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On a fourth-order Neumann problem in variable exponent spaces

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Abstract. We study the Neumann problem with Leray-Lions type operator. Using the classical variational theory, we prove the existence, uniqueness and multiplicity of solutions. As far as we know, this is the first attempt to investigate such a fourth-order problem involving Leray-Lions type operators.

1. Introduction

Our aim is to study the existence, uniqueness and multiplicity results for weak solvability of the following fourth-order problem involving Leray-Lions type operator with the Neumann boundary conditions in variable exponent spaces

$$\Delta(a(x, \Delta u)) + b(x)|u|^{p(x)-2}u = \lambda f(x, u) \text{ for } x \in \Omega, \tag{1}$$

with $a(x, \Delta u) \cdot \nu(x) = \mu g(x, u)$ for $x \in \partial\Omega$, where $\lambda, \mu \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outer unit normal vector on $\partial\Omega$, $p \in C(\bar{\Omega})$ is the variable exponent, $a = a(x, \eta) : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$, $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are the Carathéodory functions, with $A : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ given by

$$A(x, t) = \int_0^t a(x, s) ds.$$

In this paper, we shall consider the following conditions for a, A, b, f, and g :

(L_0) $a(x, -s) = -a(x, s)$ for a.e. $x \in \bar{\Omega}$ and all $s \in \mathbb{R}^N$;
(L_1) $A(x, 0) = 0$ for all $x \in \Omega$;
(L_2) There exists a constant $c_0 > 0$ such that $|a(x, \eta)| \leq c_0 \left(1 + |\eta|^{p(x)-1}\right)$ for all $x \in \Omega, \eta \in \mathbb{R}^N$;

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(L_3) $0 \le [a(x,\eta_1) - a(x,\eta_2)] \cdot (\eta_1 - \eta_2)$ for all $x \in \Omega, \eta_1, \eta_2 \in \mathbb{R}^N$, with equality if and only if $\eta_1 = \eta_2$;

(L_4) $|\eta|^{p(x)} \le a(x,\eta) \cdot \eta \le p(x)A(x,\eta)$ for all $x \in \Omega, \eta \in \mathbb{R}^N$.

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \ge b_0$ for all $x \in \Omega$;

(F) For every $q \in C_+(\bar{\Omega})$ with $q^+ < p^-$, there exist $c_1, c_2 > 0$ such that

$$|f(x,t)| \le c_1 + c_2|t|^{q(x)-1} \text{ for all } x \in \Omega, t \in \mathbb{R}; \tag{2}$$

(G) For every $r \in C_+(\bar{\Omega})$ with $r^+ < p^-$, there exist $c_3, c_4 > 0$ such that $|g(x,t)| \le c_3 + c_4|t|^{r(x)-1}$ for all $x \in \partial\Omega, t \in \mathbb{R}$.

These conditions enable us to obtain well known operators by making appropriate choices of a. Indeed, when $a(x,\eta) = |\eta|^{p(x)-2}\eta$, we get the $p(x)$-biharmonic operator of the fourth order.

Studies of problems involving such operators appear in a variety of fields, such as the clamped plate problem, elasticity theory and PDEs modeling Stokes' flows (see El Khalil, Kellati and Touzani [1], Nadirashvili [2]). When $a(x,\eta) = (1 + |\eta|^2)^{(p(x)-2)/2}\eta$, we get the generalized biharmonic mean curvature operator (see Alsaedi and Rădulescu [3]). Moreover, when we choose

$$a(x,\eta) = \left(1 + |\eta|^{p(x)}\left(1 + |\eta|^{2p(x)}\right)^{-1/2}\right)|\eta|^{p(x)-2}\eta,$$

we obtain the following differential operator

$$\Delta a(x,\Delta u) = \Delta\left[\left(1 + \frac{|\Delta u|^{p(x)}}{\sqrt{1 + |\Delta u|^{2p(x)}}}\right)|\Delta u|^{p(x)-2}\Delta u\right],$$

which describes the capillary phenomenon (see Alsaedi and Rădulescu [3], Avci [4]). We note that condition (L_0) is only needed to obtain the multiplicity of solutions. Also, we choose this kind of function a satisfying $(L_0) - (L_5)$ because we want to assure a high degree of generality in our work.

The study of fourth-order partial differential equations with constant exponent has intensively developed in recent years. It has a large variety of applications (see for example Dănet [5], Ferrero and Warnault [6], Myers [7] and the references therein). By introducing elliptic problems with variable exponent, we open the door to applications utilizing extremely nonhomogeneous materials which are nowadays becoming increasingly common in industry. One of these applications is related to the modeling of electrorheological fluids. The first significant discovery in electrorheological fluids was in 1949 by Willis Winslow. These fluids have specially viscous liquids and can significantly change their mechanical properties when they contact an electric field (see Acerbi and Mingione [8], Růžička [9]). Other known applications are related to the image restoration (see Chen, Levine and Rao [10]), elastic materials (see Boureanu [11] and Zhikov [12]), mathematical biology (see Fragnelli [13]), dielectric breakdown and electrical resistance (see Bocea and Mihăilescu [14], polycrystal plasticity (see Bocea, Mihăilescu and Popovici [15], and models of diffusion in sandpiles (see Bocea, Mihăilescu, Perez-Llanos and Rossi [16]). In order to be able to study such problems with variable exponent, we need to use the novel theory of Lebesgue and Sobolev spaces with variable exponent ($L^{p(x)}(\Omega)$, $W^{p(x)}(\Omega)$). Over the past few decades, these spaces have attracted considerable attention (see Cruz-Uribe and Fiorenza [17], Rădulescu and Repovš [18], Diening, Harjuletho, Hästö and Růžička [19]) and the references therein).

