alpha)}f(t) + \frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} \\
= \; & -\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\frac{f(a)}{(t-a)} + \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\frac{\alpha}{(t-a)}\int_0^1 (1-k)^{\alpha-1}f(kt+(1-k)a)\,dk \\
& + \frac{1-\alpha}{B(\alpha)}f(t) + \frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} \\
= \; & -\frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} + \frac{\alpha(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)(t-a)^{\alpha+1}}\int_a^t (t-s)^{\alpha-1}f(s)ds \\
& + \frac{1-\alpha}{B(\alpha)}f(t) + \frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} \\
= \; & \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds + \frac{1-\alpha}{B(\alpha)}f(t) \\
= \; & {}^{AB}_{a}I^{\alpha}\{f(t)\}.
\end{aligned}$$

Then, we can write following identity

$$\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha} f'(kt+(1-k)a)\,dk + \frac{1-\alpha}{B(\alpha)}f(t) + \frac{(t-a)^{\alpha}f(a)}{B(\alpha)\Gamma(\alpha)} = {}^{AB}_{a}I^{\alpha}\{f(t)\}. \tag{5}$$

Similarly

$$\begin{aligned}
&\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^\alpha f'(kb+(1-k)t)\,dk-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}\\
=\;&\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{k^\alpha f(kb+(1-k)t)}{(b-t)}\Big|_0^1-\int_0^1\frac{f(kb+(1-k)t)}{b-t}\,\alpha k^{\alpha-1}dk\right]\\
&-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}\\
=\;&\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{f(b)}{b-t}-\int_t^b\frac{f(s)}{(b-t)^2}\alpha\frac{(s-t)^{\alpha-1}}{(b-t)^{\alpha-1}}ds\right]\\
&-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}\\
=\;&\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{\alpha(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)(b-t)^{\alpha+1}}\int_t^b f(s)(s-t)^{\alpha-1}ds\\
&-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}\\
=\;&\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(s)(s-t)^{\alpha-1}ds-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}\\
=\;&-{}^{AB}I_b^\alpha\{f(t)\}.
\end{aligned}$$

It is easy to see that;

$$\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^\alpha f'(kb+(1-k)t)\,dk-\frac{(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{(1-\alpha)f(t)}{B(\alpha)}=-{}^{AB}I_b^\alpha\{f(t)\}. \tag{6}$$

By adding identities (5) *and* (6), *we obtain*

$$\begin{aligned}
&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^\alpha f'(kt+(1-k)a)\,dk-\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^\alpha f'(kb+(1-k)t)\,dk\\
&+\frac{(t-a)^\alpha f(a)+(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}+\frac{2(1-\alpha)f(t)}{B(\alpha)}\\
=\;&{}_a^{AB}I^\alpha\{f(t)\}+{}^{AB}I_b^\alpha\{f(t)\}.
\end{aligned}$$

So, the proof is completed. □

Theorem 2.2. *$f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left|{}_a^{AB}I^\alpha\{f(t)\}+{}^{AB}I_b^\alpha\{f(t)\}-\frac{(t-a)^\alpha f(a)+(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\le\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{|f'(t)|}{(\alpha+1)(\alpha+2)}+\frac{|f'(a)|}{(\alpha+2)}\right]\\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{|f'(b)|}{(\alpha+2)}+\frac{|f'(t)|}{(\alpha+1)(\alpha+2)}\right]
\end{aligned}$$

where $t\in[a,b]$, $\alpha\in(0,1]$, and $B(\alpha)>0$ is normalization function.

