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On the g_z-Kato decomposition and generalization of Koliha Drazin invertibility

Zakariae Aznay[a], Abdelmalek Ouahab[a], Hassan Zariouh[b]

[a] *Laboratory (L.A.N.O), Department of Mathematics, Faculty of Science, Mohammed I University, Oujda 60000 Morocco.*
[b] *Department of Mathematics (CRMEFO), and laboratory (L.A.N.O), Faculty of Science, Mohammed I University, Oujda 60000 Morocco.*

Abstract. In [24], Koliha proved that $T \in L(X)$ (X is a complex Banach space) is generalized Drazin invertible operator iff there exists an operator S commuting with T such that $STS = S$ and $\sigma(T^2S - T) \subset \{0\}$ iff $0 \notin \operatorname{acc}\sigma(T)$. Later, in [14, 34] the authors extended the class of generalized Drazin invertible operators and they also extended the class of pseudo-Fredholm operators introduced by Mbekhta [27] and other classes of semi-Fredholm operators. As a continuation of these works, we introduce and study the class of g_z-invertible (resp., g_z-Kato) operators which generalizes the class of generalized Drazin invertible operators (resp., the class of generalized Kato-meromorphic operators introduced by Živković-Zlatanović and Duggal in [35]). Among other results, we prove that T is g_z-invertible iff T is g_z-Kato with $\tilde{p}(T) = \tilde{q}(T) < \infty$ iff there exists a commuting operator S with T such that $STS = S$ and $\operatorname{acc}\sigma(T^2S - T) \subset \{0\}$ iff $0 \notin \operatorname{acc}(\operatorname{acc}\sigma(T))$. As application and using the concept of the Weak SVEP introduced at the end of this paper, we give new characterizations of Browder-type theorems.

1. Introduction

Let $T \in L(X)$, where $L(X)$ is the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space $(X, \|.\|)$. Throughout this paper T^*, $\alpha(T)$ and $\beta(T)$ means respectively, the dual of T, the dimension of the kernel $\mathcal{N}(T)$ and the codimension of the range $\mathcal{R}(T)$. The ascent and the descent of T are defined by $p(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ (with $\inf\emptyset = \infty$) and $q(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$. A subspace M of X is T-invariant if $T(M) \subset M$ and the restriction of T on M is denoted by T_M. $(M, N) \in \operatorname{Red}(T)$ if M, N are closed T-invariant subspaces and $X = M \oplus N$ ($M \oplus N$ means that $M \cap N = \{0\}$). Let $n \in \mathbb{N}$, denote by $T_{[n]} = T_{\mathcal{R}(T^n)}$ and by $m_T = \inf\{n \in \mathbb{N} : \inf\{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty\}$ the *essential degree* of T. According to [10, 28], T is called upper semi-B-Fredholm (resp., lower semi-B-Fredholm) if the *essential ascent* $p_e(T) = \inf\{n \in \mathbb{N} : \alpha(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{p_e(T)+1})$ is closed (resp., the *essential descent* $q_e(T) = \inf\{n \in \mathbb{N} : \beta(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{q_e(T)})$ is closed). If T is an upper or a lower (resp., upper and lower) semi-B-Fredholm, then T is called *semi-B-Fredholm* (resp., *B-Fredholm*) and its index is defined by $\operatorname{ind}(T) = \alpha(T_{[m_T]}) - \beta(T_{[m_T]})$. T is said to be an upper semi-B-Weyl (resp., lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) if T is an upper semi-B-Fredholm with $\operatorname{ind}(T) \leq 0$ (resp., T is a lower semi-B-Fredholm with $\operatorname{ind}(T) \geq 0$, T is a B-Fredholm with

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Email addresses: aznay.zakariae@ump.ac.ma (Zakariae Aznay), ouahab05@yahoo.fr (Abdelmalek Ouahab), h.zariouh@yahoo.fr (Hassan Zariouh)

$\mathrm{ind}(T) = 0$, T is an upper semi-B-Fredholm and $p(T_{[m_T]}) < \infty$, T is a lower semi-B-Fredholm and $q(T_{[m_T]}) < \infty$, $p(T_{[m_T]}) = q(T_{[m_T]}) < \infty$). If T is upper semi-B-Fredholm (resp., lower semi-B-Fredholm, semi-B-Fredholm, B-Fredholm, upper semi-B-Weyl, lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) with essential degree $m_T = 0$, then T is said to be an upper semi-Fredholm (resp., lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, upper semi-Browder, lower semi-Browder, Browder) operator. T is said to be bounded below if T is upper semi-Fredholm with $\alpha(T) = 0$.