The subject of the fourth order elliptic problems involving the Leray-Lions operator with variable exponent has drawn the attention of many authors, for example, Boureanu [20] who has established interesting properties which are useful in the treatment of various classes of fourth-order problems. Boureanu and Vélez-Santiago [21] studied the solvability of a higher-order problem of type (1) with subject to Navier-Stokes boundary conditions over irregular domain. Moreover, Kefi, Repovš and Saoudi [22] showed the existence and multiplicity results of weak solutions for fourth-order problems involving the Leray-Lions type operators by using the theorem of Bonanno and Marano [23]. We also mention a very interesting paper by Giri, Choudhuri and Pradhan [24]. Motivated by these results and the ideas accurately introduced by Boureanu [20], we shall investigate the weak solvabilty of problem (1) with subject to the Neumann boundary conditions.

A reasonable inquiry given the preceding knowledge is what results can be recovered when the standard p-Laplacian and p-biharmonic are replaced by a fourth-order problem employing a Leray-Lions type operator. To our knowledge, only few papers have been published on this subject (see Boureanu [20] and Bonanno [23]). Boureanu [21] established some definitions and basic properties of new fourth-order problem involving a Leray-Lions type operator with variable exponents and proved some existence results for fourth-order problems with variable exponents by using different approaches. One of the interesting aspects of our work is that we study a problem with a nonlinear boundary term that needs the application of the Trace theorem. The possibilities of the parameter being sufficiently large or small has been treated as different cases. Finally, it should be noted that the context here is different from Boureanu and Vélez-Santiago [21], due to the more complicated operator and numerous parameters.

Now, we state our main results which concern existence, uniqueness and multiplicity of solutions of problem (1) :

Theorem 1.1. *Under conditions* (L_1)-(L_4), (B), (F), *and* (G), *problem (1) has a weak solution.*

Theorem 1.2. *Under conditions* (L_1)-(L_4), (B), (F), (F_0), (G), *and* (G_0), *problem (1) has a unique weak solution.*

Theorem 1.3. *Under conditions* (L_0)-(L_4) *and* (F_1)-(F_4), *there exist an open interval* $\Lambda \subseteq (0, +\infty)$ *and a positive real number* ω *such that for each* $\lambda \in \Lambda$ *and* $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying condition* (G_1), *there exists* $\delta > 0$ *such that for each* $\mu \in [0, \delta]$, *problem (1) has at least three weak solutions with norms in* $W^{2,p(x)}(\Omega)$ *less than* ω.

Theorem 1.4. *Under conditions* (L_0)-(L_4), (F), (G), (B), (fg), (f), *and* (g), *problem (1) has an unbounded sequence of distinct weak solutions.*

We describe the structure of the paper. In Section 2, we state some notations and preliminary properties which are necessary for proving our results. In Section 3, using variational methods, we establish the existence and uniqueness result for problem (1). In Section 4, we prove the multiplicity of solutions to problem (1) by using Ricceri's Three critical points theorem and the Fountain theorem. We thank the referee for several constructive remarks.

2. Preliminaries

For simplicity, we shall use letters c_i $(i = 1, 2, \cdots, N)$ to denote positive constants in different cases. We set

$$C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) : 1 < \min_{x\in\bar{\Omega}} p(x) < \max_{x\in\bar{\Omega}} p(x) < \infty \right\}$$

and for all $p \in C_+(\bar{\Omega})$ we let

$$p^+ = \sup_{x\in\Omega} p(x), \quad p^- = \inf_{x\in\Omega} p(x).$$

Also, we denote

$$p^\star(x) = \begin{cases} Np(x)/[N - p(x)] & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N \end{cases}$$

and

$$p^\partial(x) = \begin{cases} (N-1)p(x)/[N - p(x)] & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

Finally, we define the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Proposition 2.1. *(see Fan and Zhao [25]) If $u \in L^{p(\cdot)}(\Omega)$, then:*

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1(=1;>1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1(=1;>1); \tag{3}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}; \tag{4}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}. \tag{5}$$

Remark 2.2. *Define the map $\tilde{\rho}_{p(\cdot)} : L^{p(\cdot)}(\partial\Omega) \to \mathbb{R}$ by*

$$\tilde{\rho}_{p(\cdot)}(u) = \int_{\partial\Omega} |u(x)|^{p(x)} dS,$$

where dS is a surface measure. One can easily prove relations (3)-(5) stated above.

By hypotheses (B), we have the norm

$$\|u\|_b = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + b(x)\left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\},$$

which is equivalent to $\|\cdot\|$ on $W^{2,p(x)}(\Omega)$. Therefore, we shall use $\left(W^{2,p(x)}(\Omega), \|\cdot\|_b\right)$ in the sequel.

We consider $\rho : W^{2,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(.),b}(u) = \int_\Omega \left[|\Delta u|^{p(x)} + b(x)|u|^{p(x)}\right] dx$$

and we make an important connection with the norm $\|\cdot\|_b$ by proceeding as in Boureanu, Rădulescu and Repovš [26].