Proof. *By using the identity that is given in Lemma 2.1, we can write*

$$\begin{aligned}
&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\} +^{AB} I^{\alpha}_{b}\left\{f(t)\right\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \\
= &\left|\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha} f'\left(kt+(1-k)a\right)dk - \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha} f'\left(kb+(1-k)t\right)dk\right| \\
\leq &\ \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left|f'\left(kt+(1-k)a\right)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'\left(kb+(1-k)t\right)\right|dk.
\end{aligned}$$

By using convexity of $|f'|$, *we get*

$$\begin{aligned}
&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\} +^{AB} I^{\alpha}_{b}\left\{f(t)\right\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \\
\leq &\ \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left|f'\left(kt+(1-k)a\right)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'\left(kb+(1-k)t\right)\right|dk \\
\leq &\ \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\Big[k\left|f'(t)\right| + (1-k)\left|f'(a)\right|\Big]dk \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\Big[k\left|f'(b)\right| + (1-k)\left|f'(t)\right|\Big]dk.
\end{aligned}$$

By computing the above integral, we obtain

$$\begin{aligned}
&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\} +^{AB} I^{\alpha}_{b}\left\{f(t)\right\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \\
\leq &\ \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\Big[k\left|f'(t)\right| + (1-k)\left|f'(a)\right|\Big]dk \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\Big[k\left|f'(b)\right| + (1-k)\left|f'(t)\right|\Big]dk \\
= &\ \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(t)\right| + \frac{1}{(\alpha+2)}\left|f'(a)\right|\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{(\alpha+2)}\left|f'(b)\right| + \frac{1}{(\alpha+1)(\alpha+2)}\left|f'(t)\right|\right]
\end{aligned}$$

and the proof is completed. □

Corollary 2.3. *In Theorem 2.2, if we choose* $t = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\left|{}^{AB}_{\ a}I^{\alpha} f\left(\frac{a+b}{2}\right) +^{AB} I^{\alpha}_{b} f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[f(a)+f(b)\Big] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right| \\
\leq &\ \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[2\frac{\left|f'(\frac{a+b}{2})\right|}{(\alpha+1)(\alpha+2)} + \frac{\left|f'(a)\right| + \left|f'(b)\right|}{(\alpha+2)}\right].
\end{aligned}$$

Remark 2.4. *Setting* $\alpha = 1$ *in Theorem 2.2 gives the same result as in [13], Theorem 4.*

Theorem 2.5. *$f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\leq\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(t)\right|^{q}+\left|f'(a)\right|^{q}}{2}\right)^{\frac{1}{q}}\\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(b)\right|^{q}+\left|f'(t)\right|^{q}}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $q>1$, and $B(\alpha)>0$ is normalization function.

Proof. *By Lemma 2.1, we have*

$$\begin{aligned}
&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\leq\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.
\end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned}
&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\leq\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(kt+(1-k)a)\right|^{q}dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(kb+(1-k)t)\right|^{q}dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By using convexity of $|f'|^q$, we obtain

$$\begin{aligned}
\int_0^1\left|f'(kt+(1-k)a)\right|^{q}dk&\leq\int_0^1\left[k\left|f'(t)\right|^{q}+(1-k)\left|f'(a)\right|^{q}\right]dk\\
\int_0^1\left|f'(kb+(1-k)t)\right|^{q}dk&\leq\int_0^1\left[k\left|f'(b)\right|^{q}+(1-k)\left|f'(t)\right|^{q}\right]dk.
\end{aligned}$$

By calculating the integrals that is in the above inequalities, we get desired result. □

Corollary 2.6. *In Theorem 2.5, if we choose $t=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left|{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\big[f(a)+f(b)\big]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|\\
\leq\ &\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(\frac{a+b}{2})\right|^{q}+\left|f'(a)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^{q}+\left|f'(\frac{a+b}{2})\right|^{q}}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 2.7. *Setting $\alpha=1$ in Theorem 2.5 gives the same result as in [13], Theorem 5.*

Theorem 2.8. *$f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \le\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left|f'(t)\right|^{q}+\left|f'(a)\right|^{q}}{2q}\right)\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left|f'(b)\right|^{q}+\left|f'(t)\right|^{q}}{2q}\right)\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $q>1$, and $B(\alpha)>0$ is normalization function.

Proof. By Lemma 2.1, we have

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \le\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.\end{aligned}$$

By using the Young inequality as $xy\le\frac{1}{p}x^p+\frac{1}{q}y^q$

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \le\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f'(kt+(1-k)a)\right|^{q}dk\right]\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1k^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f'(kb+(1-k)t)\right|^{q}dk\right].\end{aligned}$$

□

By a simple computation, we have the result.

