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Statistical convergence on $L-$fuzzy normed space

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Abstract. In this paper, we study the concept of statistical convergence on $L-$fuzzy normed spaces. Then we give a useful characterization for statistically convergent sequences. Furthermore, we illustrate that our method of convergence is more general than the usual convergence on $L-$fuzzy normed spaces.

1. Introduction

$L-$ fuzzy normed spaces are natural generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces [1, 3, 12–14, 16, 19, 31] based on some logical algebraic structures, which also enriches the notion of a $L-$ fuzzy metric space [7, 8, 10, 18].

There is a vast literature of studies on this structure. In particular, some properties of a variant of the statistical convergence of sequences on $L-$fuzzy normed spaces are given [6, 22]. However, generalizations of some well-known results are absent and in particular there is no literature on statistical boundedness conditions on sequences.

On the other hand, to date, many valuable studies have been conducted on statistical convergence and many features of them have been given [2, 4, 5, 9, 11, 15, 20, 21, 24–30].

In this study we give some results regarding statistical convergence of sequences on $L-$fuzzy normed spaces and investigate the relationship between statistical convergent, statistical Cauchy [17] and statistical bounded sequences, which will be newly introduced on $L-$fuzzy normed spaces.

In this regard, here we give a characterization of the statistical convergence of a sequence through the convergence of certain subsequences in the classical sense on $L-$fuzzy normed spaces. Then we introduce and discuss the notion of a statistical bounded sequence on $L-$fuzzy normed spaces. And finally we reveal some implications between statistical convergence, statistical Cauchyness and statistical boundedness of a sequence on a $L-$fuzzy normed space.

The aim of the present paper is to investigate the statistical convergence, which was first introduced by Steinhaus [23], on $L-$fuzzy normed spaces. Then we give a useful characterization for statistically convergent sequences on $L-$fuzzy normed spaces. Also we display an example illustrating that our method of convergence is more general than the usual convergence on $L-$fuzzy normed spaces.

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2. Preliminaries

Preliminaries on $L-$ fuzzy normed spaces are presented in this section.

Definition 2.1. *[22] Assume that* $K : [0,1] \times [0,1] \to [0,1]$ *is a function that satisfies the following*

1. $K(a,b) = K(a,b)$
2. $K(K(a,b),c) = K(a,K(b,c))$
3. $K(a,1) = K(1,a) = a$
4. *If* $a \leq b, c \leq d$ *then* $K(a,c) \leq K(b,d)$

is known as a $t-$ *norm.*

Example 2.2. *[22]* K_1, K_2 *and* K_3 *are the functions that given with,*

$K_1(a,b) = \min\{a,b\}$,
$K_2(a,b) = ab$,
$K_3(a,b) = \max\{a+b-1,0\}$

are the samples, which are well known of $t-$ *norms.*

Definition 2.3. *[22] Let* $\mathscr{L} = (L, \preceq)$ *be a complete lattice and let a set* A *be called the universe. An* $L-$*fuzzy set, on* A *is defined with a function*

$$X : A \to L.$$

On a set A*, the family of all* $L-$*sets is denoted by* L^A.

Two $L-$ *sets on* A *intersect and union is illustrated by,*

$$(C \cap D)(x) = C(x) \wedge D(x),$$

$$(C \cup D)(x) = C(x) \vee D(x)$$

for all $x \in A$*. Similarly, union and intersection of a family* $\{B_i : i \in I\}$ *of* $L-$ *fuzzy sets is given by*

$$\Big(\bigcup_{i\in I} B_i\Big)(x) = \bigvee_{i\in I} B_i(x)$$

$$\Big(\bigcap_{i\in I} B_i\Big)(x) = \bigwedge_{i\in I} B_i(x)$$

respectively.

0_L and 1_L are the smallest and biggest elements of the complete lattice L, respectively. On a given lattice $(L, \preceq)$, we also employ the symbols $\succeq, \prec$, and $\succ$ in the obvious meanings.