Proposition 2.3. *(see Boureanu, Rădulescu and Repovš [26]) For any $u, u_n \in W^{2,p(\cdot)}(\Omega)$, the following statements hold:*

$$\|u\|_b < (=;>1) \Leftrightarrow \rho_{p(.),b}(u) < (=;>1);$$
$$\|u\|_b \leq 1 \Rightarrow \|u\|_b^{p^+} \leq \rho_{p(.),b}(u) \leq \|u\|_b^{p^-};$$
$$\|u\|_b \geq 1 \Rightarrow \|u\|_b^{p^-} \leq \rho_{p(.),b}(u) \leq \|u\|_b^{p^+};$$
$$\|u_n\|_b \to 0(\to\infty) \quad \Leftrightarrow \rho_{p(.),b}(u_n) \to 0(\to\infty).$$

Theorem 2.4. *(see Fan and Zhao [25]) Let $q \in C(\bar{\Omega}; \mathbb{R})$ be such that $1 < q^- \leq q^+ < \infty$ and $q(x) \leq p_k^*(x)$ for all $x \in \bar{\Omega}$, where*

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \bar{\Omega}, k \geq 1$. Then there exists a continuous embedding

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

If we replace $\leq$ with $<$, then this embedding is compact.

Theorem 2.5. *(see El Amrouss, Moradi and Moussaoui [27]) Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a bounded open set with a smooth boundary. Suppose that $p \in C_+(\bar{\Omega})$ and $r \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < p^\partial(x), \text{ for all } x \in \partial\Omega.$$

Then there exists a compact boundary trace embedding $W^{2,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

3. The case $\lambda = \mu = 1$

We recall the concept of a weak solution for problem (1) :

Definition 3.1. *We call $u \in W^{2,p(\cdot)}(\Omega)$ a weak solution of (1) if for all $v \in W^{2,p(\cdot)}(\Omega)$,*

$$\int_\Omega a(x, \Delta u) \cdot \Delta v dx + \int_\Omega b(x)|u|^{p(x)-2} uv dx - \int_\Omega f(x, u) v dx - \int_{\partial\Omega} g(x, u) v dS = 0.$$

The energy functional $I : W^{2,p(\cdot)}(\Omega) \to \mathbb{R}$ associated to problem (1) is defined by

$$I(u) = \int_\Omega A(x, \Delta u) dx + \int_\Omega \frac{b(x)}{p(x)} |u|^{p(x)} dx - \int_\Omega F(x, u) dx - \int_{\partial\Omega} G(x, u) dS,$$

where

$$F(x, s) = \int_0^s f(x, t) dt, \quad G(x, s) = \int_0^s g(x, t) dt$$

and

$$\Phi(u) = \int_\Omega A(x, \Delta u) dx + \int_\Omega \frac{b(x)}{p(x)} |u|^{p(x)} dx,$$

$$\Psi(u) = -\int_\Omega F(x, u) dx, \quad J(u) = -\int_{\partial\Omega} G(x, u) dS.$$

Using the approach of Boureanu [20], we show that $I \in C^1\left(W^{2,p(\cdot)}(\Omega); \mathbb{R}\right)$ with

$$\langle I'(u), v\rangle = \int_\Omega a(x, \Delta u) \cdot \Delta v dx + \int_\Omega b(x)|u|^{p(x)-2} uv dx - \int_\Omega f(x, u) v dx - \int_{\partial\Omega} g(x, u) v dS, \tag{6}$$

for all $u, v \in W^{2,p(\cdot)}(\Omega)$.

3.1. Existence of weak solutions of problem (1)

Proof of Theorem 1.1. We show that I is coercive. It follows from (F) and (G) that

$$|F(x, t)| \le c_1 |t| + c_2 \frac{|t|^{q(x)}}{q(x)}, \quad \text{for all } x \in \Omega, t \in \mathbb{R}, \tag{7}$$

$$|G(x, t)| \le c_3 |t| + c_4 \frac{|t|^{r(x)}}{r(x)}, \quad \text{for all } x \in \partial\Omega, t \in \mathbb{R}. \tag{8}$$

By (3),(5) and Remark 2.2, we have

$$\int_\Omega F(x, u) dx \le c_1 \|u\|_{L^1(\Omega)} + \frac{c_2}{q^-}\left(\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} + \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}\right),$$

$$\int_{\partial\Omega} G(x, u) dS \le c_3 \|u\|_{L^1(\partial\Omega)} + \frac{c_4}{r^-}\left(\|u\|^{r^+}_{L^{r(\cdot)}(\partial\Omega)} + \|u\|^{r^-}_{L^{r(\cdot)}(\partial\Omega)}\right).$$

Theorems 2.4 and 2.5 imply that, for $u \in W^{2,p(\cdot)}(\Omega)$ with $\|u\|_b \ge 1$, there exist $k_1, k_2, k_3, k_4 > 0$ such that

$$\int_\Omega F(x, u) dx \le k_1 \|u\|_b + k_2 \|u\|_b^{q^+}, \tag{9}$$

$$\int_{\partial\Omega} G(x, u) dS \le k_3 \|u\|_b + k_4 \|u\|_b^{r^+}. \tag{10}$$

It follows from (L_4) and (B) that

$$\int_\Omega A(x,\Delta u)dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)}dx \geq \frac{1}{p^+}\int_\Omega \left[|\Delta u|^{p(x)} + b(x)|u|^{p(x)}\right]dx.$$

By Proposition 2.3, we know that, for $\|u\|_b \geq 1$,

$$\int_\Omega A(x,\Delta u)dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)}dx \geq \frac{1}{p^+}\|u\|_b^{p^-}. \tag{11}$$