Corollary 2.9. *In Theorem 2.8, if we choose $t=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\big[f(a)+f(b)\big]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|\\ \le\ &\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{2}{p(\alpha p+1)}+\frac{2\left|f'(\frac{a+b}{2})\right|^{q}+\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2q}\right).\end{aligned}$$

Theorem 2.10. *$f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|^q$ is a convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \le\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^{q}}{(\alpha+1)(\alpha+2)}+\frac{\left|f'(a)\right|^{q}}{(\alpha+2)}\right)^{\frac{1}{q}}\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^{q}}{(\alpha+2)}+\frac{\left|f'(t)\right|^{q}}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\end{aligned}$$

where $t \in [a,b]$, $\alpha \in (0,1]$, $q \geq 1$, and $B(\alpha) > 0$ is normalization function.

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By using convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha}\left[k\left|f'(t)\right|^q+(1-k)\left|f'(a)\right|^q\right]dk\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha}\left[k\left|f'(b)\right|^q+(1-k)\left|f'(t)\right|^q\right]dk\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^q}{(\alpha+1)(\alpha+2)}+\frac{\left|f'(a)\right|^q}{(\alpha+2)}\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q}{(\alpha+2)}+\frac{\left|f'(t)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

Corollary 2.11. *In Theorem 2.10, if we choose $t = \frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right) + {}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(a)+f(b)\right] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right| \\
\leq\ & \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|f'(\frac{a+b}{2})\right|^q}{(\alpha+1)(\alpha+2)}+\frac{\left|f'(a)\right|^q}{(\alpha+2)}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{\left|f'(b)\right|^q}{(\alpha+2)}+\frac{\left|f'(\frac{a+b}{2})\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 2.12. *Setting $\alpha = 1$ in Theorem 2.10 gives the same result as in [13], Theorem 7.*

Theorem 2.13. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|$ is a concave, then we have*

$$\begin{aligned}&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\}+{}^{AB}I^{\alpha}_{b}\left\{f(t)\right\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{1}{\alpha+2}t+\frac{\alpha+1}{\alpha+2}a\right)\right|\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{\alpha+1}{\alpha+2}b+\frac{1}{\alpha+2}t\right)\right|\end{aligned}$$

where $t\in[a,b]$, $\alpha\in(0,1]$ and $B(\alpha)>0$ is normalization function.

Proof. From Lemma 2.1 and the Jensen integral inequality, we have

$$\begin{aligned}&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\}+{}^{AB}I^{\alpha}_{b}\left\{f(t)\right\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left|f'(kb+(1-k)t)\right|dk\\ \leq\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1(1-k)^{\alpha}dk\right)\left|f'\left(\frac{\int_0^1(1-k)^{\alpha}(kt+(1-k)a)\,dk}{\int_0^1(1-k)^{\alpha}dk}\right)\right|\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1k^{\alpha}dk\right)\left|f'\left(\frac{\int_0^1k^{\alpha}(kb+(1-k)t)\,dk}{\int_0^1k^{\alpha}dk}\right)\right|.\end{aligned}$$

By computing the above integrals we have

$$\begin{aligned}&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\}+{}^{AB}I^{\alpha}_{b}\left\{f(t)\right\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{1}{\alpha+2}t+\frac{\alpha+1}{\alpha+2}a\right)\right|\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{\alpha+1}{\alpha+2}b+\frac{1}{\alpha+2}t\right)\right|.\end{aligned}$$

□

Corollary 2.14. *In Theorem 2.13, if we choose $t=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}&\left|{}^{AB}_{\ a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(a)+f(b)\right]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|\\ \leq\;&\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)\left[\left|f'\left(\frac{a+b}{2(\alpha+2)}+\frac{\alpha+1}{\alpha+2}a\right)\right|+\left|f'\left(\frac{\alpha+1}{\alpha+2}b+\frac{a+b}{2(\alpha+2)}\right)\right|\right].\end{aligned}$$

Remark 2.15. *Setting $\alpha=1$ in Theorem 2.13 gives the same result as in [13], Theorem 8.*

Theorem 2.16. *$f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|^q$ is a concave function, we have*

$$\begin{aligned}&\left|{}^{AB}_{\ a}I^{\alpha}\left\{f(t)\right\}+{}^{AB}I^{\alpha}_{b}\left\{f(t)\right\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;&\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+t}{2}\right)\right|+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{b+t}{2}\right)\right|\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $q>1$ and $B(\alpha)>0$ is normalization function.

