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Seminorm optimal dual frames for erasures

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Abstract. The main purpose of this paper is to find other measurements for constructing optimal duals that minimize the reconstruction errors, when erasures occur. Some known results are investigated with some other measurements. Moreover, we investigate extreme points of the set of all optimal duals for 1-erasure by the seminorms. Furthermore, some examples are provided for clarification.

1. Introduction and preliminaries

Let $\mathcal{H}_n$ be an n-dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H}_n)$, the Banach algebra consisting of all bounded linear operators on $\mathcal{H}_n$. A sequence $\{f_i\}_{i=1}^m$ $(m \geq n)$ in $\mathcal{H}_n$ is called a Bessel sequence if there exists $B > 0$ such that

$$\sum_{i=1}^{m} |\langle f, f_i \rangle|^2 \leq B\, \|f\|^2 \qquad (f \in \mathcal{H}).$$

The analysis operator $\theta_F : \mathcal{H}_n \to \mathbb{C}^m$ is defined by $\theta_F(f) = \{\langle f, f_i \rangle\}_{i=1}^m$, where $f \in \mathcal{H}_n$. The adjoint operator of θ_F is also given by

$$\theta_F^* : \mathbb{C}^m \longrightarrow \mathcal{H}_n, \quad \theta_F^*(\{c_i\}_{i=1}^m) := \sum_{i=1}^{m} c_i f_i$$

and is called synthesis (or pre-frame) operator, related to $\{f_i\}_{i=1}^m$.

A sequence $F = \{f_i\}_{i=1}^m$ in $\mathcal{H}_n$ is called a frame if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{m} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}_n).$$

The constants A and B are called lower and upper frame bounds, respectively which are not necessarily unique. The frame operator is defined as

$$S_X : \mathcal{H}_n \longrightarrow \mathcal{H}_n, \quad S_X f = \theta_F^* \theta_F(f) = \sum_{i=1}^{m} \langle f, f_i \rangle f_i,$$

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which is a bounded, positive, and invertible operator; see [5, Lemma 5.1.5]. For more information about frames, we refer to [3], [4] and [5].

A frame $G = \{g_i\}_{i=1}^m$ is called a dual for frame $F = \{f_i\}_{i=1}^m$ if $\theta_F^*\theta_G = I_{\mathcal{H}_n}$, where $I_{\mathcal{H}_n}$ is the identity operator on $\mathcal{H}_n$. Note that $\{S_F^{-1}f_i\}_{i=1}^m$ is a special dual frame for F, called the canonical dual of F. It is well known [7, Propositin 6.4] that $\{g_i\}_{i=1}^m$ is a dual for $\{f_i\}_{i=1}^m$ if and only if $g_i = S_F^{-1}f_i + u_i$ $(1 \leq i \leq m)$, where $\{u_i\}_{i=1}^m$ satisfies the following condition

$$\sum_{n=1}^{m} \langle f, f_i \rangle u_i = 0. \tag{1}$$

The equation

$$f = I_{\mathcal{H}_n}(f) = \theta_F^*\theta_G(f) = \sum_{i=1}^{m} \langle f, g_i \rangle f_i \qquad (f \in \mathcal{H}_n),$$

is called the reconstruction formula for the frame F. However, in application usually, some erasures occur. Therefore minimizing the maximum error takes an important role. In [9] and [10], the authors considered this issue, by using the operator norm. In [11], they used spectral radius, and in [2] this subject is investigated by using numerical radius as a measurement. Note that

$$\rho(A) \leq r(A) \leq \|A\| \leq 2r(A),$$

where $\rho(A)$ is the spectral radius and $r(A)$ is the numerical radius of A, for all $A \in \mathcal{B}(\mathcal{H}_n)$. To understand the background of this field, we refer to [6] and [8]. In addition, using several measurements may be useful for different applications. From this point of view, in this paper we investigate some available results, by considering general seminorms as the measurement.

Here, we provide some preliminaries and notions that will be used in the further results. Suppose that $I_m := \{1, ..., m\}$, and $F = \{f_i\}_{i \in I_m}$ is a frame for $\mathcal{H}_n$. If $G = \{g_i\}_{i \in I_m}$ is a dual of F and $\Lambda \subset I_m$, then the error operator E_Λ is defined as

$$E_\Lambda(f) = \sum_{i \in \Lambda} \langle f, f_i \rangle g_i = \theta_G^* D \theta_F, \quad (f \in \mathcal{H}_n),$$

where D is $k \times k$ diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and 0 otherwise. In [10], the authors defined

$$d_r(F, G) = \max\{\|\theta_G^* D \theta_F\| : D \in D_r\},$$

such that D_r is the set of all $k \times k$ diagonal matrices with r 1's and $(m - r)$ 0's, in which $|\Lambda| = r$. Moreover, $d_r(F, G)$ is the largest possible error when r-erasures occur. Indeed, G is called an optimal dual frame of F for 1-erasure if

$$d_1(F, G) = \min\{d_1(F, Y) : \text{Y is a dual of F}\}.$$

Inductively, for $r > 1$, a dual frame G is called an optimal dual of F for r-erasures if it is optimal for $(r - 1)$-erasures and

$$d_r(F, G) = \min\{d_r(F, Y) : \text{Y is a dual of F}\}.$$

Following [10], the rank-one operator $x \otimes y$ is defined as

$$(x \otimes y)(v) = \langle v, y \rangle x \qquad (\forall v \in \mathcal{H}_n). \tag{2}$$

Note that if $D \in D_1$ and $G = \{S_F^{-1}f_i + u_i\}_{i \in I_m}$ such that $U = \{u_i\}_{i \in I_m}$ satisfies (1), then for each $i \in I_m$ we have

$$\|\theta_G^* D \theta_F\| = \|(S_F^{-1}f_i + u_i) \otimes f_i\| = \|(S_F^{-1}f_i + u_i)\|\|f_i\|.$$

Thus, whenever 1-erasure optimal dual frames are considered, we always assume that $f_i \neq 0$, for all $i \in I_m$. We view U, as a vector in the orthogonal direct sum Hilbert space $\mathcal{H}_n^{(m)} := \mathcal{H}_n \oplus \cdots \oplus \mathcal{H}_n$ (m times).

In [9] and [10], the authors found conditions that the canonical dual is optimal dual for erasures. They used the operator norm as a measurement. Afterward, in [11] the authors used spectral radius for finding optimal dual for erasures. Furthermore, in [2] the numerical radius is applied. It should be noted that both spectral and numerical radius are seminorms. Therefore in this paper, we investigate the general form of seminorm, for finding optimal duals for erasures.

Let $\rho : \mathcal{B}(\mathcal{H}_n) \to [0, \infty)$ be an arbitrary seminorm. Define

$$d_r(F, Y) = \max\{\rho(E_\Lambda) : \ |\Lambda| = r\} = \max\{\rho(\theta_Y^* D\theta_F) : \ D \in D_r\},$$

$$d_1'(F, G) = \min\{d_1(F, Y) : Y \text{ is a dual of } F\},$$

and

$$d_r'(F, G) = \min\{d_r(F, Y) : Y \text{ is a dual of } F\},$$

We say G is an optimal dual for 1-erasure by the seminorm, (or a ρ-optimal dual for 1-erasure) when $d_1'(F, G) = d_1(F, G)$. Moreover, G is a ρ-optimal dual for r-erasures when it is a ρ-optimal dual for (r-1)-erasures, and $d_r'(F, G) = d_r(F, Y)$.