Definition 2.4. *[22] Let* $\mathscr{L} = (L, \preceq)$ *be a complete lattice. Then, a* $t-$ *norm is a function* $\mathscr{K} : L \times L \to L$ *that satisfies the following for all* $a, b, c, d \in L$:

1. $\mathscr{K}(a,b) = \mathscr{K}(b,a)$
2. $\mathscr{K}(\mathscr{K}(a,b),c) = \mathscr{K}(a,\mathscr{K}(b,c))$
3. $\mathscr{K}(a,1_L) = \mathscr{K}(1_L,a) = a$
4. *If* $a \preceq b$ *and* $c \preceq d$*, then* $\mathscr{K}(a,c) \preceq \mathscr{K}(b,d)$.

Definition 2.5. *[22] For sequences (a_n) and (b_n) on L such that $(a_n) \to a \in L$ and $(b_n) \to b \in L$, if the property that $\mathscr{K}(a_n, b_n) \to \mathscr{K}(a, b)$ is satisfied on L, then a t−norm $\mathscr{K}$ on a complete lattice $\mathscr{L} = (L, \preceq)$ is called continuous.*

Definition 2.6. *[22] The function $\mathscr{D} : L \to L$ is said to be as a negator on $\mathscr{L} = (L, \preceq)$ if,*

D_1) $\mathscr{D}(0_L) = 1_L$

D_2) $\mathscr{D}(1_L) = 0_L$

D_3) $a \preceq b$ *implies* $\mathscr{D}(b) \preceq \mathscr{D}(a)$ *for all* $a, b \in L$.

If in addition,

D_4) $\mathscr{D}(\mathscr{D}(a)) = a$ *for all* $a \in L$.

Then, $\mathscr{D}$ *is known as an involutive negator.*

The mapping $\mathscr{D}_s : [0,1] \to [0,1]$, on the lattice $([0,1], \leq)$ defined as $\mathscr{D}_s(x) = 1 - x$ is a well known sample of an involutive negator. This type of negator is used in the notion of standard fuzzy sets. In addition, with the order

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \iff \mu_1 \leq \mu_2 \quad and \quad \nu_1 \geq \nu_2$$

given the lattice $([0,1]^2, \preceq)$ with for all $i = 1, 2$, $(\mu_i, \nu_i) \in [0,1]^2$. Then, the function $\mathscr{D}_1 : [0,1]^2 \to [0,1]^2$,

$$\mathscr{D}_1(\mu, \nu) = (\nu, \mu)$$

in the sense of Atanassov [3], is known as an involutive negator. This type of negator is used in the notion of intuitionistic fuzzy sets.

Definition 2.7. *[22] Let $\mathscr{L} = (L, \preceq)$ be a complete lattice and V be a real vector space. Also let $\mathscr{K}$ be a continuous t− norm on $\mathscr{L}$ and ν be an L−set on $V \times (0, \infty)$ satisfying the following*

(a) $\nu(a, t) \succ 0_L$ *for all* $a \in V, t > 0$

(b) $\nu(a, t) = 1_L$ *for all* $t > 0$ *if and only if* $a = \theta$

(c) $\nu(\alpha a, t) = \nu(a, \frac{t}{|\alpha|})$ *for all* $a \in V, t > 0$ *and* $\alpha \in \mathbb{R} - \{0\}$

(d) $\mathscr{K}(\nu(a, t), \nu(b, s)) \preceq \nu(a + b, t + s)$, *for all* $a, b \in V$ *and* $t, s > 0$

(e) $\lim_{t\to\infty} \nu(a, t) = 1_L$ *and* $\lim_{t\to 0} \nu(a, t) = 0_L$ *for all* $a \in V - \{\theta\}$

(f) The functions $f_a : (0, \infty) \to L$ *which are given by* $f_a(t) = \nu(a, t)$ *are continuous.*
The triple $(V, \nu, \mathscr{K})$ is referred to as an $\mathscr{L}-$ fuzzy normed space or $\mathscr{L}-$ normed space in this context.