Proof. By using the Lemma 2.1 and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\leq\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1(1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}.
\end{aligned}$$

By using concavity of $|f'|^q$ and Jensen integral inequality, we get

$$\begin{aligned}
\int_0^1\left|f'(kt+(1-k)a)\right|^q dk &= \int_0^1 k^o\left|f'(kt+(1-k)a)\right|^q dk\\
&\leq \left(\int_0^1 k^0 dk\right)\left|f'\left(\frac{1}{\int_0^1 k^0 dk}\int_0^1(kt+(1-k)a)\,dk\right)\right|^q\\
&= \left|f'\left(\frac{a+t}{2}\right)\right|^q.
\end{aligned}$$

Similarly

$$\int_0^1\left|f'(kb+(1-k)t)\right|^q dk \leq \left|f'\left(\frac{b+t}{2}\right)\right|^q$$

so, we obtain

$$\begin{aligned}
&\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\
\leq\ &\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+t}{2}\right)\right|+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{b+t}{2}\right)\right|.
\end{aligned}$$

□

Corollary 2.17. *In Theorem 2.16, if we choose $t=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left|{}_{a}^{AB}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I_{b}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[f(a)+f(b)\Big]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|\\
\leq\ &\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|\right].
\end{aligned}$$

Remark 2.18. *Setting $\alpha=1$ in Theorem 2.16 gives the same result as in [13], Theorem 6.*

3. Simulations for Fractional Integral Operators

Although the history of fractional calculus is very old, it has had an important place in the literature in the last two decades. One of the most important reasons for this is due to the kernel used in definitions. Although the kernel used in Caputo's derivative definition has been used for many years, it is a fact that there is a singularity problem. The exponential kernel used in the Caputo-Fabrizio fractional derivative definition is non-sigular, but not non-local. The structure of fractional operators' kernels adds innovation to the field. Features such as linearity, general form, singularity and locality highlight the usage areas and efficiency of the operator. The characteristics of the operators such as producing consistent results, having memory effect feature, providing generalization and compatibility with known operators can be seen by simulations made by taking special selections of parameters and functions included in the operator. Therefore, in this article, the non-singular and non-local Atangana-Baleanu fractional integral operator is used. In the following, simulations for different α values of functions $-\frac{2\sqrt{x}}{\sqrt{\pi}}, x^2+x, (-x)^{\frac{1}{2}}$ and x^3 for Riemann-Liouville and Atangana Baleanu integral operators are shown in Figure 1-12. The harmony between the Riemann-Liouville and Atangana-Baleanu fractional integral operators can be clearly seen when the simulations are analyzed. It is possible to compare the functions integrated with the Atangana-Baleanu fractional integral operator obtained from a derivative operator with a non-local and non-singular kernel. In this way, the situation between two operators was examined.
Then, by making special choices in accordance with the conditions of Theorem 2.2, which is one of the main findings of the study, the change of the right and left sides of the inequality according to the values of the alpha parameter was examined. With Figure 13-15, the variation of the right and left sides of the inequality in Theorem 2.2, which includes the Atangana-Baleanu fractional integral operator, is revealed for the selection of different convex function and interval values.

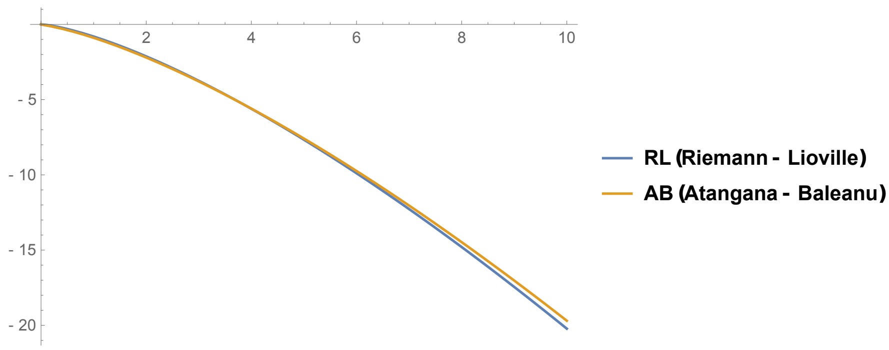

Figure 1: Numerical simulation of solutions $-\frac{2\sqrt{x}}{\sqrt{\pi}}$ for $\alpha = 0.9$