Then, applying (9), (10) and (11), for $\|u\|_b \geq 1$, we have

$$I(u) \geq \frac{1}{p^+}\|u\|_b^{p^-} - k_2\|u\|_b^{q^+} - k_4\|u\|_b^{r^+} - (k_1 + k_3)\|u\|_b. \tag{12}$$

By the assumptions on p, q and r , we obtain that $I(u) \to \infty$ when $\|u\|_b \to \infty$. Following that, we create the notations

$$\mathcal{F}(u) = \int_\Omega F(x,u)dx, \quad \mathcal{G}(u) = \int_{\partial\Omega} G(x,u)dS.$$

Given that $\mathcal{F}'$ and $\mathcal{G}'$ are entirely continuous, F and G are said to be weakly continuous. We can infer from Boureanu [20, Proposition 5] that I is a weakly lower semi continuous. Now we can apply the result in Struwe [28, Theorem 1.2]. As a consequence, we can conclude that problem (1) admits at least one weak solution. □

3.2. Uniqueness of weak solutions of problem (1)

To establish the uniqueness of solutions, we shall impose the following conditions on f and g :

(F_0) The monotonicity condition on f is satisfied, i.e. $(f(x,s) - f(x,t))(s-t) < 0$, for all $x \in \Omega$ and $s,t \in \mathbb{R}$ with $s \neq t$;

(G_0) The monotonicity condition on g is satisfied, i.e. $(g(x,s) - g(x,t))(s-t) < 0$, for all $x \in \partial\Omega$ and $s,t \in \mathbb{R}$ with $s \neq t$.

Proof of Theorem 1.2. The existence follows from Theorem 1.1. So let now u_1 and u_2 be two weak solutions to problem (1). Thanks to Definition 3.1, we can replace u by u_1 and consider $v = u_1 - u_2$ to get that

$$\begin{aligned}\int_\Omega a(x,\Delta u_1)\cdot\Delta(u_1-u_2)\,dx + \int_\Omega b(x)\,|u_1|^{p(x)-2}\,u_1\,(u_1-u_2)\,dx \\ - \int_\Omega f(x,u_1)(u_1-u_2)\,dx - \int_{\partial\Omega} g(x,u_1)(u_1-u_2)\,dS = 0.\end{aligned}$$

Next, we substitute u_2 for u in Definition 3.1 and consider $v = u_2 - u_1$ to obtain

$$\begin{aligned}\int_\Omega a(x,\Delta u_2)\cdot\Delta(u_2-u_1)\,dx + \int_\Omega b(x)\,|u_2|^{p(x)-2}\,u_2\,(u_2-u_1)\,dx \\ - \int_\Omega f(x,u_2)(u_2-u_1)\,dx - \int_{\partial\Omega} g(x,u_2)(u_2-u_1)\,dS = 0.\end{aligned}$$

After some calculation, we can deduce that

$$\begin{aligned}\int_\Omega [a(x,\Delta u_1) - a(x,\Delta u_2)]\cdot(\Delta u_1 - \Delta u_2)\,dx + \int_\Omega b(x)\left[|u_1|^{p(x)-2}\,u_1 - |u_2|^{p(x)-2}\,u_2\right](u_1-u_2)\,dx \\ - \int_\Omega [f(x,u_1) - f(x,u_2)](u_1-u_2)\,dx - \int_{\partial\Omega}[g(x,u_1) - g(x,u_2)](u_1-u_2)\,dS = 0.\end{aligned}$$

Finally, unless $u_1 = u_2$, conditions $(L_3), (F_0)$, and (G_0) indicate that all terms in the above equality are positive. As a result, we get the uniqueness of the weak solution to the problem (1).

4. The case $\lambda \geqslant 0, \mu \geqslant 0$

4.1. Multiplicity of weak solutions for problem (1)

To obtain the multiplicity of solutions, we shall need to combine the following conditions :

(F_1) For $t \in C(\bar{\Omega})$ and $t(x) < p^*(x)$ for all $x \in \bar{\Omega}$, we have

$$\sup_{(x,s)\in\Omega\times\mathbb{R}} \frac{|f(x,s)|}{1+|s|^{t(x)-1}} < +\infty;$$

(F_2) There exists a positive constant c such that $F(x,s) > 0$ for a.e. $x \in \Omega$ and all $s \in (0,c]$;

(F_3) There exist a positive constant c_5 and a function $\gamma \in C(\bar{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$, such that $|F(x,s)| \leq c_5\left(1+|s|^{\gamma(x)}\right)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

(F_4) There exist $p_1 \in C(\bar{\Omega})$ and $p^+ < p_1^- \leq p_1(x) < p^*(x)$, such that

$$\limsup_{s\to 0} \sup_{x\in\Omega} \frac{F(x,s)}{|s|^{p_1(x)}} < +\infty;$$

(G_1) For $p_2 \in C(\bar{\Omega})$ and $p_2(x) < p^{\partial}(x)$ for all $x \in \bar{\Omega}$, we have

$$\sup_{(x,s)\in\partial\Omega\times\mathbb{R}} \frac{|g(x,s)|}{1+|s|^{p_2(x)-1}} < +\infty.$$