Example 2.8. *Let $V = \mathbb{R}^2$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\nu : \mathbb{R}^2 \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with*

$$\nu((a, b), t) = \{r \in \mathbb{R}^+ : \max\{| ra |, | rb |\} < t\}.$$

Since the fuzzy norm ν satisfies the Definition 2.7., the triple $(\mathbb{R}^2, \mathscr{P}(\mathbb{R}^+), \nu)$ is a $\mathscr{L}-$ fuzzy normed space.

Definition 2.9. *[22] A sequence (a_n) is said to be Cauchy sequence in a $\mathscr{L}-$ fuzzy normed space $(V, \nu, \mathscr{K})$ if, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n > n_0$*

$$\nu(a_n - a_m, t) \succ \mathscr{D}(\epsilon)$$

where $\mathscr{D}$ is a negator on $\mathscr{L}$, for each $\epsilon \in L - \{0_L\}$ and $t > 0$.

Definition 2.10. *A sequence $a = (a_n)$ is said to be bounded with respect to fuzzy norm ν in a $\mathscr{L}-$ fuzzy normed space $(V, \nu, \mathscr{K})$, provided that, for each $r \in L - \{0_L, 1_L\}$ and $t > 0$,*

$$\nu(a_n, t) \succ \mathscr{D}(r)$$

for all $n \in \mathbb{N}$.

3. Statistical convergence on $\mathscr{L}-$fuzzy normed space

We will look at statistical convergence on $\mathscr{L}-$fuzzy normed spaces in this section. Before we go any further, we should review some statistical convergence terminology [6]. If K is a subset of $\mathbb{N}$, the set of natural numbers, then its natural density, denoted by $\delta(K)$, is

$$\delta(K) := \lim_{n} \frac{1}{n} \left| \{k \leq n : k \in K\} \right|$$

whenever the limit exists, with $|A|$ denoting the cardinality of the set A.

If the set $K(\epsilon) = \{k \leq n : |x_k - \ell| > \epsilon\}$ has the natural density zero, i.e.

$$\lim_{n} \frac{1}{n} \left| \{k \leq n : |x_k - \ell| > \epsilon\} \right| = 0$$

then a number sequence $x = (x_k)$ is said to be statistically convergent to the ℓ. In this scenario, we will write $st - \lim x = \ell$.

Although every convergent sequence is statistically convergent to the same limit, the converse is not always true.

In $\mathscr{L}-$fuzzy normed space, we study the principle of statistical convergence of sequences.

Definition 3.1. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_n)$ is statistically convergent to $\ell \in V$ with respect to ν fuzzy norm, provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta(\{k \in \mathbb{N} : \nu(x_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\}) = 0.$$

In this scenario, we will write $st_{\mathscr{L}} - \lim x = \ell$.

Lemma 3.2. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\epsilon \in L - \{0_L\}$ and $t > 0$:*

(a) $st_{\mathscr{L}} - \lim x_n = \ell$.

(b) $\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \nsucc \mathscr{N}(\epsilon)\}) = 0$.

(c) $\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \succ \mathscr{N}(\epsilon)\}) = 1$.

(d) $st - \lim \nu(x_n - \ell, t) = 1_L$.

Proof. The equivalences between (a), (b) and (c) follow directly from the definitions.

$(a) \iff (d)$: Note that $st_{\mathscr{L}} - \lim x_n = \ell$ means that, for all $\varepsilon \in L - \{0_L\}$ and $t > 0$ we have

$$\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \nsucc \mathscr{N}(\epsilon)\}) = 0.$$

On the other hand, a local base for the open neighborhoods of $1_L \in L$ with respect to the order topology on the lattice $\mathscr{L} = (L, \leq)$, are the sets

$$(a, 1_L] = \{x \in L : a < x \leq 1_L\}$$

for each $a \in L - \{1_L\}$. $st - \lim \nu(x_n - \ell, t) = 1_L$ if and only if, for any given $a \in L - \{1_L\}$,

$$\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \notin (a, 1_L]\}) = 0$$

or equivalently

$$\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \nsucc a\}) = 0.$$

Note that, the two statements

$\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \nsucc \mathscr{N}(\epsilon)\}) = 0$ for all $\varepsilon \in L - \{0_L\}$