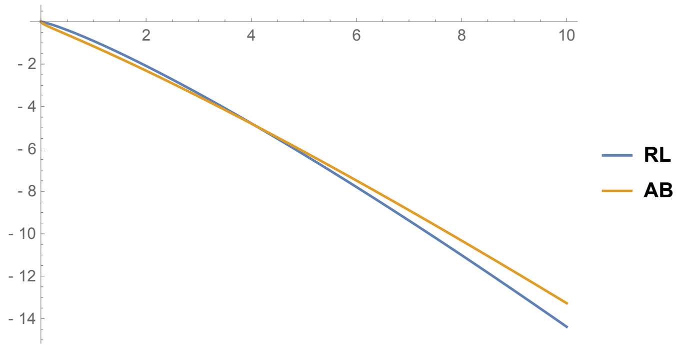

Figure 2: Numerical simulation of solutions $-\frac{2\sqrt{x}}{\sqrt{\pi}}$ for $\alpha = 0.7$

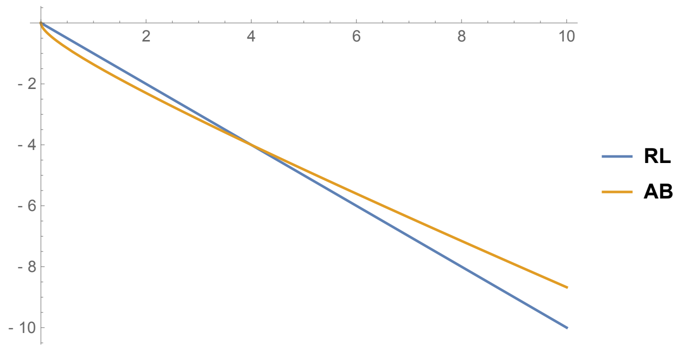

Figure 3: Numerical simulation of solutions $-\frac{2\sqrt{x}}{\sqrt{\pi}}$ for $\alpha = 0.5$