By (2), for 1-erasure we have

$$\max\{\rho(\theta_G^* D\theta_F)\} = \max\{\rho((S_F^{-1} f_i + u_i) \otimes f_i)\}.$$

This paper is organized as follows.

In section 2, the set of all ρ-optimal duals is studied. Mainly, we present the necessary or sufficient conditions, under which the canonical dual is a (unique) ρ-optimal dual or not. At the end of this section, the extreme points of the set of all ρ-optimal duals are investigated.

In section 3, we generalize some of the available results about arbitrary duals, for the case where general seminorms, under some circumstances, are the measurements.

In section 4, we provide some examples to illustrate our results, in the previous sections.

2. Seminorm canonical optimal dual

We denote by

$$D_{OF\rho} = \{G; G \text{ is a } \rho\text{-optimal dual of } F \text{ for erasures}\},$$

the set of all ρ-optimal dual frames for erasures.

We commence our results with a generalization of [10, Lemma 2.1], as follows.

Lemma 2.1. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and $F = \{f_i\}_{i=1}^m$ be a frame in $\mathcal{H}_n$, in which $f_i \neq 0$ for all $i \in I_m$. Then ρ-optimal dual frame for F exists for 1-erasure. Moreover, the set of all ρ-optimal dual frames of F for r-erasures is a convex, closed and bounded subset of $\mathcal{H}_n^{(m)}$.*

Proof. It is obvious that $D_{OF\rho}$ is nonempty. Moreover, the mapping

$$F(U) = \max\{\rho(\theta_G^* D\theta_F)\},$$

is continuous. It follows that $D_{OF\rho}$ is closed and bounded. We show that $D_{OF\rho}$ is convex. To this end, let

$$\max \rho(F, G_1) = \max \rho(F, G_2) = z,$$

for ρ-optimal duals G_1 and G_2. It is easily verified that $G = \lambda G_1 + (1 - \lambda)G_2$ ($\lambda \in [0, 1]$) is a dual of F. Furthermore, G is ρ-optimal, because

$$\begin{aligned} \rho(\theta_G^* D\theta_F) &= \rho(\lambda\theta_{G_1}^* D\theta_F + (1 - \lambda)\theta_{G_2}^* D\theta_F) \\ &\leq \rho(\theta_{G_1}^* D\theta_F) + (1 - \lambda)\rho(\theta_{G_2}^* D\theta_F) \\ &= \lambda z + (1 - \lambda)z = z. \end{aligned}$$

On the other hand, by the optimality of G_1 and G_2, we have

$$z \leq \max \rho(\theta_G^* D\theta_F).$$

Therefore

$$\max \rho(\theta_G^* D\theta_F) = \max \rho(F, G_1) = \max \rho(F, G_2).$$

□

By using induction, we obtain the next result, immediately.

Corollary 2.2. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$, in which $f_i \neq 0$ for all $i \in I_m$. Then ρ-optimal dual frames for F exist for any r-erasures. Moreover, the set of all ρ-optimal dual frames of F for r-erasures is a convex, closed and bounded subset of $\mathcal{H}_n^{(m)}$.*

Before proceeding to the next results, we introduce some notions that will be required. Let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$, with the frame operator S. Following [9], we have

$$c = \max \|S_F^{-1} f_i\|\|f_i\|, \quad \Lambda_1 = \{i \in I_m; \|S_F^{-1} f_i\|\|f_i\| = c\}, \quad \Lambda_2 = I_m - \Lambda_1,$$

and

$$H_1 = \operatorname{span}\{f_i\}_{i\in\Lambda_1}; \quad H_2 = \operatorname{span}\{f_i\}_{i\in\Lambda_2}.$$

Now suppose that ρ is a seminorm on $\mathcal{B}(\mathcal{H}_n)$. Let

$$c_\rho = \max \rho(S_F^{-1} f_i \otimes f_i), \quad \Lambda_1^\rho = \{i \in I_m; \rho(S_F^{-1} f_i \otimes f_i) = c\}, \quad \Lambda_2^\rho = I_m - \Lambda_1^\rho.$$

In the next theorem, we generalize [9, Theorem 1.1], for an arbitrary seminorm ρ on $\mathcal{B}(\mathcal{H}_n)$.

Theorem 2.3. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$, with the frame operator S_F. Suppose that ρ is a seminorm on $\mathcal{B}(\mathcal{H}_n)$ such that $\rho(\cdot) \leq \|\cdot\|$, $c_\rho = c$ and $\Lambda_1^\rho \subseteq \Lambda_1$. If the canonical dual is the unique ρ-optimal dual, then $H_1 \cap H_2 = \{0\}$ and $\{f_i\}_{i\in\Lambda_2^\rho}$ is linearly independent.*

Proof. Let $\{f_i\}_{i\in\Lambda_2^\rho}$ be linearly dependent. Thus there exists $u_i(i \in \Lambda_2^\rho$, not all zero) in H_2 such that

$$\sum_{i\in\Lambda_2^\rho} \langle f, f_i\rangle u_i = 0, \ (f \in \mathcal{H}).$$

Suppose that $u_i = 0$, for all $i \in \Lambda_1^\rho$ and $U := \{u_i\}_{i\in I_m}$. Consequently, $\theta_{tU}^* \theta_F = 0$, for any scalar t. It follows that $\{S_F^{-1} f_i + tu_i\}_{i\in I_m}$ is a dual of F. For each $i \in \Lambda_1^\rho$, we have

$$\rho(S_F^{-1} f_i + tu_i \otimes f_i) = \rho(S_F^{-1} f_i \otimes f_i) = c_\rho.$$

There exists $t > 0$, small enough such that $\rho(S_F^{-1} f_i + tu_i \otimes f_i) < c_\rho$. Indeed, for any $i \in \Lambda_2^\rho$ define

$$\begin{aligned} f_i &: \mathbb{R} \longrightarrow \mathbb{R} \\ f_i(t) &= \rho(S_F^{-1} f_i + tu_i \otimes f_i) \\ f_i(0) &= \rho(S_F^{-1} f_i \otimes f_i) < c_\rho. \end{aligned} \quad (3)$$

