Filomat 37:7 (2023), 2067–2075
https://doi.org/10.2298/FIL2307067Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Roman domination problem of some Johnson graphs

Tatjana Zec[a]

[a]*Department of Mathematics and Computer Science, Faculty of Natural Sciences and Mathematics, University of Banja Luka, Mladena Stojanovića 2, Banja Luka 78000, Republic of Srpska, Bosnia and Hercegovina*

Abstract. A Roman domination function (RDF) on a graph G with a set of vertices $V = V(G)$ is a function $f : V \to \{0, 1, 2\}$ which satisfies the condition that each vertex $v \in V$ such that $f(v) = 0$ is adjacent to at least one vertex u such that $f(u) = 2$. The minimum weight value of an RDF on graph G is called the *Roman domination number* (RDN) of G and it is denoted by $\gamma_R(G)$. An RDF for which $\gamma_R(G)$ is achieved is called a $\gamma_R(G)$-function. This paper considers Roman domination problem for Johnson graphs $J_{n,2}$ and $J_{n,3}$. For $J_{n,2}$, $n \geqslant 4$ it is proved that $\gamma_R(J_{n,2}) = n - 1$. New lower and upper bounds for $J_{n,3}, n \geqslant 6$ are derived using results on the minimal coverings of pairs by triples. These bounds quadratically depend on dimension n.

1. Introduction

Let $G = G(V, E)$ represent a simple graph where $V = V(G)$ is a set of vertices and $E = E(G)$ is a set of edges. The number of vertices $|V|$ is called the order of graph G. For vertex $v \in V$, its closed neighborhood denoted by $N[v]$ is the set of all its neighbour vertices including v as well, i.e., $N[v] = \{u \in V | uv \in E\} \cup \{v\}$.

The Roman domination problem (RDP) is introduced in [1]. The function $f : V \mapsto \{0, 1, 2\}$ is said to be a Roman domination function (RDF) if it satisfies the following condition:

$$(\forall v \in V) f(v) = 0 \Rightarrow (\exists u \in N[v]) f(u) = 2. \tag{1}$$

Following the definition of RDF f, it can be concluded that the following must hold

$$(\forall v \in V) \sum_{u \in N[v]} f(u) \geqslant 1. \tag{2}$$

Note that every RDF f determines the partitioning $V = (V_0, V_1, V_2)$ of the set V, where $V_i = \{v \in V | f(v) = i\}$, $i \in \{0, 1, 2\}$. The term *weight of RDF f* is defined as $f(G) = \sum_{v \in V} f(v)$.

Given the above notation, it holds that $f(G) = 2|V_2| + |V_1|$. The minimum weight value of RDF on graph G is known as the *Roman domination number* (RDN) of G, denoted by $\gamma_R(G)$. The RDF for which $\gamma_R(G)$ is achieved is called a $\gamma_R(G)$-function.

The following property will be used in the proof of Theorem 2.3.

2020 *Mathematics Subject Classification*. 05C69; 05C76
Keywords. Graph theory, Graph domination, Roman domination, Johnson graphs
Received: 11 March 2022; Accepted: 13 April 2022
Communicated by Dragan S. Djordjević
Email address: tatjana.zec@pmf.unibl.org (Tatjana Zec)

Property 1.1. *[2] Let G be a graph of order n with the maximum degree $\Delta = \Delta(G)$. Then, it holds that*

$$\gamma_R(G) \geqslant \frac{2n}{\Delta + 1}. \tag{3}$$

Definition 1.2. *Let $[n]$ denote the set $\{1, 2, ..., n\}$, $n \in \mathbb{N}$ and let k be an integer, such that $1 \leqslant k \leqslant n$. The Johnson graph $J_{n,k}$ is a graph whose vertices are modelled as the k-element subset of $[n]$. Two vertices are adjacent in $J_{n,k}$ iff the corresponding sets have exactly $k-1$ elements in common.*

The order of graph $J_{n,k}$ is $\binom{n}{k}$ and it is a regular graph with degree equal to $k(n-k)$. For $k = 1$, $J_{n,1}$ is the complete graph K_n with n vertices and $\gamma_R(J_{n,1}) = \gamma_R(K_n) = 2$. It is known that $J_{n,2}$ is isomorphic to the triangular graph $T(n)$. An example of the Johnson graph is shown in Figure 1.