$\delta(\{n \in \mathbb{N} : \nu(x_n - \ell, t) \not\succ a\}) = 0$ for all $a \in L - \{1_L\}$

are equivalent since for each $\varepsilon \in L - \{0_L\}$ we can choose $a \in L - \{1_L\}$ as $a = \mathscr{N}(\varepsilon)$ and conversely for each $a \in L - \{1_L\}$ we can choose $\varepsilon \in L - \{0_L\}$ as $\varepsilon = \mathscr{N}(a)$, so that $a = \mathscr{N}(\mathscr{N}(a)) = \mathscr{N}(\varepsilon)$. This proves that (a) is equivalent to (d). □

Theorem 3.3. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}$−fuzzy normed space. If* $\lim x = \ell$ *then* $st_{\mathscr{L}} - \lim x = \ell$.

Proof. Let $\lim x = \ell$. Then for every $\epsilon \in L - \{0_L\}$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\nu(x_n - \ell, t) \succ \mathscr{N}(\epsilon)$$

for all $k \geq k_0$. Therefore, there are only a finite number of terms in

$$\{n \in \mathbb{N} : \nu(x_n - \ell, t) \not\succ \mathscr{N}(\epsilon)\}.$$

We can see right away that any finite subset of the natural numbers has density zero. Hence,

$$\delta\{n \in \mathbb{N} : \nu(x_n - \ell, t) \not\succ \mathscr{N}(\epsilon)\} = 0.$$

□

As seen in the following example, the converse of the Theorem is not true.

Example 3.4. *Let $V = \mathbb{R}^2$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\nu : \mathbb{R}^2 \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with*

$$\nu((a, b), t) = \{r \in \mathbb{R}^+ : \max\{| ra |, | rb |\} < t\}.$$

Then, $(\mathbb{R}^2, \mathscr{P}(\mathbb{R}^+), \nu)$ is a $\mathscr{L}$−normed space. On this space, consider the sequence given by the rule $x_k = (sgn(k \sin k + k - 1), sgn(k \cos k + k - 1))$. Then $st_{\mathscr{L}} - \lim x = (1, 1) \in \mathbb{R}^2$, while it can be conjectured that the sequence itself is not convergent.

Theorem 3.5. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}$−fuzzy normed space. If a sequence $x = (x_n)$ is statistically convergent with respect to the $\mathscr{L}$−fuzzy norm ν, then $st_{\mathscr{L}}$−limit is unique.*

Proof. Suppose that $st_{\mathscr{L}} - \lim x = \ell_1$ and $st_{\mathscr{L}} - \lim x = \ell_2$. For any given $\epsilon \in L - \{0_L\}$ and $t > 0$, we can choose a $r \in L - \{0_L\}$ such that

$$\mathscr{K}(\mathscr{D}(r), \mathscr{D}(r)) \succ \mathscr{D}(\epsilon).$$

Define the following sets

$$K_1 = \{n \in \mathbb{N} : \nu(x_n - \ell_1, t)) \not\succ \mathscr{D}(r)\}$$

and

$$K_2 = \{n \in \mathbb{N} : \nu(x_n - \ell_2, t)) \not\succ \mathscr{D}(r)\}$$

for any $t > 0$. Since for elements of the set $K(\epsilon, t) = K_1(\epsilon, t) \cup K_2(\epsilon, t)$ we have

$$\nu(\ell_1 - \ell_2, t) \geq \mathscr{K}(\nu(x_n - \ell_1, \frac{t}{2}), \nu(x_n - \ell_2, \frac{t}{2})) \succ \mathscr{K}(\mathscr{D}(r), \mathscr{D}(r)) \succ \mathscr{D}(\epsilon).$$