The degree of stable iteration of T is defined by $\mathrm{dis}(T) = \inf \Delta(T)$, where

$$\Delta(T) = \{m \in \mathbb{N} : \alpha(T_{[m]}) = \alpha(T_{[r]}),\ \forall r \in \mathbb{N}\ r \geq m\}.$$

T is said to be semi-regular if $\mathcal{R}(T)$ is closed and $\mathrm{dis}(T) = 0$, and is said to be quasi-Fredholm if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-regular, see [25, 27]. Note that every semi-B-Fredholm operator is quasi-Fredholm [10, Proposition 2.5].

According to [1], T is said to have the SVEP at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$. T is said to have the SVEP on $A \subset \mathbb{C}$ if T has the SVEP at every $\lambda \in A$, and is said to have the SVEP if it has the SVEP on $\mathbb{C}$. It is easily seen that $T \oplus S$ has the SVEP at λ if and only if T and S have the SVEP at λ, see [1, Theorem 2.15]. Moreover,

$$p(T - \lambda I) < \infty \Longrightarrow \text{ T has the SVEP at } \lambda \quad (A)$$

$$q(T - \lambda I) < \infty \Longrightarrow T^* \text{ has the SVEP at } \lambda, \quad (B)$$

and these implications become equivalences if $T - \lambda I$ has topological uniform descent [1, Theorem 2.97, Theorem 2.98]. For definitions and properties of operators which have topological uniform descent, see [18].

Definition 1.1. *[1] (i) The local spectrum of T at $x \in X$ is the set defined by*

$$\sigma_T(x) := \left\{ \begin{array}{l} \lambda \in \mathbb{C} : \text{ for all open neighborhood } U_\lambda \text{ of } \lambda \text{ and analytic function} \\ f : U_\lambda \longrightarrow X \text{ there exists } \mu \in U_\lambda \text{ such that } (T - \mu I)f(\mu) \neq x. \end{array} \right\}$$

(ii) If F is a complex closed subset, then the local spectral subspace of T associated to F is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subset F\}.$$

A Banach space operator S is said to be nilpotent of degree d if $S^d = 0$ and $S^{d-1} \neq 0$ [with the degree of the null operator takes 0 if it acts on the space $\{0\}$ and takes 1 otherwise]. S is a quasi-nilpotent (resp., Riesz, meromorphic) operator if $S - \lambda I$ is invertible (resp., Browder, Drazin invertible) for all non-zero complex λ. Note that S is nilpotent $\Longrightarrow$ S is quasi-nilpotent $\Longrightarrow$ S is Riesz $\Longrightarrow$ S is meromorphic. Denote by $\mathcal{K}(T)$ the analytic core of T (see [27]):

$$\mathcal{K}(T) = \{x \in X : \exists \epsilon > 0, \exists (u_n)_n \subset X \text{ such that } x = u_0, Tu_{n+1} = u_n \text{ and } \|u_n\| \leq \epsilon^n \|x\|\ \forall n \in \mathbb{N}\},$$

and by $\mathcal{H}_0(T)$ the quasi-nilpotent part of T : $\quad \mathcal{H}_0(T) = \{x \in X : \lim\limits_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0\}$.

In [23, Theorem 4, 1958], Kato proved that if T is a semi-Fredholm operator, then T is of Kato-type of degree d, that is there exists $(M, N) \in \mathrm{Red}(T)$ such that:

(i) T_M is semi-regular.
(ii) T_N is nilpotent of degree d.