Since f_i $(i \in \Lambda_2^\rho)$, is continuous, there exists $\delta_i > 0$ such that $(-\delta_i, \delta_i) \subseteq f_i^{-1}((-\infty, c))$. It follows that $f_i(t) < c_\rho$ $(i \in \Lambda_2^\rho)$. Now for all $t \in \cap_{i\in\Lambda_2^\rho}(-\delta_i, \delta_i)$, we have

$$\rho(S_F^{-1} f_i + tu_i \otimes f_i) < c_\rho.$$

Thus $\{S_F^{-1}f_i + tu_i\}_{i\in I_m}$ is ρ-optimal, which is a contradiction. Now assume that $H_1 \cap H_2 \neq \{0\}$. There exists a nonzero element $x \in H_1 \cap H_2$ such that

$$x = \sum_{i\in\Lambda_1^\rho} c_i f_i = \sum_{i\in\Lambda_2^\rho} c_i f_i.$$

Since $H_1 = \text{span}\{f_i\}_{i\in\Lambda_1^\rho}$, thus there exist a linear independent set $\{f_{i_l}\}_{l=1}^j$, $(i_l \in \Lambda_1^\rho)$ and also nonzero constants $\{c_{i_l}\}_{l=1}^j$ such that

$$\sum_{j=1}^{l} c_{i_j} f_{i_j} + \sum_{i\in\Lambda_2^\rho} c_i f_i = 0.$$

Moreover, since $\{f_{i_l}\}_{l=1}^j$ $((i_l) \in \Lambda_1^\rho)$ is linearly independent, there exists also $h \in \mathcal{H}_n$ such that $\langle S^{-1}(\overline{c_{i_j}} f_{i_j}), h\rangle < 0$, based on Proof of [1, Proposition 2.2]. Suppose that $\Omega = \{i_l\}_{l=1}^j \cup \Lambda_2^\rho$ and $u_i = 0$, for $i \notin \Omega$. Let $u_i = \overline{c_i}h$, for $i \in \Omega$ and $U := \{u_i\}_{i=1}^m$. Then by some simple calculations we obtain $\theta_U^*\theta_F = 0$. By the same way as (3), one can find $t > 0$, small enough such that for any $i \in \Lambda_2^\rho$, $\rho((S_F^{-1}f_i + tu_i) \otimes f_i) < c_\rho$. For each $i \in \Lambda_1^\rho - \{i_l\}_{l=1}^j$, we have

$$\rho((S_F^{-1}f_i + tu_i) \otimes f_i) = \rho((S_F^{-1}f_i) \otimes f_i) = c_\rho.$$

Again, there exists $t > 0$ small enough such that for all $i \in \{i_l\}_{l=1}^j$,

$$\begin{aligned}
(\rho((S_F^{-1}f_i + tu_i) \otimes f_i))^2 &\leq \|(S_F^{-1}f_i + tu_i)\|^2\|f_i\|^2 \\
&= (\|S_F^{-1}f_i\|\|f_i\|)^2 + \left(t^2\|u_i\|^2 + 2t\langle S_F^{-1}f_i, u_i\rangle\right)\|f_i\|^2 \\
&= c^2 + \left(t^2\|u_i\|^2 + 2t\langle S_F^{-1}f_i, u_i\rangle\right)\|f_i\|^2 \\
&< c^2 = (c_\rho)^2.
\end{aligned}$$

Therefore $\rho((S_F^{-1}f_i + tu_i) \otimes f_i) \leq c_\rho$, for all $i \in I_m$ and so the claim is achieved. □

The next result is a generalization of Proposition 3.1 from [9].

Proposition 2.4. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$ $(m > n)$, with the frame operator S. Suppose ρ is an arbitrary seminorm on $\mathcal{B}(\mathcal{H}_n)$ such that $\{f_i\}_{i\in\Lambda_1^\rho}$ is linearly independent and $H_1 \cap H_2 = \{0\}$. Then the canonical dual is a ρ-optimal dual, but not unique.*

Proof. Assume that $\{g_i\}_{i\in I_m} = \{S_F^{-1}f_i + u_i\}_{i\in I_m}$ is an arbitrary dual for F. Thus (1) holds and so for all $f \in \mathcal{H}_n$

$$\sum_{i\in\Lambda_1^\rho} \langle f, u_i\rangle f_i + \sum_{i\in\Lambda_2^\rho} \langle f, u_i\rangle f_i = 0.$$

Since $\{f_i\}_{i\in\Lambda_1^\rho}$ is linearly independent, then for all $i \in \Lambda_1^\rho$ and $f \in \mathcal{H}_n$, $\langle f, u_i\rangle = 0$, which implies that $u_i = 0$, $(i \in \Lambda_1^\rho)$. Moreover, we have

$$\begin{aligned}
\max_{i\in I_m} \rho((S_F^{-1}f_i + u_i) \otimes f_i) &\geq \max_{i\in\Lambda_1^\rho} \rho((S_F^{-1}f_i + u_i) \otimes f_i) \\
&= \max_{i\in\Lambda_1^\rho} \rho(S_F^{-1}f_i \otimes f_i) \\
&= \max_{i\in I_m} \rho(S_F^{-1}f_i \otimes f_i).
\end{aligned}$$

Therefore $\{S_F^{-1}f_i\}_{i\in I_m}$ is a ρ-optimal dual for 1-erasure and so for any r-erasures. Now we prove it is not unique. Since $m > n$, there exists a dual $\{S_F^{-1}f_i + u_i\}_{i\in I_m}$ for F. Based on the previous part of the proof $u_i \neq 0$, for some $i \in \Lambda_2^\rho$. There exists $t > 0$, small enough such that for any $i \in \Lambda_2^\rho$,

$$\rho((S_F^{-1}f_i + tu_i) \otimes f_i) < c_\rho. \tag{4}$$

Indeed, for all $i \in \Lambda_2^\rho$, define

$$\begin{aligned} f_i &: \mathbb{R} \longrightarrow \mathbb{R} \\ f_i(t) &= \rho(S_F^{-1} f_i + tu_i \otimes f_i) \\ f_i(0) &= \rho(S_F^{-1} f_i \otimes f_i) < c_\rho, \end{aligned} \tag{5}$$

and by the same way in the proof of Theorem 2.3, one can find desired $t > 0$, satisfying (4). Furthermore, for all $i \in \Lambda_1^\rho$ we have

$$\rho((S_F^{-1} f_i + tu_i) \otimes f_i) = \rho(S_F^{-1} f_i \otimes f_i) = c_\rho.$$

Therefore $\{S_F^{-1} f_i + tu_i\}_{i \in I_m}$ is also ρ-optimal. □

Proposition 3.2 in [9], leads us to the following result.

Proposition 2.5. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$, $(m > n)$, with the frame operator S and ρ be a seminorm on $\mathcal{B}(\mathcal{H}_n)$ such that $\rho(\cdot) \leq \|\cdot\|$. Suppose that $\{f_i\}_{i \in \Lambda_1^\rho}$ is linearly independent and there exists a sequence of scalars $\{c_i\}_{i \in I_m}$ such that $\sum_{i \in I_m} c_i x_i = 0$, $\{c_i\}_{i \in \Lambda_1^\rho} \neq 0$, $c_\rho = c$ and $\Lambda_1^\rho \subseteq \Lambda_1$. Then the canonical dual is not ρ-optimal for any erasures.*

Proof. Since $\{f_i\}_{i \in \Lambda_1^\rho}$ is linearly independent, there exists $h \in \mathcal{H}_n$ such that $\langle S_F^{-1}(c_i f_i), h\rangle < 0$. Let $u_i = \overline{c_i} h$. Then by using $\sum_{i \in I_m} c_i x_i = 0$ and some simple calculations, we obtain for any arbitrary t that $\theta_{tU}^* \theta_F = 0$. There exists $t > 0$, small enough such that for all $i \in \Lambda_1^\rho$, we have

$$\rho((S_F^{-1} f_i + tu_i) \otimes f_i) < c_\rho.$$

Indeed, by the assumption we have

$$\begin{aligned} (\rho((S_F^{-1} f_i + tu_i) \otimes f_i))^2 &\leq (\|S_F^{-1} f_i + tu_i\| \|f_i\|)^2 \\ &= c^2 + \left(t^2 \|u_i\|^2 + 2t \langle S_F^{-1} f_i, u_i\rangle\right) \|f_i\|^2 \\ &= (c_\rho)^2 + \left(t^2 \|u_i\|^2 + 2t \langle S_F^{-1} f_i, u_i\rangle\right) \|f_i\|^2 \\ &< (c_\rho)^2. \end{aligned}$$

Now for each $i \in \Lambda_2^\rho$, define f_i exactly similar to the ones, given in (5) and by the same way, one can show that there exists t such that for any $i \in \Lambda_2^\rho$, $\rho((S_F^{-1} f_i + tu_i) \otimes f_i) < c_\rho$. Therefore there exists the sequence $\{u_i\}_{i \in I_m}$ ($\{u_i\} \neq 0$) such that $\{S_F^{-1} f_i + tu_i\}_{i \in I_m}$ is ρ-optimal. □

Similar to [9, Corollary 3.3], we have the next result.