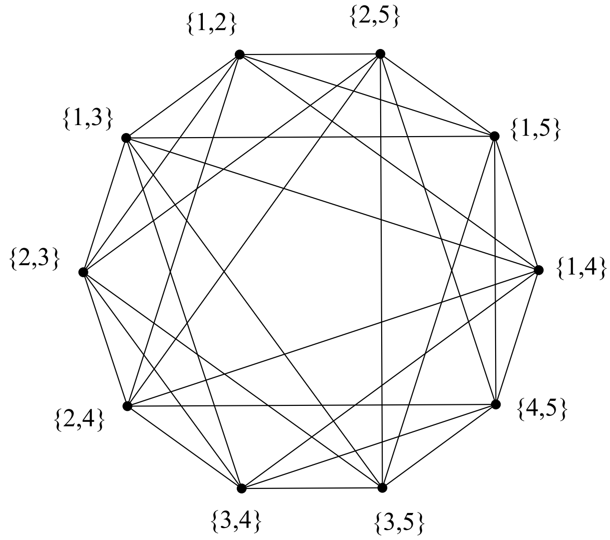

Figure 1: Johnson graph $J_{5,2}$

The following corollary follows directly from Property 1.1.

Corollary 1.3. *For the Johnson graph $J_{n,k}$ it holds that $\gamma_R(J_{n,k}) \geqslant \frac{2\binom{n}{k}}{k(n-k)+1}$.*
Specially, for $k = 2$, $\gamma_R(J_{n,2}) \geqslant \frac{n(n-1)}{2n-3}$.

1.1. Previous work

The basic properties of the Roman domination problem have been proposed in [1, 2]. The RDN for several types of the regular graph was studied in [3]. It was proved that some classes of circulant generalized Petersen and Cartesian product graphs are Roman graphs; graph G is Roman if $\gamma_R(G) = 2\gamma(G)$, where $\gamma(G)$ is the domination number of graph G. Bounds on RDN $\gamma_R(G)$ in terms of the diameter of the graph and the girth of general graphs are presented in [4]. For nontrivial connected graph G of order $n \geqslant 3$ and maximum degree Δ Chellali et al. [5] proposed the following two lower bounds: $\gamma_R(G) \geqslant \frac{\Delta+1}{\Delta}\gamma(G)$ and $\gamma_R(G) \geqslant 2\gamma_{ve}(G)$, where $\gamma_{ve}(G)$ denotes the *vertex-edge domination number* of G. In [6] the bounds on the sum $\gamma_R(G) + \frac{\gamma(G)}{2}$, and the cardinalities $|V_0|, |V_1|$, and $|V_2|$ are given for RDF. It was proved that for the connected graph G of order $n \geqslant 3$, it holds that $\gamma_R(G) + \frac{\gamma(G)}{2} \leqslant n$. Further, if $f = (V_0, V_1, V_2)$ is an RDF, then $\frac{n}{5} + 1 \leqslant |V_0| \leqslant n - 1, 0 \leqslant |V_1| \leqslant \frac{4n}{5} - 2$ and $1 \leqslant |V_2| \leqslant \frac{2n}{5}$. The RDP for several types of graphs of convex polytopes was considered in [7], where the RDNs are proved for following graphs: $A_n, R_{3k}, R_{3k+1}, T_{8k}, T_{8k+2}, T_{8k+3}, T_{8k+5}$, and T_{8k+6}. For the graphs $R_{3k+2}, T_{8k+1}, T_{8k+4}$, and T_{8k+7} there were new upper and lower bounds. Li [8] obtained that for every nontrivial connected graph G it holds that $\gamma_R(G) \geqslant \frac{\Delta+2\delta}{\Delta+\delta}\gamma(G)$, where $\delta = \delta(G)$ represents the minimum degree in G. Liu et al. obtained in [9] the upper bound $\gamma_R(G) \leqslant \frac{2n}{3}$ for graph G of order n and minimum degree $\delta \geqslant 3$. For a detailed review of the results regarding the Roman domination problem, the reader should refer to [10].

There is a number of papers about the properties of Johnson graphs: competition number and edge clique number in [11], metric dimension in [12], distance property in [13], automorphism groups in [14, 15], chromatic number in [16], and equidistant dimension in [17].

2. New results for Johnson graphs

This section presents the main findings of this research. Theorem 2.1 provides the exact value of RDN for graphs $J_{n,2}, n \geqslant 4$. Theorem 2.3 presents the upper bound for value $\gamma_R(J_{n,3}), n \geqslant 6$. Observation 2.4 gives the lower bound for $\gamma_R(J_{n,3}), n \geqslant 6$, obtained by using the well known result exposed in Property 1.1.