The main tool employed to prove Theorem 1.3 is the variational method, used to find critical points of the functional $H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u)$ on $W^{2,p(x)}(\Omega)$, where

$$\Phi(u) = \int_\Omega A(x,\Delta u)dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)}dx, \tag{13}$$

$$\Psi(u) = -\int_\Omega F(x,u)dx, \tag{14}$$

$$J(u) = -\int_{\partial\Omega} G(x,u)d\sigma. \tag{15}$$

and

$$F(x,u) = \int_0^u f(x,s)ds, G(x,u) = \int_0^u g(x,s)ds.$$

In this case, we define the weak solution of problem (1) on $W^{2,p(x)}(\Omega)$ as

$$\begin{aligned}&\int_\Omega a(x,\Delta u)\Delta v dx + \int_\Omega b(x)|u|^{p(x)-2}uv dx\\ &= \lambda\int_\Omega f(x,u)v dx + \mu\int_{\partial\Omega} g(x,u)v d\sigma \quad \text{for all } v \in W^{2,p(x)}(\Omega).\end{aligned}$$

To prove Theorem 1.3, we shall use the Three critical points theorem (see Ricceri [30, Proposition 3.1]).

Proposition 4.1. *Let X be a nonempty set and Φ, Ψ real functions on X. Assume that there exist $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \quad \Phi(x_1) > r, \qquad \sup_{x\in\Phi^{-1}(]-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then for each h satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(h+\Psi(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(h+\Psi(x))).$$

Here, we have $X = W^{2,p(x)}(\Omega)$. Using the techniques of Boureanu [20], we can prove the following properties (we shall avoid the details here).

Proposition 4.2. *Let $\Omega\subset\mathbb{R}^N (N\geq 2)$ be a bounded domain with a smooth boundary, $\Phi : X\to\mathbb{R}$ the functional defined by (13,) and $a : \Omega\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that the conditions (L_2) and (B) are satisfied. Then Φ is well-defined and of class C^1, with the Gâteaux derivative*

$$\langle\Phi'(u), v\rangle = \int_\Omega a(x,\Delta u)\Delta v dx + \int_\Omega b(x)|u|^{p(x)-2}uvdx.$$

Theorem 4.3. *Assume that the mapping a satisfies conditions (L_0)-(L_4) . Then*

1. *Φ' is continuous and strictly monotone.*
2. *Φ' is of (S_+) type.*
3. *Φ' is a homeomorphism.*

Proposition 4.4. *Let $\Omega\subset\mathbb{R}^N (N\geq 2)$ be a bounded domain with a smooth boundary, $\Phi : X\to\mathbb{R}$ as defined by (13) and $a : \Omega\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that conditions $(L_1), (L_2)$, and (B) are satisfied. Then Φ is (sequentially) weakly lower semicontinuous, that is, for any $u\in X$ and any subsequence $(u_n)_n\subset X$ such that $u_n\rightharpoonup u$ in X, the following holds*

$$\Phi(u)\leq\liminf_{n\to\infty}\Phi(u_n).$$

Proof of Theorem 1.3:
(i) Let $u, v\in X$ be such that

$$\langle\Phi'(u), v\rangle = \int_\Omega\left[a(x,\Delta u)\Delta v + b(x)|u|^{p(x)-2}uv\right]dx,$$

$$\langle\Psi'(u), v\rangle = -\int_\Omega f(x,u)vdx,\ \ \langle J'(u), v\rangle = -\int_{\partial\Omega} g(x,u)vd\sigma.$$

By Theorem 4.3 and Proposition 4.4, Φ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X'.

By (F) and (G), Ψ and J are continuously Gâteaux differentiable functionals. Furthermore, by the compactness of the embedding $W^{2,p(x)}(\Omega)\hookrightarrow L^{p(x)}(\Omega)$ and the trace embedding $W^{2,p(x)}(\Omega)\hookrightarrow L^{p(x)}(\partial\Omega)$, we can conclude that Ψ' and J' are compact. As a result, Φ is bounded on each bounded subset of X .

By condition (L_4), if $\|u\|_b\geq 1$, then we have

$$\begin{aligned}\Phi(u) &= \int_\Omega A(x,\Delta u)dx + \int_\Omega\frac{1}{p(x)}b(x)|u|^{p(x)}dx\\ &\geq\int_\Omega\frac{1}{p(x)}\left(|\Delta u|^{p(x)} + b(x)|u|^{p(x)}\right)dx\geq\frac{1}{p^+}\rho(u)\geq\frac{1}{p^+}\|u\|_b^{p^-}.\end{aligned}$$

Using relations (7) and (8), we can deduce that

$$\begin{aligned}\lambda\Psi(u) &= -\lambda\int_\Omega F(x,u)dx\geq -\lambda\int_\Omega c_5\left(1+|u|^{\gamma(x)}\right)dx\\ &\geq -\lambda c_5\left(|\Omega| + \max\|u\|_{\gamma(x)}^{\gamma^+}, \|u\|_{\gamma(x)}^{\gamma^-}\right)\geq -c_5'\left(1+\max\|u\|_{\gamma(x)}^{\gamma^+}, \|u\|_{\gamma(x)}^{\gamma^-}\right)\geq -c_5''\left(1+\|u\|_b^{\gamma^+}\right)\end{aligned}$$

for any $u \in X$. Consequently, $\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|_b^{p^-} - c_5''\left(1 + \|u\|_b^{\gamma^+}\right)$.