Corollary 2.6. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}_n$, $(m = n + 1)$, with the frame operator S. Suppose that Λ_1^ρ has only one element and ρ is an arbitrary seminorm on $\mathcal{B}(\mathcal{H}_n)$. If $\{f_i\}_{i \in \Lambda_2^\rho}$ is linearly independent, then the canonical dual is not ρ-optimal for any erasures.*

2.1. Extreme points of the set of ρ-optimal duals

In Lemma 2.1, it was shown that $D_{OF\rho}$ is a nonempty convex and compact set. Thus by the Kerin-Milman theorem, ext$(D_{OF\rho})$ is nonempty and $D_{OF\rho}$ is the closed convex hull of its extreme points. Indeed, $D_{OF\rho} = \overline{co}(ext(D_{OF\rho}))$. In this section, we generalize [1, Theorem 3.2], for some seminorms on $\mathcal{B}(\mathcal{H}_n)$, as a measurement.

Theorem 2.7. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i \in I_m}$ be a frame for $\mathcal{H}_n$, ρ be a seminorm on $\mathcal{B}(\mathcal{H}_n)$ and the canonical dual be a ρ-optimal dual for F. If $\{S_F^{-1} f_i\}_{i \in I_m} \in ext(D_{OF\rho})$, then $\{f_i\}_{i \in \Lambda_2^\rho}$ is linearly independent. Moreover, if there exists $M > 1$ such that $M\|\cdot\| \leq \rho(\cdot)$ and also $c_\rho = c$,$\Lambda_1^\rho \subseteq \Lambda_1$, and $\{f_i\}_{i \in \Lambda_2^\rho}$ is linearly independent, then $\{S_F^{-1} f_i\}_{i \in I_m} \in ext(D_{OF\rho})$.*

Proof. Let $\{S_F^{-1}f_i\}_{i\in I_m}\in ext(D_{OF\rho})$ and $\{f_i\}_{i\in\Lambda_2^\rho}$ be linearly dependent. Then there exists the sequence $\{u_i\}_{i\in\Lambda_2^\rho}$, not all zero such that

$$\sum_{i\in\Lambda_2^\rho}\langle f,f_i\rangle u_i=0 \qquad (f\in\mathcal{H}_n).$$

Let $u_i=0$, for $i\in\Lambda_1^\rho$ and $U:=\{u_i\}_{i\in I_m}$. Thus for any $t>0$, we have

$$\max_{i\in\Lambda_1^\rho}\rho((S_F^{-1}f_i+tu_i)\otimes f_i)=\max_{i\in\Lambda_1^\rho}\rho(S_F^{-1}f_i\otimes f_i)=c_\rho.$$

Take $t>0$, small enough such that

$$\max_{i\in\Lambda_2^\rho}\rho((S_F^{-1}f_i+tu_i)\otimes f_i)<c_\rho \text{ and } \max_{i\in\Lambda_2^\rho}\rho((S_F^{-1}f_i-tu_i)\otimes f_i)<c_\rho.$$

Indeed, define

$$\begin{aligned} f_i &: \mathbb{R}\longrightarrow\mathbb{R}\\ f_i(t) &= \rho(S_F^{-1}f_i+tu_i\otimes f_i)\\ f_i(0) &= \rho(S_F^{-1}f_i\otimes f_i)<c_\rho,\ \ (i\in\Lambda_2^\rho). \end{aligned}$$

Thus $0\in f_i^{-1}(-\infty,c_\rho)$ and consequently there exists $\delta_i>0$ such that $(-\delta_i,\delta_i)\subseteq f_i^{-1}(-\infty,c_\rho)$. For each $t\in\cap_{i\in\Lambda_2^\rho}(-\delta_i,\delta_i)$, we have $f_i(t)<c_\rho$, for all $i\in\Lambda_2^\rho$. Moreover, define

$$\begin{aligned} g_i &: \mathbb{R}\longrightarrow\mathbb{R}\\ g_i(t) &= \rho(S_F^{-1}f_i-tu_i\otimes f_i)\\ g_i(0) &= \rho(S_F^{-1}f_i\otimes f_i)<c_\rho,\ \ (i\in\Lambda_2^\rho). \end{aligned}$$

By the same way, there exists $\gamma_i>0$ such that $(-\gamma_i,\gamma_i)\subseteq g_i^{-1}(-\infty,c_\rho)$. For all $t\in\cap_{i\in\Lambda_2^\rho}(-\delta_i,\delta_i)\bigcap\cap_{i\in\Lambda_2^\rho}(-\gamma_i,\gamma_i)$, the desired inequalities are satisfied. Consequently, $\{S_F^{-1}f_i+tu_i\}_{i\in I_m}$, and $\{S_F^{-1}f_i-tu_i\}_{i\in I_m}$ are both ρ-optimal. Furthermore,

$$S_F^{-1}f_i=\frac{S_F^{-1}f_i+tu_i+S_F^{-1}f_i-tu_i}{2},$$

which is a contradiction.

Now suppose that $\{S_F^{-1}f_i\}_{i\in I_m}\notin ext(D_{OF\rho})$ and $\{f_i\}_{i\in\Lambda_2^\rho}$ is linearly independent. Thus there exist $\{g_i\}_{i\in I_m},\{h_i\}_{i\in I_m}\in D_{OF\rho}$ such that

$$g_i=S_F^{-1}f_i+u_i, h_i=S_F^{-1}f_i+v_i \quad\text{and}\quad S_F^{-1}f_i=\frac{g_i+h_i}{2}.$$

Note that $\{u_i\}_{i\in I_m}$ and $\{v_i\}_{i\in I_m}$ satisfy (1). It follows that $u_i=-v_i$. Moreover,

$$\max_{i\in I_m}\rho((S_F^{-1}f_i+u_i)\otimes f_i)=\max_{i\in I_m}\rho((S_F^{-1}f_i-u_i)\otimes f_i)=c_\rho.$$