2.1. Roman domination number for graph $J_{n,2}$

Theorem 2.1. *For $n \geqslant 4$ it holds that $\gamma_R(J_{n,2}) = n - 1$.*

Proof. **Step 1:** $\gamma_R(J_{n,2}) \geqslant n - 1$

Let $\overline{f}$ be an arbitrary RDF and suppose that $|V_2| = m$, for some $m \in [0, \binom{n}{2}]$. For $m = 0$, i.e., when the set V_2 is empty, it holds that $V_1 = V$, as function $\overline{f}$ is an RDF. Thus it follows $\overline{f}(J_{n,2}) = 2|V_2| + |V_1| = |V| = \binom{n}{2} \geqslant n - 1$. If $m \geqslant \frac{n-1}{2}$, it holds that $\overline{f}(J_{n,2}) = 2|V_2| + |V_1| \geqslant 2\frac{n-1}{2} = n - 1$.

This also holds that for $0 < m < \frac{n-1}{2}$, which can be proved as follows. As previously stated, each vertex corresponds to its underlying 2-subset. Therefore, if there are m vertices, this will correspond to at most $2m$ distinct underlying numbers from $[n]$. This is a scenario in which all corresponding 2-subsets are non-overlapping. Since $\overline{f}$ is an RDF, all vertices non-adjacent to a vertex from V_2 must belong to V_1.

Since among all vertices from set V_2 there are at most $2m$ elements, each 2-element subset formed from the rest of (at least) $n - 2m$ elements will not be adjacent to any vertex from V_2. There are at least $\binom{n-2m}{2}$ such subsets and they must all belong to V_1. Therefore, $|V_1| \geqslant \binom{n-2m}{2}$. It follows that

$$\begin{aligned}\overline{f}(J_{n,2}) = 2|V_2| + |V_1| &\geqslant 2m + \frac{(n-2m)(n-2m-1)}{2} \\ &= \frac{n^2 - n(4m+1) + 4m^2 + 6m}{2}.\end{aligned}$$

Let us prove that $\frac{n^2-n(4m+1)+4m^2+6m}{2} \geqslant n - 1$, i.e. $n^2 - n(4m + 3) + 4m^2 + 6m + 2 \geqslant 0$. The solutions of $n^2 - n(4m + 3) + 4m^2 + 6m + 2 = 0$ are equal to

$$n_1 = 2m + 1 \text{ and } n_2 = 2m + 2,$$

so the inequality $n^2 - n(4m + 3) + 4m^2 + 6m + 2 \geqslant 0$ holds for each $n \in ([1, n_1] \cup [n_2, +\infty)) \cap \mathbb{N}$. Since $m \in \mathbb{N}$, $([1, n_1] \cup [n_2, +\infty)) \cap \mathbb{N} = \mathbb{N}$.

By this, for any $n \geqslant 4$ it holds that $\overline{f}(J_{n,2}) \geqslant n - 1$, implying $\gamma_R(J_{n,2}) \geqslant n - 1$.

Step 2: $\gamma_R(J_{n,2}) \leqslant n - 1$

Let the function f be defined by partition (V_0, V_1, V_2), shown in Table 1.

n	V_2	V_1	V_0
$2l$	$\{2i-1, 2i\},\ i = 1, \ldots, \frac{n-2}{2}$	$\{n-1, n\}$	$V \setminus (V_1 \cup V_2)$
$2l+1$	$\{2i-1, 2i\},\ i = 1, \ldots, \frac{n-1}{2}$	$\emptyset$	$V \setminus V_2$

Table 1: Definition of RDF f on the graph $J_{n,2}$

From the definition of function f, it holds that

- for odd n: $f(J_{n,2}) = 2 \cdot \frac{n-1}{2} = n - 1$ (illustrated in Figure 2),
- for even n: $f(J_{n,2}) = 2 \cdot \frac{n-2}{2} + 1 = n - 1$ (illustrated in Figure 3).