Since $\gamma^+ < p^-$, we get

$$\lim_{\|u\|_b \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty, \quad \text{for all} \quad u \in X, \lambda \in [0, +\infty).$$

(ii) Let $u_0 = 0$. Invoking Proposition 4.1, condition (L_2), and the definition of F, we get $\Phi(u_0) = -\Psi(u_0) = 0$. By virtue of (F_4), there exist $\eta \in [0, 1], c_6 > 0$, such that

$$F(x, s) \leq c_6 |s|^{p_1(x)} \leq c_6 |s|^{p_1^-}, \quad \text{for all } s \in [-\eta, \eta] \text{ and a.e. } x \in \Omega.$$

Invoking condition (F_3), we can determine a constant M such that

$$F(x, s) < M|s|^{p_1^-}, \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

On the other hand, by virtue of the Sobolev embedding theorem, $W^{2,p(x)}(\Omega) \hookrightarrow L^{p_1^-}(\Omega)$ is continuous, so we have

$$-\Psi(u) = \int_\Omega F(x, u)dx < M \int_\Omega |u|^{p_1^-} dx \leq c_7 \|u\|_b^{p_1^-} \leq c_8 r^{p_1^-/p^+}$$

when $\|u\|_b^{p^+}/p^+ \leq r$. Since $p_1^- > p^+$, we obtain

$$\lim_{r \to 0^+} \frac{1}{r} \sup_{\|u\|_b^+/p^+ \leq r} \{-\Psi(u)\} = 0. \tag{16}$$

Next, let $u_1 \in C^1(\Omega)$ be a positive function in Ω, with $\max_{\bar{\Omega}} u_1 \leq c$. Then, $u_1 \in X$ and $\Phi(u_1) > 0$. Invoking condition (F_2), we get

$$-\Psi(u_1) = \int_\Omega F(x, u_1(x))\, dx > 0.$$

Therefore, by (16), we can find $r \in \left(0, \min\left\{\Phi(u_1), \frac{1}{p^+}\right\}\right)$ such that

$$\sup_{\frac{\|u\|_b^{p^+}}{p^+} \leq r} \{-\Psi(u)\} < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

Now, let $u \in \Phi^{-1}((-\infty, r])$. Then

$$\int_\Omega \left(p(x)A(x, \Delta u) + b(x)|u|^{p(x)}\right) dx \leq rp^+ < 1.$$

It follows from Proposition 2.3 that $\|u\|_b < 1$ and we can conclude that

$$\frac{1}{p^+}\|u\|_b^{p^+} \leq \frac{1}{p^+}\rho(u) \leq \int_\Omega \left(p(x)A(x, \Delta u) + b(x)|u|^{p(x)}\right) dx < r.$$

Therefore, we can infer that $\Phi^{-1}((-\infty, r]) \subset \left\{u \in X : \frac{1}{p^+}\|u\|_b^{p^+} < r\right\}$, and so

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \{-\Psi(u)\} < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

According to Proposition 4.1, (ii) is proved, hence problem (1) indeed has at least three solutions. □

4.2. Existence of an unbounded sequence of distinct weak solutions of problem (1)

We shall impose the following additional conditions:

(f) f satisfies the (AR) condition, that is, there exist $\theta_1 > p^+$ and $l_1 > 0$ such that

$$0 < \theta_1 F(x,t) \le t f(x,t) \text{ for all } |t| > l_1 \text{ and a.e. } x \in \Omega,$$

and $\operatorname{ess\,inf}_{x\in\Omega} F(\cdot, t_0) > 0$, where $F(x,t) = \int_0^t f(x,s)ds$;

(g) g satisfies the (AR) condition, that is, there exist $\theta_2 > p^+$ and $l_1 > 0$ such that

$$0 < \theta_2 G(x,t) \le t g(x,t) \text{ for all } |t| > l_1 \text{ and a.e. } x \in \partial\Omega,$$

and $\operatorname{ess\,inf}_{x\in\partial\Omega} G(\cdot, t_0) > 0$, where $G(x,t) = \int_0^t g(x,s)ds$;

(fg) $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, odd with respect to the second variable.

In order to prove Theorem 1.4, we shall invoke the Fountain theorem (see Willem [31]). Let

$$\langle f_n, e_m \rangle = \delta_{n,m} = \chi_{\{m=n\}}, X = \text{span}\{e_n : \quad n = 1, 2, \ldots\}$$

and

$$X^\star = \text{span}\{f_n : \quad n = 1, 2, \ldots\},$$

where $(e_n)_{n=1}^\infty \subset X$ and $(f_n)_{n=1}^\infty \subset X^\star$. We take $X = W^{2,p(x)}(\Omega)$ and for $i = 1, 2, \ldots$ we denote

$$X_i = \text{span}\{e_i\}, \quad Y_i = \bigoplus_{j=1}^{i} X_j \quad \text{and} \quad Z_i = \bigoplus_{j=i}^{\infty} X_j. \tag{17}$$

Theorem 4.5. *(Fountain theorem, see Willem [31]) Assume that $\Phi \in C^1(X, \mathbb{R})$ is even and that for each $i = 1, 2, \ldots$, there exist $\rho_i > \gamma_i > 0$ such that*

(H_1) $\inf_{u\in Z_i, \|u\|_X=\gamma_i} \Phi(u) \to \infty$ *as* $i \to \infty$;

(H_2) $\max_{u\in Y_i, \|u\|_X=\rho_i} \Phi(u) \le 0$;

(H_3) Φ *satisfies the (PS)* $_c$ *condition for every $c > 0$, that is, any sequence $(u_n)_n \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in $X^\star$ as $n \to \infty$ contains a subsequence converging to a critical point of Φ.*

Then Φ has a sequence of critical values tending to $+\infty$.