By the hypothesis, there exists a $M>1$ such that

$$M\|S_F^{-1}f_i+u_i\otimes f_i\|\leq\rho(S_F^{-1}f_i+u_i\otimes f_i),$$

and

$$M\|S_F^{-1}f_i-u_i\otimes f_i\|\leq\rho(S_F^{-1}f_i-u_i\otimes f_i).$$

Thus

$$\begin{aligned} (\|S_F^{-1}f_i+u_i\|\|f_i\|)^2 &\leq \frac{(c_\rho)^2}{M^2},\\ (\|S_F^{-1}f_i-u_i\|\|f_i\|)^2 &\leq \frac{(c_\rho)^2}{M^2}. \end{aligned}$$

Thus for all $i \in \Lambda_1 \supseteq \Lambda_1^{\rho}$, we have $(c_\rho)^2 = c^2 = \|S_F^{-1} f_i\|^2 \|f_i\|^2$ and so

$$\begin{aligned} \|u_i\|^2 - 2Re\langle S_F^{-1} f_i, u_i\rangle &\leq \frac{(c_\rho)^2}{\|f_i\|^2}(\frac{1}{M^2} - 1) < 0, \\ \|u_i\|^2 - 2Re\langle S_F^{-1} f_i, u_i\rangle &\leq \frac{(c_\rho)^2}{\|f_i\|^2}(\frac{1}{M^2} - 1) < 0. \end{aligned}$$

Consequently by [10, lemma 2.7], $u_i = 0$ for all $i \in \Lambda_1^{\rho}$ and so

$$0 = \sum_{i\in\Lambda_2^{\rho}} \langle f, u_i - v_i\rangle f_i,$$

which is in contradiction with the linear independence of $\{f_i\}_{i\in\Lambda_2^{\rho}}$. □

3. Seminorm arbitrary optimal duals

In [1], the authors generalized some results, related to canonical duals for arbitrary duals. In this section, we investigate and generalize Proposition 2.1, and Proposition 2.2 of [1], for arbitrary ρ-optimal duals. First, we recall some standard notions.

Let $F = \{f_i\}_{i\in I_m}$ be a frame for $\mathcal{H}_n$ and $G = \{g_i\}_{i\in I_m}$ be an arbitrary dual of F. Then

$$\Lambda_1^g = \{i \in I_m : \|g_i\|\|f_i\| = c^g\}, \quad \Lambda_2^g = I_m - \Lambda_1$$

and

$$H_1 = \operatorname{span}\{f_i\}_{i\in\Lambda_1^g}, \quad H_2 = \operatorname{span}\{f_i\}_{i\in\Lambda_2^g}.$$

Now for any seminorm ρ, let

$$(\Lambda_1^g)^{\rho} = \{i \in I_m : \rho(g_i \otimes f_i) = c_\rho^g\}, \quad (\Lambda_2^g)^{\rho} = I_m - (\Lambda_1^g)^{\rho},$$

and

$$H_1^g = \operatorname{span}\{f_i\}_{i\in(\Lambda_1^g)^{\rho}}, \quad H_2^g = \operatorname{span}\{f_i\}_{i\in(\Lambda_2^g)^{\rho}}.$$

The following proposition is a generalization of Proposition 2.4, likewise [1, Proposition 2.1] which generalizes [9, Proposition 3.1].

Proposition 3.1. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i\in I_m}$ be a frame for $\mathcal{H}_n$, $(m > n)$, $G = \{g_i\}_{i\in I_m}$ be a dual of F and ρ be an arbitrary seminorm on $\mathcal{B}(\mathcal{H}_n)$. Suppose that $\{f_i\}_{i\in(\Lambda_1^g)^{\rho}}$ is linearly independent such that $H_1^g \cap H_2^g = \{0\}$. Then G is a ρ-optimal dual for F for erasures but not unique.*

Proof. We follow some arguments, similar to [1, Proposition 2.1]. Assume that $G = \{S^{-1} f_i + v_i\}_{i\in I_m}$, where $\{v_i\}_{i\in I_m}$ satisfies (1). Hence for all $f \in \mathcal{H}_n$,

$$\sum_{i\in(\Lambda_1^g)^{\rho}} \langle f, f_i\rangle u_i + \sum_{i\in(\Lambda_2^g)^{\rho}} \langle f, f_i\rangle u_i = 0.$$

Since $H_1^g \cap H_2^g = 0$, thus

$$\sum_{i\in(\Lambda_1^g)^{\rho}} \langle f, f_i\rangle u_i = 0 = \sum_{i\in(\Lambda_2^g)^{\rho}} \langle f, f_i\rangle u_i = 0.$$

By the assumption, $\{f_i\}_{i\in(\Lambda_1^g)^\rho}$ is linearly independent. It follows that $\langle f, v_i\rangle = 0$ $(f \in \mathcal{H}_n)$ and so $v_i = 0$, for all $i \in (\Lambda_1^g)^\rho$. Thus for any arbitrary sequence $\{h_i\}_{i\in I_m}$, satisfying (1), we have $h_i = 0$ $(i \in (\Lambda_1^g)^\rho)$. Consequently,

$$\begin{aligned}\max_{i\in I_m}\rho((S_F^{-1}f_i + h_i)\otimes f_i) &= \max_{i\in I_m}\rho((S_F^{-1}f_i + h_i - v_i + v_i)\otimes f_i)\\ &\geq \max_{i\in(\Lambda_1^g)^\rho}\rho((S_F^{-1}f_i + h_i - v_i + v_i)\otimes f_i)\\ &= \max_{i\in(\Lambda_1^g)^\rho}\rho((S_F^{-1}f_i + v_i)\otimes f_i)\\ &= \max_{i\in I_m}\rho((S_F^{-1}f_i + v_i)\otimes f_i).\end{aligned}$$

Thus G is a ρ-optimal dual. Now, suppose that $\{S_F^{-1}f_i + u_i\}_{i\in I_m}$ is a dual frame for F such that $u_i \neq 0$, for all $i \in (\Lambda_2^g)^\rho$. There exists $\epsilon > 0$ small enough such that for any $i \in (\Lambda_2^g)^\rho$, $\rho((S_F^{-1}f_i + v_i + \epsilon u_i)\otimes f_i) < c_\rho^g$. Indeed, define

$$\begin{aligned}f_i &: \mathbb{R}\longrightarrow\mathbb{R}\\ f_i(t) &= \rho(S_F^{-1}f_i + v_i + \epsilon u_i\otimes f_i)\\ f_i(0) &= \rho(S_F^{-1}f_i + v_i\otimes f_i) < c_\rho^g,\ \ (i \in (\Lambda_2^g)^\rho),\end{aligned}$$

and follow the same arguments similar to the proof of Theorem 2.3. Thus for all $i \in (\Lambda_1^g)^\rho$, we obtain

$$\rho(S_F^{-1}f_i + v_i + \epsilon u_i\otimes f_i) = \rho(S_F^{-1}f_i\otimes f_i) = c_\rho^g.$$

Therefore $\{S_F^{-1}f_i + v_i + \epsilon u_i\}_{i\in I_m}$ is also ρ-optimal and so the proof is completed. □

In the following proposition, $\{g_i\}_{i\in(\Lambda_1^g)^\rho}$ is linearly independent instead of $\{f_i\}_{i\in(\Lambda_1^g)^\rho}$. In [1], the authors generalized Proposition 2.2, for the general dual frames. In the next proposition, we generalize Proposition 2.5, similarly.