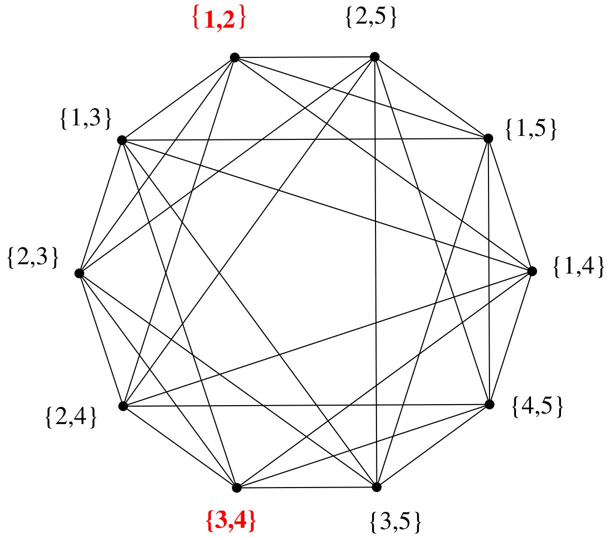

Figure 2: $n = 5$. The vertices modelled by the red colored sets belong to the set V_2

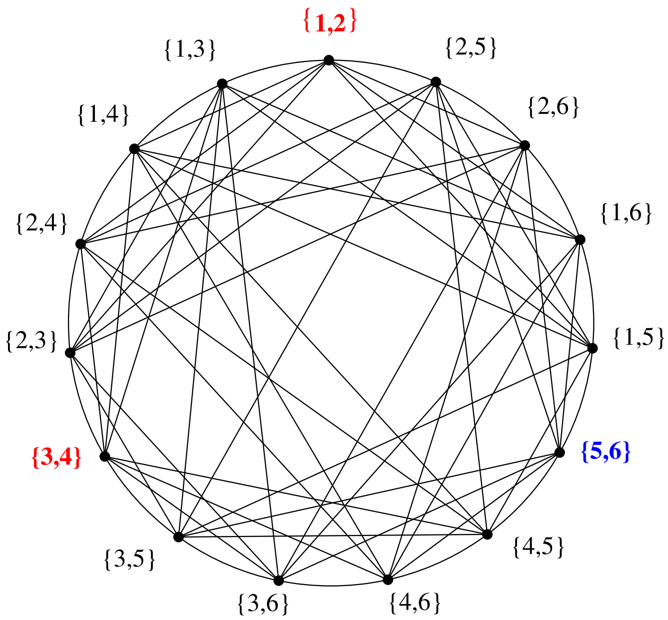

Figure 3: $n = 6$. The vertices modelled by the red colored sets belong to the set V_2 and the vertex modelled by the blue colored set belongs to the set V_1

It is necessary to prove that the function f is an RDF, i.e., condition (1) is satisfied.

Let $\{a, b\}, a < b$ be an arbitrary vertex from V_0 in the following two cases under consideration.

Case 1: a is odd.

Then $\{a, a + 1\} \in V_2$. Notice that $b \neq a + 1$, since $\{a, b\} \in V_0$. Then, $|\{a, b\} \cap \{a, a + 1\}| = 1$, implying that vertices $\{a, b\}$ and $\{a, a + 1\}$ are adjacent.

Case 2: a is even.

Then $\{a - 1, a\} \in V_2$. As $b \neq a - 1$. It follows that $|\{a, b\} \cap \{a - 1, a\}| = 1$, which means that vertices $\{a, b\}$ and $\{a - 1, a\}$ are adjacent.

Therefore, function f satisfies the condition (1), i.e., it is an RDF. $\square$

2.2. Roman domination number for graph $J_{n,3}$

First, let introduce the concept of covering all 2-element subsets with 3-element subsets, which will be used in the proof for the upper bound of value $\gamma_R(J_{n,3})$, as shown in Theorem 2.3.

For analysis of such concept of covering, the reader is reffered to [18]. Let P_m be the set of $m \geqslant 3$ elements. For this set, let the collection C_m of 3-element subsets of P_m be the set which satisfies the condition

$$(\forall \text{ 2-element subset } u \text{ of the set } P_m)(\exists v \in C_m)\ u \subset v. \tag{4}$$

Such collection of minimum cardinality is called *minimal P_m-covering of pairs by triples* and let it be denoted as C_m. In [18] it was shown that

$$|C_m| = \begin{cases} m^2/6, & m = 6l, \\ m(m-1)/6, & m = 6l+1 \text{ or } m = 6l+3, \\ (m^2+2)/6, & m = 6l+2 \text{ or } m = 6l+4, \\ (m^2-m+4)/6 & m = 6l+5. \end{cases} \tag{5}$$

holds. The methodology of constructing an RDF for graph $J_{n,3}$ is as follows.

For given $n \in \mathbb{N}$, $n \geqslant 3$, the set $[n]$ is separated into two partitions: P_{m_1} and P_{m_2}, with m_1 and $m_2 = n - m_1$ elements, where $0 \leqslant m_1, m_2 \leqslant n$ and $m_1, m_2 \notin \{1, 2\}$. Note that it is possible that one of the partitions can be empty. The arrangement of elements can be arbitrary. Let the collections C_{m_1} and C_{m_2} be the minimal triple coverings of pair sets P_{m_1} and P_{m_2}, respectively.
Note that $C_{m_1}, C_{m_2} \subseteq V(J_{n,3})$.