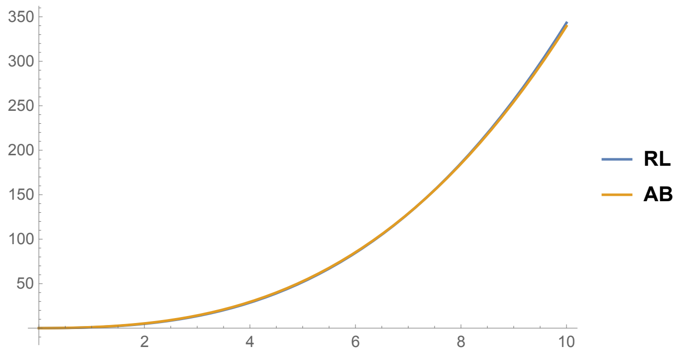

Figure 4: Numerical simulation of solutions $x^2 + x$ for $\alpha = 0.9$

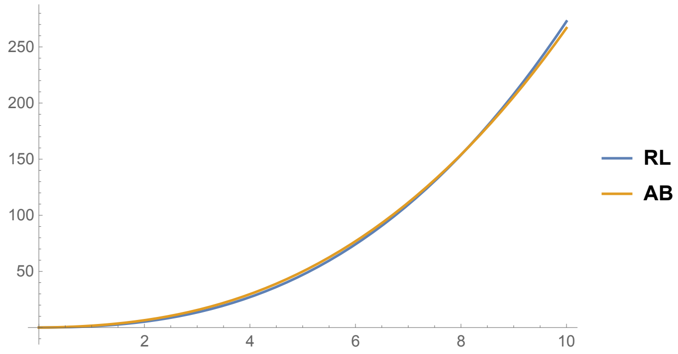

Figure 5: Numerical simulation of solutions $x^2 + x$ for $\alpha = 0.7$

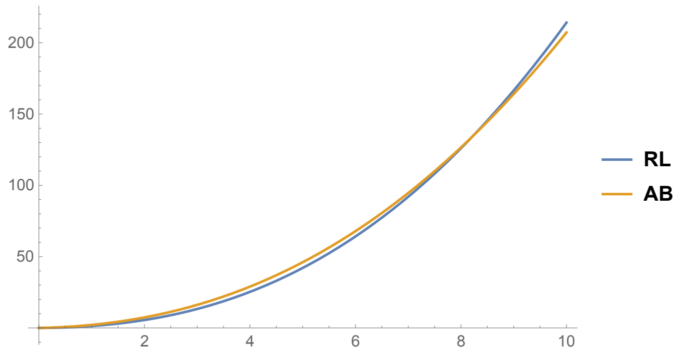

Figure 6: Numerical simulation of solutions $x^2 + x$ for $\alpha = 0.5$

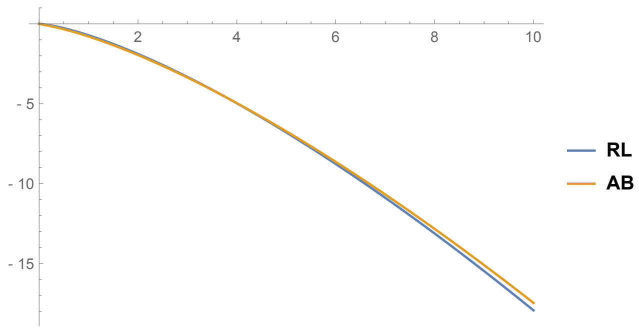

Figure 7: Numerical simulation of solutions $(-x)^{\frac{1}{2}}$ for $\alpha = 0.9$

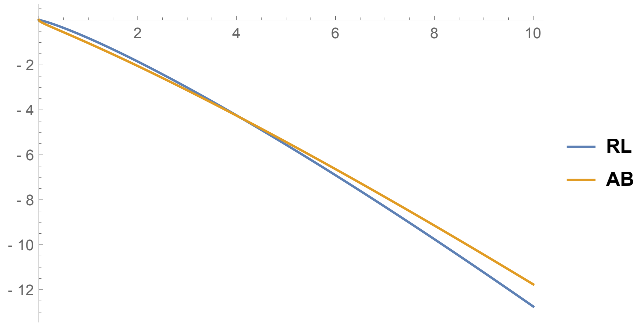

Figure 8: Numerical simulation of solutions $(-x)^{\frac{1}{2}}$ for $\alpha = 0.7$

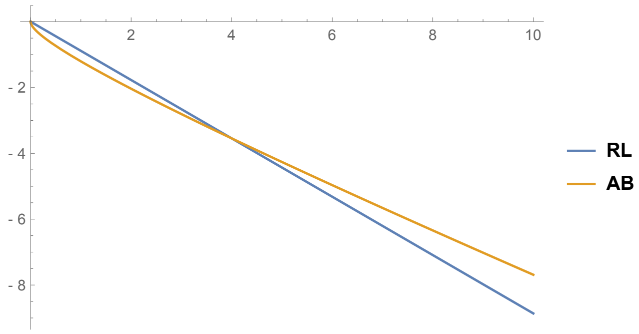

Figure 9: Numerical simulation of solutions $(-x)^{\frac{1}{2}}$ for $\alpha = 0.5$