Proposition 3.2. *Let $\mathcal{H}_n$ be a finite dimensional Hilbert space with the dimension n, and let $F = \{f_i\}_{i\in I_m}$ be a frame for $\mathcal{H}_n$, $G = \{g_i\}_{i\in I_m}$ be a dual of F and ρ be a seminorm on $\mathcal{B}(\mathcal{H}_n)$ such that $\rho(\cdot) \leq \|\cdot\|$. Suppose that $\{g_i\}_{i\in(\Lambda_1^g)^\rho}$ is linearly independent and there exists a sequence of scalars $\{c_i\}_{i\in I_m}$ such that $\sum_{i\in I_m} c_i x_i = 0$, $\{c_i\}_{i\in(\Lambda_1^g)^\rho} \neq 0$, $c^g = c_\rho^g$ and $(\Lambda_1^g)^\rho \subseteq \Lambda_1^g$. Then G is not a ρ-optimal dual for any erasures.*

Proof. We have

$$\{g_i\}_{i\in I_m} = \{S_F^{-1}f_i + u_i\}_{i\in I_m},$$

such that $\{u_i\}_{i\in I_m}$ satisfies (1). Since $\{g_i\}_{i\in(\Lambda_1^g)^\rho}$ is linearly independent, there exists $h \in \mathcal{H}_n$ such that for all $i \in (\Lambda_1^g)^\rho$, we have $\langle c_i g_i, h\rangle < 0$. Now let $v_i = t\overline{c_i}h + u_i$, $(i \in I_m, t \in \mathbb{R})$. Thus $\{S_F^{-1}f_i + v_i\}_{i\in I_m}$ is a dual of F. There exists $t > 0$, small enough such that $\rho(S_F^{-1}f_i + v_i\otimes f_i) < c_\rho^g$, for all $i \in (\Lambda_1^g)^\rho$. Indeed,

$$\begin{aligned}(\rho((S_F^{-1}f_i + v_i)\otimes f_i))^2 &\leq (\|S_F^{-1}f_i + v_i\|\|f_i\|)^2\\ &= (\|g_i + t\overline{c_i}h\|\|f_i\|)^2\\ &= (c^g)^2 + \left(t^2\|\overline{c_i}h\|^2 + 2t\langle g_i, \overline{c_i}h\rangle\right)\|f_i\|^2\\ &< (c_\rho^g)^2.\end{aligned}$$

Now for each $i \in (\Lambda_2^g)^\rho$, define

$$\begin{aligned}f_i &: \mathbb{R}\longrightarrow\mathbb{R}\\ f_i(t) &= \rho(S_F^{-1}f_i + t\overline{c_i}h + u_i\otimes f_i)\\ f_i(0) &= \rho(S_F^{-1}f_i + u_i\otimes f_i)\\ &= \rho(g_i\otimes f_i) < c_\rho^g.\end{aligned}$$

Again one can find t such that $\rho(S_F^{-1}f_i + v_i) < c_\rho^g$ $(i \in I_m)$. Therefore $\{g_i\}_{i\in I_m}$ is not a ρ-optimal dual. $\square$

4. Examples

In this section, we provide some examples, for clarifying our results.

Example 4.1. *Let $\mathcal{H} = \mathbb{R}^2$ and $F = \{f_i\}_{i=1}^3 = \{e_1, e_2, e_2\}$ be a frame. Then*

$$S_F = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow S_F^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

and

$$S_F^{-1}f_i = \left\{e_1, \frac{1}{2}e_2, \frac{1}{2}e_2\right\}.$$

Define

$$\begin{aligned} \rho \quad &: \quad \mathcal{B}(\mathbb{R}^2) \to [0, \infty) \\ \rho(T) \quad &= \quad \|T(e_1)\|_2. \end{aligned}$$

It is obvious that ρ is a seminorm on $\mathcal{B}(\mathbb{R}^2)$. Consequently,

$$\left\{\rho(S_F^{-1}f_i \otimes f_i)\right\}_{i=1}^3 = \{1, 0, 0\}.$$

It is clear that

$$c_\rho = 1, \quad \Lambda_1^\rho = \{i = 1\}, \quad \Lambda_2^\rho = \{i = 2, 3\},$$

Moreover, $\{f_i\}_{i\in\Lambda_1^\rho} = \{e_1\}$ is linearly independent and $H_1 \cap H_2 = \{0\}$. Therefore $\{S_F^{-1}f_i\}_{i=1}^3$ is a ρ-optimal dual but not unique, by Proposition 2.4. We also check it again as follows. Let $\{g_i\}_{i=1}^3 = \{S_F^{-1}f_i + u_i\}_{i=1}^3$ be a dual of F. Then $\{u_i\}_{i=1}^3$ satisfies (1) *and we have*

$$\begin{aligned} \{u_i\}_{i=1}^3 \quad &= \quad \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -a \\ -b \end{bmatrix}\right\}, \\ \text{and} \quad \{S_F^{-1}f_i + u_i\}_{i=1}^3 \quad &= \quad \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ \frac{1}{2} + b \end{bmatrix}, \begin{bmatrix} -a \\ \frac{1}{2} - b \end{bmatrix}\right\}. \end{aligned}$$

By some simple calculations, we obtain for all $a, b \in \mathbb{R}$ that

$$\max\{\rho((S_F^{-1}f_i + u_i) \otimes f_i)\}_{i=1}^3 = \max\{\rho(S_F^{-1}f_i \otimes f_i)\}_{i=1}^3 = 1.$$

Therefore $\{S_F^{-1}f_i\}_{i=1}^3$ is optimal but not unique, as we expected.

We provide an example, for illuminating Proposition 2.5.

Example 4.2. *Let $\mathcal{H} = \mathbb{R}^2$ and $F = \{f_i\}_{i=1}^4 = \{e_1, e_2, e_2, e_2 - e_1\}$ be a frame. Then*

$$S_F = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow S_F^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

and

$$S_F^{-1}f_i = \left\{\frac{3}{5}e_1 + \frac{1}{5}e_2, \frac{1}{5}e_1 + \frac{2}{5}e_2, \frac{1}{5}e_1 + \frac{2}{5}e_2, \frac{-2}{5}e_1 + \frac{1}{5}e_2\right\}.$$

Suppose that ρ is defined, as in Example 4.1. Thus

$$\left\{\rho(S_F^{-1}f_i\otimes f_i)\right\}_{i=1}^4=\left\{\sqrt{\frac{2}{5}},0,0,\sqrt{\frac{1}{5}}\right\}.$$

It is clear that

$$c_\rho=\sqrt{\frac{2}{5}},\quad \Lambda_1^\rho=\{i=1\},\quad \Lambda_2^\rho=\{i=2,3,4\},$$

and $\{f_i\}_{i\in\Lambda_1^\rho}=\{e_1\}$ is linearly independent. If $\{c_i\}_{i=1}^4=\{1,0,-1,1\}$, then

$$\sum_{i=1}^4 c_if_i=0,\ \textit{and}\ \{c_i\}_{i\in\Lambda_1^\rho}=c_1\neq 0.$$

Moreover,

$$\left\{\|S_F^{-1}f_i\|\|f_i\|\right\}_{i=1}^4=\left\{\sqrt{\frac{2}{5}},\sqrt{\frac{1}{5}},\sqrt{\frac{1}{5}},\sqrt{\frac{1}{5}}.\sqrt{2}\right\}.$$