it can be concluded that $\ell_1 = \ell_2$. □

Note that from the definition of a $\mathscr{L}$−normed space, one have $\nu(x, t) \succ 0_L$ for all $x \in V$ and $t \in (0, \infty)$. In contrast to this, one also have $\lim_{t\to\infty} \nu(x, t) = 0_L$. In particular, defining $a_n = \nu(x, \frac{1}{n})$ will give a sequence (a_n) on L such that $a_n \neq 0_L$ for all positive integer n, while $(a_n) \to 0_L$ on L. Now say $b_n := \mathscr{D}(a_n)$. Then for each n, $b_n \neq 1_L$, since otherwise one would have

$$a_n = \mathscr{D}(\mathscr{D}(a_n)) = \mathscr{D}(b_n) = \mathscr{D}(1_L) = 0_L,$$

which is a contradiction. Being decreasing mapping and bijective by the identity $\mathscr{D}(\mathscr{D}(x)) = x$, the involutive negator $\mathscr{D}$ is order continuous, so that $(b_n) \to \mathscr{N}(0_L) = 1_L$.

Since $(a_n) \to 0_L$, for every open basic neighborhood $A_c = \{x \in L : x \prec c\}$ of 0_L, where $c \in L - \{0_L\}$, there exist an $n_0 = n_0(c) \in \mathbb{N}$ such that $a_n \in A_c$ for all $n \geq n_0$. Saying $i_1 = 1, i_2 = n_0(a_1)+1 = n_0(a_{i_1})+1, i_3 = n_0(a_{i_2})+1$ and $i_{k+1} = n_0(a_{i_k}) + 1$ in general, we have decreasing subsequence of (a_n).

The discussion above guarantees that given any $\mathscr{L}-$normed space, it is always possible to find a sequence (a_n) in $L - \{0_L\}$ such that $(a_n) \to 0_L$ so that $\mathscr{D}(a_n) \to 1_L$. In particular, we can always find an increasing sequence (ϵ_n) in $L - \{0_L\}$ such that $\mathscr{D}(\epsilon_n) \to 1_L$.

Theorem 3.6. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, $st_{\mathscr{L}} - \lim x = \ell$ if and only if there exists a subset K of $\mathbb{N}$ such that $\delta(K) = 1$ and $\lim_{n\to\infty} x_{k_m} = \ell$ for $k_m \in K$ and $k_m < k_{m+1}$ for all $m \in \mathbb{N}$.*

Proof. Suppose that $st_{\mathscr{L}} - \lim x = \ell$. Let (ϵ_n) be a decreasing sequence in $L - \{0_L\}$ such that $\mathscr{D}(\epsilon_n) \to 1_L$ in L, and for any $t > 0$ and $i \in \mathbb{N}$, let

$$K(i) = \{k \leq n : \nu(x_k - l, t) \succ \mathscr{D}(\epsilon_i)\}$$

Then, for any $t > 0$ and $i \in \mathbb{N}$,

$$K(i + 1) \subset K(i).$$

Since $st_{\mathscr{L}} - \lim x = \ell$, it is obvious that

$$\delta(K(i)) = 1, (i \in \mathbb{N} \text{ and } t > 0).$$

Now let p_1 be an arbitrary number of $K(1)$. Then there is a number $p_2 \in K(2)$, $(p_2 > p_1)$, such that for all $n > p_2$,

$$\frac{1}{n}|\{k \leq n : \nu(x_k - \ell, t) \succ \mathscr{D}(\epsilon_2)\}| > \frac{1}{2}.$$

Further, there is a number $p_3 \in K(3)$, $(p_3 > p_2)$ such that for all $n > p_3$,

$$\frac{1}{n}|\{k \leq n : \nu(x_k - l, t) \succ \mathscr{N}(\epsilon_3)\}| > \frac{2}{3}$$

and so on. So, we can construct, by induction, an increasing index sequence $(p_i)_{i\in\mathbb{N}}$ of the natural numbers such that $p_i \in K(i)$ and that the following statement holds for all $n > p_i$ $(i \in \mathbb{N})$:

$$\frac{1}{n}|\{k \leq n : \nu(x_k - l, t) \succ \mathscr{D}(\epsilon_i)\}| > \frac{i-1}{i}.$$