The following lemma will be used in the proof of Theorem 2.3.

Lemma 2.2. *Let collections C_{m_1} and C_{m_2} be the minimal triple coverings of pair sets P_{m_1} and P_{m_2}, respectively. The function defined by the following partitioning (V_0, V_1, V_2)*

- $V_2 = C_{m_1} \cup C_{m_2}$,
- $V_1 = \emptyset$,
- $V_0 = V \setminus V_2$

is an RDF.

Proof. Let $\{a, b, c\}$ be an arbitrary vertex from V_0. Then at least two of its elements belong to either P_{m_1} or P_{m_2}. Without the loss of generality, suppose that $a, b \in P_{m_1}$. Then, according to condition (4), there is a vertex $u \in C_{m_1}$ such that $\{a, b\} \subset u$. Notice that $c \notin u$, since $\{a, b, c\} \notin V_2$. Therefore, $|\{a, b, c\} \cap u| = 2$, which means that vertices $\{a, b, c\}$ and u are adjacent. Since $\{a, b, c\} \in V_0$ is arbitrarily chosen and since it has a neighbor in set V_2, it is proved that the function defined by partition (V_0, V_1, V_2) is an RDF. $\square$

Theorem 2.3. *For $n \geqslant 6$ it holds $\gamma_R(J_{n,3}) \leqslant \xi_n$, where ξ_n is defined as follows:*

$$\xi_n = \begin{cases} (n^2-2n+12)/6, & n = 12l, \\ (n^2-n)/6, & n = 12l+1, \\ (n^2-2n)/6, & n = 12l+2, \\ (n^2-n+6)/6, & n = 12l+3, \\ (n^2-2n+4)/6, & n = 12l+4, \\ (n^2-n+4)/6, & n = 12l+5, \\ (n^2-2n)/6, & n = 12l+6, \\ (n^2-n+6)/6, & n = 12l+7, \\ (n^2-2n+12)/6, & n = 12l+8, \\ (n^2-n+12)/6, & n = 12l+9, \\ (n^2-2n+16)/6, & n = 12l+10, \\ (n^2-n+10)/6, & n = 12l+11. \end{cases} \tag{6}$$

Proof. As it is described in Lemma 2.2, partitioning P_{m_1}, P_{m_2} of the set $[n]$ and the corresponding coverings C_{m_1} and C_{m_2} define an RD function. Among all such partitionings, the one which defines the RDF of minimum weight will be determined.

Now, consider all possible cardinalities for P_{m_1} and P_{m_2} and the corresponding RD functions. For that purpose, let $A_n = \{0, 1, \dots \lfloor \frac{n}{2} \rfloor - 3\} \cup \{\lfloor \frac{n}{2} \rfloor\}$. Without the loss of generality, suppose that $m_1 \leqslant m_2$.

Each partitioning P_{m_1}, P_{m_2} determines an RDF f_t for some $t \in A_n$, where

$$\begin{aligned} & m_1 = \frac{n}{2} - t \text{ and } m_2 = \frac{n}{2} + t, \text{ if } n \text{ is even,} \\ & m_1 = \frac{n-1}{2} - t \text{ and } m_2 = \frac{n-1}{2} + t + 1, \text{ if } n \text{ is odd.} \end{aligned} \tag{7}$$

By the definition of function f_t, it holds that $f_t(J_{n,3}) = 2(|C_{m_1}| + |C_{m_2}|)$.
As f_t is an RDF, it is obvious that $\gamma_R(J_{n,3}) \leqslant f_t(J_{n,3})$ for each $t \in A_n$. Let us now determine $\min\limits_{t \in A_n}\{f_t(J_{n,3})\}$ and show that it is equal to ξ_n.

From (5), values $|C_{m_1}|$ and $|C_{m_2}|$ (and therefore the value of f_t) depend on m_1 modulo 6 and m_2 modulo 6, respectively, while from (7) both m_1 and m_2 depend on $\frac{n}{2}$. It is thus, necessary to differ 12 possible cases depending on n modulo 12.