Thus

$$c=\sqrt{\frac{2}{5}}=c_\rho\ \textit{and}\ \Lambda_1=\{i=1,4\}\supseteq\Lambda_1^\rho.$$

Therefore $\{S_F^{-1}f_i\}_{i=1}^4$ is not a ρ-optimal dual by Proposition 2.5, but we again check it for clarification. Let $\{g_i\}_{i=1}^4=\{S_F^{-1}f_i+u_i\}_{i=1}^4$ be a dual of F. Then $\{u_i\}_{i=1}^4$ satisfies (1) and we have

$$\begin{aligned}\{u_i\}_{i=1}^3 &= \left\{\begin{bmatrix}a\\ b\end{bmatrix},\begin{bmatrix}-\frac{a}{2}\\ -\frac{b}{2}\end{bmatrix},\begin{bmatrix}-\frac{a}{2}\\ -\frac{b}{2}\end{bmatrix},\begin{bmatrix}a\\ b\end{bmatrix}\right\},\\ \textit{and}\quad \{S_F^{-1}f_i+u_i\}_{i=1}^4 &= \left\{\begin{bmatrix}\frac{3}{5}+a\\ \frac{1}{5}+b\end{bmatrix},\begin{bmatrix}\frac{1}{5}-\frac{a}{2}\\ \frac{2}{5}-\frac{b}{2}\end{bmatrix},\begin{bmatrix}\frac{1}{5}-\frac{a}{2}\\ \frac{2}{5}-\frac{b}{2}\end{bmatrix},\begin{bmatrix}-\frac{2}{5}+a\\ \frac{1}{5}+b\end{bmatrix}\right\}.\end{aligned}$$

Simple calculations imply that for all $a,b\in\mathbb{R}$

$$\{\rho((S_F^{-1}f_i+u_i)\otimes f_i)\}_{i=1}^4=\left\{\sqrt{(\frac{3}{5}+a)^2+(\frac{1}{5}+b)^2},0,0,\sqrt{(\frac{-2}{5}+a)^2+(\frac{1}{5}+b)^2}\right\}.$$

If $a=0,b=-\frac{1}{5}$, then

$$\max\{\rho((S_F^{-1}f_i+u_i)\otimes f_i)\}_{i=1}^4=\frac{3}{5}<c_\rho.$$

Thus $\{S_F^{-1}f_i\}_{i=1}^4$ is not a ρ-optimal dual.

Remark 4.3. *It is worth noting that c_ρ is not always equal to c. For instance in the last example, define*

$$\begin{aligned}\rho &: \mathcal{B}(\mathbb{R}^2)\to[0,\infty)\\ \rho(T) &= \|T(e_2)\|_2.\end{aligned}$$

It is obvious that ρ is a seminorm on $\mathcal{B}(\mathbb{R}^2)$. Moreover,

$$\{\rho(S_F^{-1}f_i\otimes f_i)\}_{i=1}^4=\left\{0,\sqrt{\frac{1}{5}},\sqrt{\frac{1}{5}},\sqrt{\frac{1}{5}}\right\}.$$

Therefore $c_\rho=\sqrt{\frac{1}{5}}$, whereas $c=\sqrt{\frac{2}{5}}$.

One may ask that whether the conditions $c_\rho = c$ and $\Lambda_1^\rho \subseteq \Lambda_1$ are necessary and sufficient. This question will be answered in the following example.

Example 4.4. *Let $\mathcal{H} = \mathbb{R}^2$, and $F = \{f_i\}_{i=1}^4 = \{2e_1, \sqrt{2}e_2, e_2\}$ be a frame. Then*

$$S_F = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow S_F^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and so

$$S_F^{-1} f_i = \left\{ \frac{1}{2} e_1, \frac{\sqrt{2}}{3} e_2, \frac{1}{3} e_2 \right\}.$$

Suppose that ρ is defined, as in Remark 4.3. Thus

$$\left\{ \rho(S_F^{-1} f_i \otimes f_i) \right\}_{i=1}^3 = \left\{ 0, \frac{2}{3}, \frac{1}{3} \right\}.$$

It is clear that

$$c_\rho = \frac{2}{3}, \quad \Lambda_1^\rho = \{i = 2\}, \quad \Lambda_2^\rho = \{i = 1, 3\}.$$

Moreover, $\{f_i\}_{i \in \Lambda_1^\rho} = \{\sqrt{2}e_2\}$ is linearly independent. If $\{c_i\}_{i=1}^3 = \{0, 1, -\sqrt{2}\}$, then

$$\sum_{i=1}^3 c_i f_i = 0, \text{ and } \{c_i\}_{i \in \Lambda_1^\rho} = c_2 \neq 0$$

and

$$c = \max \left\{ \|S^{-1} f_i\| \|f_i\| \right\}_{i=1}^3 = 1 \neq c_\rho \text{ and } \Lambda_1 = \{i = 1\} \neq \Lambda_1^\rho.$$

Let $\{g_i\}_{i=1}^3 = \{S_F^{-1} f_i + u_i\}_{i=1}^3$ be a dual of F. Then $\{u_i\}_{i=1}^3$ satisfies (1) *and so we obtain*

$$\begin{aligned} \{u_i\}_{i=1}^3 &= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -\sqrt{2}a \\ -\sqrt{2}b \end{bmatrix} \right\}, \\ \text{and} \quad \{S_F^{-1} f_i + u_i\}_{i=1}^3 &= \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b + \frac{\sqrt{2}}{3} \end{bmatrix}, \begin{bmatrix} -\sqrt{2}a \\ -\sqrt{2}b + \frac{1}{3} \end{bmatrix} \right\}. \end{aligned}$$

By simple calculations one can get for all $a, b \in \mathbb{R}$ that

$$\{\rho((S_F^{-1} f_i + u_i) \otimes f_i)\}_{i=1}^3 = \left\{ 0, \sqrt{2} \sqrt{a^2 + (b + \frac{\sqrt{2}}{3})^2}, \sqrt{2a^2 + (-\sqrt{2}b + \frac{1}{3})^2} \right\}.$$

For example if $a = 0, b = -\frac{1}{4\sqrt{2}}$, then

$$\max\{\rho((S_F^{-1} f_i + u_i) \otimes f_i)\}_{i=1}^3 < c_\rho.$$

Thus $\{S_F^{-1} f_i\}_{i=1}^3$ is not a ρ-optimal dual. Therefore, although the conditions $c_\rho = c$ and $\Lambda_1 = \Lambda_1^\rho$ are sufficient, they are not necessary.

The following example emphasizes that all cases of equality or inequality, occur for Λ_1 and Λ_1^ρ, or c and c_ρ.