Now we will build an increasing index sequence:

$$K := \{k \leq n : 1 < k < p_1\} \cup \Big[\bigcup_{i\in\mathbb{N}} \{k \in K(i) : p_i \leq k < p_{i+1}\}\Big]$$

Therefore, it is obvious that $\delta(K) = 1$. Now let $\varepsilon \succ 0_L$ and choose a positive integer i such that $\varepsilon_i \prec \varepsilon$. Such a number i always exists since $(\varepsilon_k) \to 0_L$. Assume that $k \geq p_i$ and $k \in K$. Then, according to the definition of K, a number $a \geq i$ exists such that $p_a \leq k < p_{a+1}$ and $k \in K(i)$. Therefore, for every $\varepsilon \succ 0_L$

$$\nu(x_k - l, t) \succ \mathscr{D}(\varepsilon_i) \succ \mathscr{D}(\varepsilon)$$

for all $k \geq p_i$ and $k \in K$ and

$$\mathscr{L} - \lim_{m\in K} x_{k_m} = \ell.$$

In the opposite case, assume there is an increasing index sequence $K = (k_m)_{m\in\mathbb{N}}$ of pairs of natural numbers such that $\delta(K) = 1$ and $\mathscr{L} - \lim_{m\in K} x_{k_m} = \ell$. Hence, for every $\varepsilon \succ 0_L$ there is a number n_0 such that for each $k \geq n_0$ the inequality $\nu(x_k - \ell, t) \succ \mathscr{D}(\varepsilon)$ holds. Now define

$$M(\varepsilon) := \{k \leq n : \nu(x_k - \ell, t) \nsucc \mathscr{D}(\varepsilon)\}.$$

Therefore, there exists an $n_0 \in \mathbb{N}$ such that

$$M(\varepsilon) \subseteq \mathbb{N} - (K - \{k_m : m \leq n_0\}).$$

Since $\delta(K) = 1$, we get $\delta(\mathbb{N} - (K - \{k_m : m \leq n_0\})) = 0$, which yields that $\delta(M(\varepsilon)) = 0$. That means, $st_{\mathscr{L}} - \lim x = \ell$. □

4. Statistically Cauchy and Statistically Bounded Sequences

Definition 4.1. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}$−fuzzy normed space. Then a sequence $x = (x_n)$ is said to be statistically Cauchy with respect to fuzzy norm ν, provided that*

$$\delta(\{n \in \mathbb{N} : \nu(x_n - x_m, t) \nsucc \mathscr{D}(\epsilon)\}) = 0$$

for each $\epsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$.

Definition 4.2. *Let $(V, \nu, \mathscr{K})$ be a $\mathscr{L}$−fuzzy normed space. Then a sequence $x = (x_n)$ is said to be statistically bounded with respect to fuzzy norm ν, provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that*

$$\delta(\{n \in \mathbb{N} : \nu(x_n, t) \nsucc \mathscr{D}(r)\}) = 0.$$

Theorem 4.3. *Every bounded sequence on a $\mathscr{L}$−fuzzy normed space $(V, \nu, \mathscr{K})$, is statistically bounded.*

Proof. Let (x_n) be a bounded sequence on $(V, \nu, \mathscr{K})$. Then there exist $t > 0$ and $r \in L - \{0_L, 1_L\}$ such that $\nu(x_n, t) \succ \mathscr{D}(r)$. In that case we have,

$$\{n \in \mathbb{N} : \nu(x_n, t) \nsucc \mathscr{D}(r)\} = \emptyset$$

which yields

$$\delta(\{n \in \mathbb{N} : \nu(x_n, t) \nsucc \mathscr{D}(r)\}) = 0.$$

Thus (x_n) is statistically bounded. □

However the converse of this theorem does not hold in general as seen in the example below.