Proof of Theorem 1.4.

(H_1) For each $i \in \mathbb{N}^*$ there exists $\gamma_i > 0$ such that $\inf_{u\in Z_i, \|u\|_b=\gamma_i} I(u) \to \infty \quad$ as $i \to \infty$.

We have already proved that for $\|u\|_b \ge 1$ we have

$$I(u) \ge \frac{1}{p^+}\|u\|_b^{p^-} - \lambda k_2 \|u\|_b^{q^+} - \mu k_4 \|u\|_b^{r^+} - (\lambda k_1 + \mu k_3)\|u\|_b. \tag{18}$$

Since $p^- > q^+$ and $r^+ < p^-$, we can choose $(\gamma_i)_i$ such that $\gamma_i \to \infty$ as $i \to \infty$. Consequently, since $q^+ > 1$, (18) yields that $I(u) \to \infty$ as $\gamma_i = \|u\|_b \to \infty$.

(H_2) The proof is similar as in Boureanu [21, Theorem 1]. For each $i \in \mathbb{N}^*$, there exist $\rho_i > \gamma_i$ such that $\max_{u\in Y_i, \|u\|_b=\rho_i} I(u) \leq 0$. To establish this, we set $A(x,t) = \int_0^1 a(x,st)tds$ and by (L_2) and the Hölder-type inequality, we get

$$\Phi(u) \leq c_0|\Omega|\|\Delta u\|_{L^{p(\cdot)}(\Omega)} + (p^-)^{-1} c_0\rho_{p(.),b}(u).$$

Then, for $u \in X$ with $\|u\|_b > 1$, invoking Proposition 2.3, there exists the constants $c_9, c_{10} > 0$, such that $\Phi(u) \leq c_9\|u\|_b + c_{10}\|u\|_b^{p^+}$. Now, by the (AR) condition on (f) and (g), we deduce that there exist $c_{11}, c_{12}, c_{13} > 0$ such that

$$I(u) \leq c_9\|u\|_b + c_{10}\|u\|_b^{p^+} - \lambda c_{11}\|u\|^{\theta_1}_{L^{\theta_1}(\Omega)} - \mu c_{12}\|u\|^{\theta_2}_{L^{\theta_2}(\Omega)} + c_{13}.$$

We put $\theta_3 = \inf\{\theta_1, \theta_2\}$, and deduce that

$$I(u) \leq c_9\|u\|_b + c_{10}\|u\|_b^{p^+} - \lambda c_{11}\|u\|^{\theta_3}_{L^{\theta_3}(\Omega)} - \mu c_{12}\|u\|^{\theta_3}_{L^{\theta_3}(\Omega)} + c_{13}.$$

Since $\theta_3 > p^+$and Y_i is finite-dimensional, all norms are equivalent on Y_i, so we have completed the verification of (H_2).

(H_3) Let $M \in \mathbb{R}$ and $(u_n)_n \subset X$ be such that

$$|I(u_n)| < M \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } X^\star \text{ as } n \to \infty. \tag{19}$$

We first show that $(u_n)_n$ is bounded. We argue by contradiction and we assume that, up to a subsequence, $\|u_n\| \to \infty$ as $n \to \infty$. Then, using (19) and (L_4), we can take $\tau \in (p^+, \theta)$, where $\theta = \max\{\theta_1, \theta_2\}$. Then for sufficiently large n, we have

$$\begin{aligned}
M + 1 + \|u_n\| \geq & I(u_n) - \frac{1}{\tau}\langle I'(u_n), u_n\rangle \\
\geq & \left(\frac{1}{p^+} - \frac{1}{\tau}\right)\rho_{p(.),b}(u_n) - \lambda\int_\Omega \left(F(x,u_n) - \frac{1}{\tau}f(x,u_n)u_n\right)dx \\
\geq & \int_{\partial\Omega}\left(G(x,u_n) - \frac{1}{\tau}g(x,u_n)u_n\right)dS \\
\geq & \left(\frac{1}{p^+} - \frac{1}{\tau}\right)\rho_{p(.),b}(u_n) - \lambda\int_{\{x\in\Omega: \quad |u_n(x)|>l_1\}}\left(F(x,u_n) - \frac{1}{\tau}f(x,u_n)u_n\right)dx \\
& - \lambda|\Omega|\sup\left\{\left|F(x,t) - \frac{1}{\tau}f(x,t)t\right| : x\in\Omega, |t|\leq l_1\right\} \\
& - \mu\int_{\{x\in\partial\Omega: \quad |u_n(x)|>l_1\}}\left(G(x,u_n) - \frac{1}{\tau}g(x,u_n)u_n\right)dx \\
& - \mu|\partial\Omega|\sup\left\{\left|G(x,t) - \frac{1}{\tau}g(x,t)t\right| : x\in\partial\Omega, |t|\leq l_1\right\}.
\end{aligned}$$

Using Proposition 2.3 and (AR) condition on f and g, we deduce that, for sufficiently large n,

$$\begin{aligned}
M + 1 + \|u_n\| \geq & \left(\frac{1}{p^+} - \frac{1}{\tau}\right)\|u_n\|_b^{p^-} - \lambda|\Omega|\sup\left\{\left|F(x,t) - \frac{1}{\tau}f(x,t)t\right| : x\in\Omega, |t|\leq l_1\right\} \\
& - \mu|\partial\Omega|\sup\left\{\left|G(x,t) - \frac{1}{\tau}g(x,t)t\right| : x\in\partial\Omega, |t|\leq l_1\right\}.
\end{aligned}$$