Let $\varphi_{n,i}$ denote the minimal value over all functions f_t for $m_1 \equiv i \pmod 6$, $i = \overline{0,5}$, i.e.,

$$\varphi_{n,i} = \min_{\substack{t \in A_n \\ m_1 \equiv i \ (\mathrm{mod}\ 6)}} \{f_t(J_{n,3})\}, \ i = \overline{0,5}.$$

Then $\min\limits_{t \in A_n}\{f_t(J_{n,3})\}$ is equal to $\min\limits_{i=\overline{0,5}}\{\varphi_{n,i}\}$.

Two cases will be consider in detail: $n = 12l$ and $n = 12l + 6$. The rest of the cases can be proved in a similar way.

Case 1: $m_1 \equiv 0 \pmod 6$.
As $n \equiv 0 \pmod 6$ and $n = m_1 + m_2$, it follows that $m_2 \equiv 0 \pmod 6$. Therefore, from (5) it follows

$$f_t(J_{n,3}) = 2\left(\frac{m_1^2}{6} + \frac{m_2^2}{6}\right) = \frac{n^2 + 4t^2}{6}.$$

Subcase 1.1: $n = 12l$.
From $m_1 = 6l - t$ it follows that $t \equiv 0 \pmod 6$, so $\varphi_{n,0}$ is obtained for $t = 0$ i.e. $\varphi_{n,0} = \frac{n^2}{6}$.

Subcase 1.2: $n = 12l + 6$.
Here $m_1 = 6l + 3 - t$, so $t \equiv 3 \pmod 6$. Therefore, $\varphi_{n,0}$ is obtained for $t = 3$ and $\varphi_{n,0} = \frac{n^2+36}{6}$.

Case 2: $m_1 \equiv 1 \pmod 6$.

Hence, $m_2 \equiv 5 \pmod 6$. According to (5), it holds that

$$f_t(J_{n,3}) = 2\left(\frac{m_1(m_1-1)}{6} + \frac{m_2^2 - m_2 + 4}{6}\right) = \frac{n^2 + 4t^2 - 2n + 8}{6}.$$

Subcase 2.1: $n = 12l$.

Here $t \equiv 5 \pmod 6$ and $\varphi_{n,1} = \frac{n^2-2n+108}{6}$ is obtained for $t = 5$.

Subcase 2.2: $n = 12l + 6$.

It holds that $t \equiv 2 \pmod 6$ and $\varphi_{n,1} = \frac{n^2-2n+24}{6}$.

Case 3: $m_1 \equiv 2 \pmod 6$.

It follows that $m_2 \equiv 4 \pmod 6$. Again, from (5) it follows that

$$f_t(J_{n,3}) = 2\left(\frac{m_1^2+2}{6} + \frac{m_2^2+2}{6}\right) = \frac{n^2 + 4t^2 + 8}{6}.$$

Subcase 3.1: $n = 12l$.

Here it holds that $t \equiv 4 \pmod 6$, so $\varphi_{n,2} = \frac{n^2+72}{6}$ for $t = 4$.

Subcase 3.2: $n = 12l + 6$.

Here $t \equiv 1 \pmod 6$ and $\varphi_{n,2} = \frac{n^2+12}{6}$

Case 4: $m_1 \equiv 3 \pmod 6$.

It follows that $m_2 \equiv 3 \pmod 6$, so (5) implies that

$$f_t(J_{n,3}) = 2\left(\frac{m_1(m_1-1)}{6} + \frac{m_2(m_2-1)}{6}\right) = \frac{n^2 + 4t^2 - 2n}{6}.$$

Subcase 4.1: $n = 12l$.

Here $t \equiv 3 \pmod 6$. Therefore $\varphi_{n,3} = \frac{n^2-2n+36}{6}$ and it is obtained for $t = 3$.

Subcase 4.2: $n = 12l + 6$.

It follows that $t \equiv 0 \pmod 6$, so $\varphi_{n,3} = \frac{n^2-2n}{6}$, which is obtained for $t = 0$.

Case 5: $m_1 \equiv 4 \pmod 6$.

Here it holds that $m_2 \equiv 2 \pmod 6$, so according to (5), f_t has the same value as in Case 3.

Subcase 5.1: $n = 12l$.

For these values of n it holds that $t \equiv 2 \pmod 6$. Thus $\varphi_{n,4} = \frac{n^2+24}{6}$, and it is obtained for $t = 2$.

Subcase 5.2: $n = 12l + 6$.

Here $t \equiv 5 \pmod 6$ and $\varphi_{n,5} = \frac{n^2+108}{6}$ is obtained for $t = 5$.

Case 6: $m_1 \equiv 5 \pmod 6$.