Example 4.5. *Let $\mathcal{H} = \mathbb{R}^2$, and $F = \{f_i\}_{i=1}^4 = \{e_1, e_2, e_2, e_2 - 2e_1\}$ be a frame. Then*

$$S_F = \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix} \Rightarrow S_F^{-1} = \begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{5}{11} \end{bmatrix}$$

and so

$$S_F^{-1} f_i = \left\{ \frac{3}{11}e_1 + \frac{2}{11}e_2, \frac{2}{11}e_1 + \frac{5}{11}e_2, \frac{2}{11}e_1 + \frac{5}{11}e_2, \frac{-4}{11}e_1 + \frac{1}{11}e_2 \right\}.$$

Suppose that ρ is the seminorm, defined as in Example 4.1. Thus

$$\left\{\rho(S_F^{-1} f_i \otimes f_i)\right\}_{i=1}^4 = \left\{ \frac{\sqrt{13}}{11}, 0, 0, 2\frac{\sqrt{17}}{11} \right\}.$$

It is easily verified that

$$c_\rho = 2\frac{\sqrt{17}}{11}, \quad \Lambda_1^\rho = \{i = 4\}, \quad \Lambda_2^\rho = \{i = 1, 2, 3\}.$$

Moreover, $\{f_i\}_{i \in \Lambda_1^\rho} = \{e_2 - 2e_1\}$ is linearly independent. In the case where $\{c_i\}_{i=1}^4 = \{2, -1, 0, 1\}$, we have

$$\sum_{i=1}^4 c_i f_i = 0, \text{ and } \{c_i\}_{i \in \Lambda_1^\rho} = c_4 \neq 0$$

and

$$c = \max\left\{\|S_F^{-1} f_i\| \|f_i\|\right\}_{i=1}^4 = \frac{\sqrt{85}}{11} \neq c_\rho \text{ and } \Lambda_1 = \{i = 4\} \supseteq \Lambda_1^\rho.$$

Let $\{g_i\}_{i=1}^4 = \{S_F^{-1} f_i + u_i\}_{i=1}^4$ be a dual of F. Then $\{u_i\}_{i=1}^4$ satisfies (1) *and we obtain*

$$\begin{aligned} \{u_i\}_{i=1}^3 &= \left\{ \begin{bmatrix} 2a \\ 2b \end{bmatrix}, \begin{bmatrix} -\frac{a}{2} \\ -\frac{b}{2} \end{bmatrix}, \begin{bmatrix} -\frac{a}{2} \\ -\frac{b}{2} \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\}, \\ \text{and} \quad \{S^{-1} f_i + u_i\}_{i=1}^4 &= \left\{ \begin{bmatrix} \frac{3}{11} + 2a \\ \frac{2}{11} + 2b \end{bmatrix}, \begin{bmatrix} \frac{2}{11} - \frac{a}{2} \\ \frac{5}{11} - \frac{b}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{11} - \frac{a}{2} \\ \frac{5}{11} - \frac{b}{2} \end{bmatrix}, \begin{bmatrix} -\frac{4}{11} + a \\ \frac{1}{11} + b \end{bmatrix} \right\}. \end{aligned}$$

By simple calculations one can obtain for all $a, b \in \mathbb{R}$ that

$$\{\rho((S^{-1} f_i + u_i) \otimes f_i)\}_{i=1}^4 = \left\{ \sqrt{(\frac{3}{11} + 2a)^2 + (\frac{2}{11} + 2b)^2},\ 0,\ 0,\ 2\sqrt{(\frac{-4}{11} + a)^2 + (\frac{1}{11} + b)^2} \right\}.$$

For example if $a = 0, b = -\frac{1}{11}$, then

$$\max\{\rho((S_F^{-1} f_i + u_i) \otimes f_i)\}_{i=1}^4 = \frac{8}{11} < c_\rho.$$

Thus $\{S_F^{-1} f_i\}_{i=1}^4$ is not a ρ-optimal dual.

Here, we show by an example that the converse of Theorem 2.3 may not be hold in the general form of alternate dual.

Example 4.6. *Let $\mathcal{H} = \mathbb{R}^2$ and $F = \{f_i\}_{i=1}^3 = \{2e_1, \sqrt{2}e_2, e_2\}$ be a frame. Then*

$$S_F = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow S_F^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and so

$$S_F^{-1} f_i = \left\{ \frac{1}{2} e_1, \frac{\sqrt{2}}{3} e_2, \frac{1}{3} e_2 \right\}.$$

Define

$$\begin{aligned} \rho \quad &: \quad \mathcal{B}(\mathbb{R}^2) \to [0, \infty) \\ \rho(T) \quad &= \quad \frac{1}{\sqrt{2}} \|T(e_1) + T(e_2)\|_2. \end{aligned}$$

It is clear that ρ is a seminorm on $\mathcal{B}(\mathbb{R}^2)$ and

$$\{(S_F^{-1} f_i \otimes f_i) f\}_{i=1}^3 = \{\langle f, f_i \rangle S_F^{-1} f_i\}_{i=1}^3.$$

Thus

$$\left\{\rho(S_F^{-1} f_i \otimes f_i)\right\}_{i=1}^3 = \left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{6} \right\}.$$

Consequently,

$$\max \left\{\rho(S_F^{-1} f_i \otimes f_i)\right\}_{1=1}^3 = \frac{\sqrt{2}}{2} = c_\rho, \ \Lambda_1^\rho = \{i = 1\}, \ \Lambda_2^\rho = \{i = 2, 3\}.$$

Furthermore, $\{f_i\}_{i \in \Lambda_1^\rho} = \{2e_1\}$ is linearly independent and $H_1 \cap H_2 = \{0\}$. By Proposition 2.4, $\{S_F^{-1} f_i\}_{i=1}^3$ is a ρ-optimal dual. Now assume that $\{g_i\}_{i=1}^3 = \{S_F^{-1} f_i + u_i\}_{i=1}^3$ is a dual of F. Thus $\{u_i\}_{i=1}^3$ satisfies (1) *and we have*

$$\begin{aligned} \{u_i\}_{i=1}^3 \quad &= \quad \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -\sqrt{2}a \\ -\sqrt{2}b \end{bmatrix} \right\}, \\ \text{and} \quad \{S_F^{-1} f_i + u_i\}_{i=1}^3 \quad &= \quad \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ \frac{\sqrt{2}}{3} + b \end{bmatrix}, \begin{bmatrix} -\sqrt{2}a \\ \frac{1}{3} - \sqrt{2}b \end{bmatrix} \right\}. \end{aligned}$$

Therefore

$$\left\{\rho((S_F^{-1} f_i + u_i) \otimes f_i)\right\}_{i=1}^3 = \left\{ \frac{\sqrt{2}}{2}, \sqrt{a^2 + (\frac{\sqrt{2}}{3} + b)^2}, \sqrt{a^2 + (\frac{1}{3\sqrt{2}} - b)^2} \right\}.$$

For all $a > \frac{\sqrt{2}}{2}$, we have

$$\max \left\{\rho(S_F^{-1} f_i + u_i) \otimes f_i)\right\}_{1=1}^3 > \max \left\{(\rho(S_F^{-1} f_i \otimes f_i)\right\}_{1=1}^3.$$

Hence for all $a > \frac{\sqrt{2}}{2}$, $\{S^{-1} f_i + u_i\}_{i=1}^3$ is not a ρ-optimal dual, while when $b = -\frac{1}{6\sqrt{2}}$, then $\Lambda_1^g = \{i = 2, 3\}$ and $\Lambda_2^g = \{i = 1\}$. Also $\{f_i\}_{i \in \Lambda_2^g} = \{2e_1\}$ is linearly independent and $H_1^g \cap H_2^g = \{0\}$.

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