Later, these operators are characterized by Labrousse [25, 1980] in the case of Hilbert space. The important results obtained by Kato and Labrousse opened the field to many researchers to work in this direction [7, 11, 14, 16, 27, 33–35]. In particular, Berkani [7] showed that T is B-Fredholm (resp., B-Weyl) if and only if there exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is Fredholm (resp., Weyl) and T_N is nilpotent. On the other hand,

it is well known [16] that T is Drazin invertible if and only if there exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is invertible and T_N is nilpotent.
If the condition (ii) "T_N is nilpotent" mentioned in the Kato's decomposition is replaced by "T_N is quasi-nilpotent" (resp., "T_N is Riesz", "T_N is meromorphic"), we find the pseudo-Fredholm [27] (resp., generalized Kato-Riesz [34], generalized Kato-meromorphic [35]) decomposition. By the same argument the pseudo B-Fredholm [32, 33] (resp., generalized Drazin-Riesz Fredholm [11, 34], generalized Drazin-meromorphic Fredholm [35]) decomposition are obtained by substituting in the B-Fredholm decomposition the condition "T_N is nilpotent" by "T_N is quasi-nilpotent" (resp., "T_N is Riesz", "T_N is meromorphic"). Similarly, the Drazin decomposition has been generalized [24, 34, 35].

We summarize in the following definition several known decompositions.

Definition 1.2. *[5, 7, 10–12, 14, 27, 33–35] T is said to be*
(i) of Kato-type of order d [resp., quasi upper semi-B-Fredholm, quasi lower semi-B-Fredholm, quasi B-Fredholm, quasi upper semi-B-Weyl, quasi lower semi-B-Weyl, quasi semi-B-Weyl] if there exists $(M, N) \in Red(T)$ such that T_M is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl] and T_N is nilpotent of degree d. We write $(M, N) \in KD(T)$ if it is a Kato-type decomposition.
(ii) Pseudo-Fredholm [resp., upper pseudo semi-B-Fredholm, lower pseudo semi-B-Fredholm, pseudo B-Fredholm, upper pseudo semi-B-Weyl, lower pseudo semi-B-Weyl, pseudo B-Weyl, left generalized Drazin invertible, right generalized Drazin invertible, generalized Drazin invertible] if there exists $(M, N) \in Red(T)$ such that T_M is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, bounded below, surjective, invertible] and T_N is quasi-nilpotent. We write $(M, N) \in GKD(T)$ if it is a pseudo-Fredholm type decomposition.
(iii) Generalized Kato-Riesz [resp., generalized Drazin-Riesz upper semi-Fredholm, generalized Drazin-Riesz lower semi-Fredholm, generalized Drazin-Riesz Fredholm, generalized Drazin-Riesz upper semi-Weyl, generalized Drazin-Riesz lower semi-Weyl, generalized Drazin-Riesz Weyl, generalized Drazin-Riesz bounded below, generalized Drazin-Riesz surjective, generalized Drazin-Riesz invertible] if there exists $(M, N) \in Red(T)$ such that T_M is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, bounded below, surjective, invertible] and T_N is Riesz.
(iv) Generalized Kato-meromorphic [resp., generalized Drazin-meromorphic upper semi-Fredholm, generalized Drazin-meromorphic lower semi-Fredholm, generalized Drazin-meromorphic Fredholm, generalized Drazin-meromorphic upper semi-Weyl, generalized Drazin-meromorphic lower semi-Weyl, generalized Drazin-meromorphic Weyl, generalized Drazin-meromorphic bounded below, generalized Drazin-meromorphic surjective, generalized Drazin-meromorphic invertible] if there exists $(M, N) \in Red(T)$ such that T_M is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, bounded below, surjective, invertible] and T_N is meromorphic.