Example 4.4. *Let $V = \mathbb{R}$ and $\mathscr{L} = (L, \leq)$ where L is the set of non-negative extended real numbers, that is $L = [0, \infty]$. Then $0_L = 0, 1_L = \infty$. Define a $\mathscr{L}$−fuzzy norm ν on V by $\nu(x, t) = \frac{t}{|x|}$ for $x \neq 0$ and $\nu(0, t) = \infty$ for each $t \in (0, \infty)$. Consider the t− norm $\mathscr{K}(a, b) = \min\{a, b\}$ on $\mathscr{L}$. Given the sequence,*

$$x_n = \begin{cases} n, & \text{if } n \text{ is a prime number} \\ \frac{1}{\tau(n)-2}, & \text{otherwise} \end{cases}$$

where $\tau(n)$ denotes the number of positive divisors of n. Note that (x_n) is not bounded since for each $t > 0$ and $r \in L - \{0, \infty\}$, for any prime number n such that $rt \leq n$ we have

$$\nu(x_n, t) = \nu(n, t) = \frac{t}{|\, n \,|} = \frac{t}{n} \nsucc \frac{1}{r} = \mathscr{D}(r).$$

However for $t = 1$ and any non-prime integer n, $r = 2$ satisfies

$$\nu(x_n, 1) = \nu(\frac{1}{\tau(n) - 2}, 1) = \frac{1}{|\frac{1}{\tau(n)-2}|} = |\tau(n) - 2| > \frac{1}{2} = \mathscr{D}(r)$$

since $\tau(n) \neq 2$ for any non-prime n, and since the density of prime numbers converges zero by Prime Number Theorem we have,

$$\delta(\{n \in \mathbb{N} : \nu(x_n, 1) \nsucc \mathscr{D}(2)\}) = 0$$

suggesting that (x_n) is statistically bounded.

Theorem 4.5. *Every statistically Cauchy sequence on a $\mathscr{L}-$fuzzy normed space $(V, \nu, \mathscr{K})$ is statistically bounded.*

Proof. Let (x_n) be a statistically Cauchy sequence on $(V, \nu, \mathscr{K})$. Then for each $\epsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$,

$$\delta(\{n \in \mathbb{N} : \nu(x_n - x_m, t) \not> \mathscr{D}(\epsilon)\}) = 0.$$

Then

$$\delta(\{n \in \mathbb{N} : \nu(x_n - x_m, t) > \mathscr{D}(r)\}) = 1.$$

Consider a number $n \in \mathbb{N}$ such that $\nu(x_n - x_m, 1) > \mathscr{D}(\epsilon)$. Then for $t = 2$

$$\nu(x_n, 2) = \nu(x_n - x_m + x_m, 2) > \mathscr{K}(\nu(x_n - x_m, 1), \nu(x_m, 1)) > \mathscr{K}(\mathscr{D}(\epsilon), \nu(x_m, 1)).$$

Say $r := \mathscr{D}(\mathscr{K}(\mathscr{D}(\epsilon), \nu(x_m, 1)))$. Then

$$\nu(x_n, 2) > \mathscr{K}(\mathscr{D}(\epsilon), \nu(x_m, 1)) = \mathscr{D}(r),$$

which implies

$$\delta(\{n \in \mathbb{N} : \nu(x_n, 2) > \mathscr{D}(r)\}) = 1$$

or equivalently

$$\delta(\{n \in \mathbb{N} : \nu(x_n, 2) \not> \mathscr{D}(r)\}) = 0$$

giving statistically boundedness of (x_n). □

5. Conclusion

In the framework of the present study, some properties related to the statistical convergence of sequences are investigated on L-fuzzy normed spaces, a structure which generalize various structures such as normed spaces, fuzzy normed spaces and IF-normed spaces and offer a flexible framework. Also some new concepts have been defined and some of the relationships between them are exemplified. These findings can be synthesized with the lattice structure and the normed space structure and thus they can make it possible to benefit from the convenience provided by a variation of the notion of norm on a wider family of topological vector space spaces. In another study, similar inquiries have been conducted for some mathematical tools such as double sequences, and some other research is being done to bring the results into a framework that is more general in some aspects and more elegant, at the same time.

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