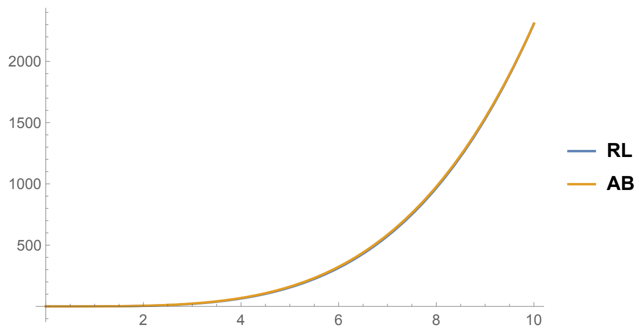

Figure 10: Numerical simulation of solutions x^3 for $\alpha = 0.9$

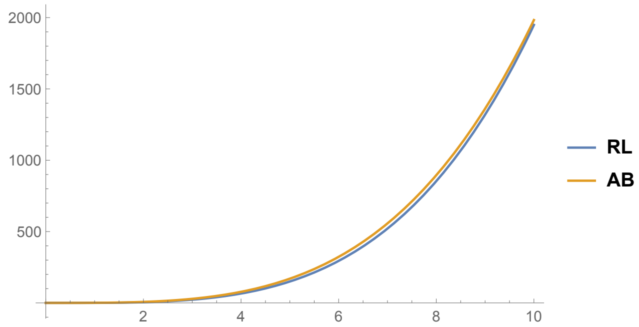

Figure 11: Numerical simulation of solutions x^3 for $\alpha = 0.7$

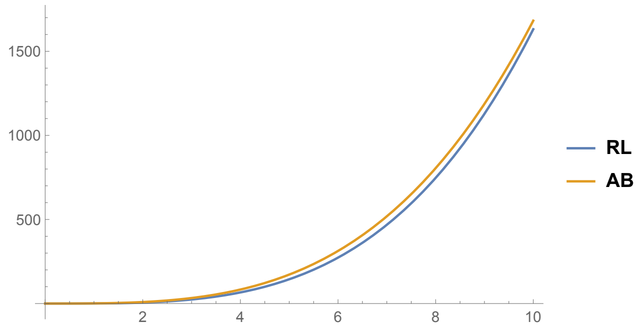

Figure 12: Numerical simulation of solutions x^3 for $\alpha = 0.5$

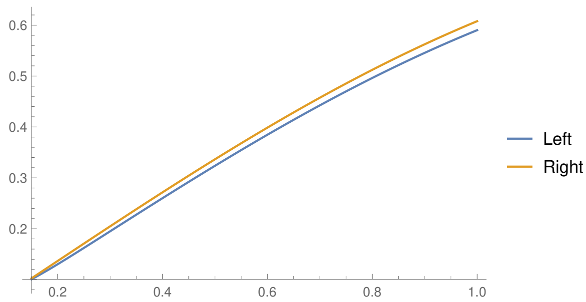

Figure 13: Comparison of right and left sides of the inequality that is given in Theorem 2.2 for $-\frac{2\sqrt{x}}{\sqrt{\pi}}$ and $0 \leq \alpha \leq 1$.

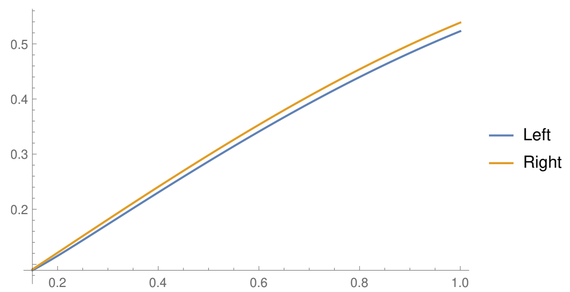

Figure 14: Comparison of right and left sides of the inequality that is given in Theorem 2.2 for $(-x)^{\frac{1}{2}}$ and $0 \le \alpha \le 1$.

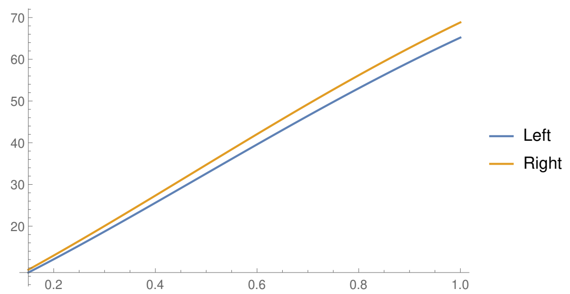

Figure 15: Comparison of right and left sides of the inequality that is given in Theorem 2.2 for x^3 and $0 \le \alpha \le 1$.

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