Dividing by $\|u_n\|_b^{p^-}$ in the above inequality, we obtain a contradiction. This implies that $(u_n)_n$ is bounded in X. Therefore $u_n \rightharpoonup u$ in X, where u is a critical point of I, since $I'(u_n) \to 0$ in $X^\star$ and we have that $\lim_{n\to\infty} |\langle I'(u_n), u_n - u\rangle| = 0$. By (F), Hölder's type inequality, and Theorem 2.4, we can deduce that

$$\lim_{n\to\infty}\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| = 0$$

and by Theorem 2.5, we have

$$\lim_{n\to\infty}\left|\int_{\partial\Omega} g(x,u_n)(u_n-u)\,dx\right| = 0.$$

According to Theorem 4.3 (ii), functional $\Phi' : X \to X^\star$ is of type (S+). We also know that I is even because of condition (fg). Therefore the proof of Theorem 1.4 is finally completed. □

References

[1] A. El Khalil, S. Kellati, A. Touzani, On the principal frequency curve of the p-biharmonic operator, Arab J. Math. Sci. 17 (2011), 89–99.

[2] N.-S. Nadirashvili, Rayleigh's conjecture on the principal frequency of the clamped plate, Arch. Ration. Mech. Anal. 129 (1995), 1–10.

[3] R. Alsaedi, V. Rădulescu, Generalized biharmonic problems with variable exponent and Navier boundary condition, Electron. J. Diff. Eqns., Conference 25 (2018), 27–37.

[4] M. Avci, Ni-Serrin type equations arising from capillarity phenomena with non-standard growth, Bound Value Probl. 55 (2013), 1–13.

[5] C.-P. Dănet, Two maximum principles for a nonlinear fourth order equation from thin plate theory, Electronic J. Qualitative Theory of Diff. Eq. 31 (2014), 1–9.

[6] A. Ferrero, G. Warnault, On solutions of second and fourth order elliptic equations with power-type nonlinearities, Nonlinear Anal. 70 (2009), 2889–2902.

[7] T.G. Myers, Thin films with high surface tension, SIAM Review 40 (1998), 441–462.

[8] E. Acerbi, G. Mingione, Regularity results for electrorheological fluids, the stationary case, C. R. Acad. Sci. Paris 334 (2002), 817–822.

[9] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer Verlag Berlin, 2002.

[10] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (2006), 1386–1406.

[11] M.-M. Boureanu, A. Matei, M. Sofonea, Nonlinear problems with $p(.)$-growth conditions and applications to antiplane contact models, Adv. Nonl. Studies 14 (2014), 295–313.

[12] V.-V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. 29 (1987), 33–66.

[13] G. Fragnelli, Positive periodic solutions for a system of anisotropic parabolic equations, J. Math. Anal. Appl. 367 (2010), 204–228.

[14] M. Bocea, M. Mihăilescu, Γ-convergence of power-law functionals with variable exponents, Nonlinear Anal. 73 (2010), 110–121.

[15] M. Bocea, M. Mihăilescu, C. Popovici, On the asymptotic behavior of variable exponent power-law functionals and applications, Ricerche Mat. 59 (2010), 207–238.

[16] M. Bocea, M. Mihăilescu, M. Perez-Llanos, J.-D. Rossi, Models for growth of heterogeneous sandpiles via Mosco convergence, Asympt. Anal. 78 (2012), 11–36.

[17] D.-V. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces. Foundations and Harmonic Analysis, in Contributions to Appl. Numer. Harmon. Anal. Birkhauser/Springer, Heidelberg, 2013.

[18] V.D. Rădulescu, D.D. Repovš, Partial Diferential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, Chapman and Hall/CRC, Hoboken, 2015.

[19] L. Diening, P. Harjuletho, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponent, Lect. Notes Math., Springer-Verlag, Berlin Heidelberg, 2011.

[20] M.-M. Boureanu, Fourth-order problems with Leray-Lions type operators in variable exponent spaces, Discrete Contin. Dyn. Syst. S, 12 (2019), 231–243.

[21] M.-M. Boureanu, A. Vélez-Santiago, Applied higher-order elliptic problems with nonstandard growth structure, Appl. Math. Lett. 123 (2022), 107603.

[22] K. Kefi, D.D. Repovš, K. Saoudi, On weak solutions for fourth-order problems involving the Leray-Lions type operators, Math. Methods Appl. Sci. 44 (2021), 13060–13068.

[23] G. Bonanno, S.-A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal. 89 (2010), 1–10.

[24] R.K. Giri, D. Choudhuri, S. Pradhan, Existence and concentration of solutions for a class of elliptic PDEs involving p-biharmonic operator, Mat. Vesnik 70(2) (2018), 147–154.

[25] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424–446.

[26] M.-M. Boureanu, V. Rădulescu, D. Repovš, On a $p(x)$-biharmonic problem with no-flux boundary condition, Comput. Math. Appl. 72 (2016), 2505–2515.

[27] A.R. El Amrouss, F. Moradi, M. Moussaoui, Existence and multiplicity of solutions for a $p(x)$-biharmonic problem with Neumann boundary conditions, Bol. Soc. Paran. Mat. 40 (2022), 1–15.

[28] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 2008.
[29] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (2009), 3084–3089.
[30] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, Math. Comput. Modelling 32 (2000), 1485–1494.
[31] M. Willem, Minimax Theorems, Birkhauser, Boston, 1996.