It holds that $m_2 \equiv 1 \pmod 6$. Therefore, from (5) it follows that

$$f_t(J_{n,3}) = 2\left(\frac{m_1^2 - m_1 + 4}{6} + \frac{m_2(m_2-1)}{6}\right) = \frac{n^2 + 4t^2 - 2n + 8}{6}.$$

Subcase 6.1: $n = 12l$.

It follows that $t \equiv 1 \pmod 6$ and $\varphi_{n,5} = \frac{n^2-2n+12}{6}$ is obtained for $t = 1$.

Subcase 6.2: $n = 12l + 6$.

It holds that $t \equiv 4 \pmod 6$ and $\varphi_{n,5} = \frac{n^2-2n+72}{6}$ is obtained for $t = 4$.

This implies that the value $\min_{x\in A_n}\{f_t(J_{n,3})\} = \min_{i=\overline{0,5}}\{\varphi_{n,i}\}$ for $n = 12l$ is equal to $\varphi_{n,5} = \frac{n^2-2n+12}{6}$, while for $n = 12l + 6$ it is $\varphi_{n,3} = \frac{n^2-2n}{6}$. Therefore, Step 2 for cases $n = 12l$ and $n = 12l + 6$ are proved.

A detailed proof for other cases is omitted since it is analogous to previous cases. Table 2.2 provides details about the lowest upper bounds. The first column contains all cases of number n, depending on the remainder of division by 12. The second and third columns, denoted respectively by $\overline{m}_1$ and $\overline{t}$, contain the

n	$\overline{m}_1$	$\bar{t}$	$f_{\bar{t}}$	$\min_{i=0,5}\{\varphi_{n,i}\}$
$12l$	$6l-1$	1	$\frac{n^2+4t^2-2n+8}{6}$	$\frac{n^2-2n+12}{6}$
$12l+1$	$6l$	0	$\frac{n^2+4t^2-n+2t}{6}$	$\frac{n^2-n}{6}$
$12l+2$	$6l+1$	0	$\frac{n^2+4t^2-2n}{6}$	$\frac{n^2-2n}{6}$
$12l+3$	$6l,\ 6l+1$	$1,0$	$\frac{n^2+4t^2-n+2t}{6}, \frac{n^2+4t^2-n+6t+6}{6}$	$\frac{n^2-n+6}{6}$
$12l+4$	$6l+1$	1	$\frac{n^2+4t^2-2n}{6}$	$\frac{n^2-2n+4}{6}$
$12l+5$	$6l+2$	0	$\frac{n^2+4t^2-n+2t+4}{6}$	$\frac{n^2-n+4}{6}$
$12l+6$	$6l+3$	0	$\frac{n^2+4t^2-2n}{6}$	$\frac{n^2-2n}{6}$
$12l+7$	$6l+3$	0	$\frac{n^2+4t^2-n+6t+6}{6}$	$\frac{n^2-n+6}{6}$
$12l+8$	$6l+3$	1	$\frac{n^2+4t^2-2n+8}{6}$	$\frac{n^2-2n+12}{6}$
$12l+9$	$6l+3,\ 6l+4$	$1,0$	$\frac{n^2+4t^2-n+6t+2}{6}, \frac{n^2+4t^2-n+2t+12}{6}$	$\frac{n^2-n+12}{6}$
$12l+10$	$6l+3, 6l+5$	$2,0$	$\frac{n^2+4t^2-2n}{6}, \frac{n^2+4t^2-2n+16}{6}$	$\frac{n^2-2n+16}{6}$
$12l+11$	$6l+4,\ 6l+5$	$1,0$	$\frac{n^2+4t^2-n+2t+4}{6}, \frac{n^2+4t^2-n+6t+10}{6}$	$\frac{n^2-n+10}{6}$

Table 2: Detailed overview of the lowest upper bounds, for 12 different cases

cardinality of P_{m_1} and the value of parameter t, for which function f_t achieves the lowest value. The fourth column contains the value of function f_t for $\bar{t}$. The last column contains the lowest RDF value among all considered RDFs.

Note that in cases $n \in \{12l+3, 12l+9, 12l+10, 12l+11\}$, the corresponding value ξ_n is obtained for two different values of m_1. □

The following observation proposes the lower bound for $\gamma_R(J_{n,3})$.

Observation 2.4. *For each $n \geqslant 6$ it holds that $\gamma_R(J_{n,3}) \geqslant \left\lceil \frac{n(3n-1)}{27} + 1 \right\rceil$.*

Proof. From Corollary 1.3 it follows that

$$\gamma_R(J_{n,3}) \geqslant \frac{2\binom{n}{3}}{3(n-3)+1} = \frac{n(n-1)(n-2)}{3(3n-8)}.$$

It is easy to prove that

$$\frac{n(3n-1)}{27} + 1 > \frac{n(n-1)(n-2)}{3(3n-8)} > \frac{n(3n-1)}{27},$$

which concludes the proof. □

Remark 2.5. *For $n \in \{3,4,5\}$ it can be easily determined that*

$$\gamma_R(J_{n,3}) = \begin{cases} 1, & n=3, \\ 2, & n=4, \\ 4, & n=5. \end{cases}$$

Remark 2.6. *From Table 2.2, column $\bar{t}$, one can observe that $\min_{t\in A_n}\{f_t(J_{n,3})\}$ is obtained for small values of t, i.e. when the cardinalities of two partitions P_{m_1} and P_{m_2} are balanced.*

3. Conclusions

This paper considered the Roman domination number for Johnson graphs with $k \in \{2, 3\}$. It was proved that $\gamma_R(J_{n,2})$ is equal to $n - 1$. Moreover, new lower and upper bounds for $J_{n,3}$ were proposed, proving that the Roman domination number for $J_{n,3}$ quadratically depends on dimension n.

The major goal for future work isdetermination some other Roman domination problems on Johnson graphs. This includes determining total Roman domination number, signed (total) Roman domination number, etc.

Acknowledgements

The author would like to thank Milanka Treml for her suggestions related to the results of the paper [18].

References

[1] E. J. Cockayne, P. A. Dreyer Jr, S. M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, Discrete mathematics 278 (1-3) (2004) 11–22.
[2] E. Cockayne, P. Grobler, W. Gründlingh, J. Munganga, J. Vuuren, Protection of a graph, Utilitas Mathematica 67 (2005) 19–32.
[3] F. Xueliang, Y. Yuansheng, J. Baoqi, Roman domination in regular graphs, Discrete Mathematics 309 (6) (2009) 1528–1537.
[4] B. Mobaraky, S. Sheikholeslami, Bounds on Roman domination numbers of graphs, Matematički vesnik 60 (4) (2008) 247–253.
[5] M. Chellali, T. W. Haynes, S. T. Hedetniemi, Lower bounds on the Roman and independent Roman domination numbers, Applicable Analysis and Discrete Mathematics 10 (1) (2016) 65–72.
[6] O. Favaron, H. Karami, R. Khoeilar, S. M. Sheikholeslami, On the Roman domination number of a graph, Discrete Mathematics 309 (10) (2009) 3447–3451.
[7] A. Kartelj, M. Grbić, D. Matić, V. Filipović, The Roman domination number of some special classes of graphs-convex polytopes, Applicable Analysis and Discrete Mathematics 15 (2) (2021) 393–412.
[8] Z. Li, A note on the bounds of Roman domination numbers, AIMS Mathematics 6 (4) (2021) 3940–3946.
[9] C.-H. Liu, G. J. Chang, Upper bounds on Roman domination numbers of graphs, Discrete Mathematics 312 (7) (2012) 1386–1391.
[10] M. Chellali, N. J. Rad, S. M. Sheikholeslami, L. Volkmann, Roman domination in graphs, in: Topics in domination in graphs, Springer, 2020, pp. 365–409.
[11] S.-R. Kim, B. Park, Y. Sano, The competition numbers of Johnson graphs, Discussiones Mathematicae Graph Theory 30 (3) (2010) 449–459.
[12] R. F. Bailey, J. Cáceres, D. Garijo, A. González, A. Márquez, K. Meagher, M. L. Puertas, Resolving sets for Johnson and Kneser graphs, European Journal of Combinatorics 34 (4) (2013) 736–751.
[13] A. Dabrowski, L. S. Moss, The Johnson graphs satisfy a distance extension property, Combinatorica 20 (2) (2000) 295–300.
[14] A. Ganesan, On the automorphism group of a Johnson graph, arXiv preprint arXiv:1412.5055.
[15] M. Ramras, E. Donovan, The automorphism group of a Johnson graph, SIAM Journal on Discrete Mathematics 25 (1) (2011) 267–270.
[16] T. Etzion, S. Bitan, On the chromatic number, colorings, and codes of the Johnson graph, Discrete applied mathematics 70 (2) (1996) 163–175.
[17] A. González, C. Hernando, M. Mora, The equidistant dimension of graphs, arXiv preprint arXiv:2107.10805.
[18] M. Fort Jr, G. Hedlund, Minimal coverings of pairs by triples., Pacific Journal of Mathematics 8 (4) (1958) 709–719.