As a continuation of the studies mentioned above, we define new classes of operators: one of them named g_z-Kato which generalizes the class of generalized Kato-meromorphic operators. We prove that the g_z-Kato spectrum $\sigma_{g_zK}(T)$ is compact and $\operatorname{acc}\sigma_{pf}(T) \subset \sigma_{g_zK}(T)$. Moreover, we show that if T is g_z-Kato, then $\alpha(T_M)$, $\beta(T_M)$, $p(T_M)$ and $q(T_M)$ are independent of the choice of the decomposition $(M, N) \in g_zKD(T)$. An other class named g_z-invertible which generalizes the class of generalized Drazin invertible operators introduced by Koliha. As a characterization of g_z-invertible operator, we prove that T is g_z-invertible iff $0 \notin \operatorname{acc}(\operatorname{acc}\sigma(T))$ iff there exists a Drazin invertible operator S such that $TS = ST$, $STS = S$ and $T^2S - T$ is zeroloid. These characterizations are analogous to those proved by Koliha [24] which established that T is generalized Drazin invertible operator iff $0 \notin \operatorname{acc}\sigma(T)$ iff there exists an operator S such that $TS = ST$, $STS = S$ and $T^2S - T$ is quasi-nilpotent. As application, using the new spectra studied in the present work and the concept of the Weak SVEP introduced at the end of this paper, we give new characterizations of Browder-type theorems.

The next list summarizes some notations and symbols that we will need later.

$r(T)$	: the spectral radius of T
$\operatorname{iso} A$	: isolated points of a complex subset A
$\operatorname{acc} A$	: accumulation points of a complex subset A
$\overline{A}$	: the closure of a complex subset A
A^C	: the complementary of a complex subset A
$B(\lambda, \epsilon)$	: the open ball of radius ϵ centered at λ
$D(\lambda, \epsilon)$	: the closed ball of radius ϵ centered at λ
(B)	: the class of operators satisfying Browder's theorem ($T \in (B)$ if $\sigma_w(T) = \sigma_b(T)$)
(B_e)	: the class of operators satisfying essential Browder's theorem [4] ($T \in (B_e)$ if $\sigma_e(T) = \sigma_b(T)$)
(aB)	: the class of operators satisfying a-Browder's theorem ($T \in (aB)$ if $\sigma_{uw}(T) = \sigma_{ub}(T)$)

$\sigma(T)$: spectrum of T	$\sigma_{pf}(T)$: pseudo-Fredholm spectrum of T
$\sigma_a(T)$: approximate points spectrum of T	$\sigma_{pbf}(T)$: pseudo B-Fredholm spectrum of T
$\sigma_s(T)$: surjective spectrum of T	$\sigma_{upbf}(T)$: upper pseudo semi-B-Fredholm spectrum of T
$\sigma_{se}(T)$: semi-regular spectrum of T	$\sigma_{lpbf}(T)$: lower pseudo semi-B-Fredholm spectrum of T
$\sigma_e(T)$: essential spectrum of T	$\sigma_{pbw}(T)$: pseudo B-Weyl spectrum of T
$\sigma_{uf}(T)$: upper semi-Fredholm spectrum of T	$\sigma_{upbw}(T)$: upper pseudo semi-B-Weyl spectrum of T
$\sigma_{lf}(T)$: lower semi-Fredholm spectrum of T	$\sigma_{lpbw}(T)$: lower pseudo semi-B-Weyl spectrum of T
$\sigma_w(T)$: Weyl spectrum of T	$\sigma_{gd}(T)$: generalized Drazin invertible spectrum of T
$\sigma_{uw}(T)$: upper semi-Weyl spectrum of T	$\sigma_{lgd}(T)$: left generalized Drazin invertible spectrum of T
$\sigma_{lw}(T)$: lower semi-Weyl spectrum of T	$\sigma_{rgd}(T)$: right generalized Drazin invertible spectrum of T
$\sigma_b(T)$: Browder spectrum of T	$\sigma_d(T)$: Drazin spectrum of T
$\sigma_{bf}(T)$: B-Fredholm spectrum of T	$\sigma_{bw}(T)$: B-Weyl spectrum of T

2. The g_z-Kato decomposition

We begin this section by the following definition of zeroloid operators.

Definition 2.1. *We say that $T \in L(X)$ is a zeroloid operator if $\operatorname{acc}\sigma(T) \subset \{0\}$.*

The next remark summarizes some properties of zeroloid operators.

Remark 2.2. *(i) A zeroloid operator has at most a countable spectrum.*
(ii) Since $\operatorname{acc}\sigma(T) \subset \sigma_d(T)$ for every $T \in L(X)$, then every meromorphic operator is zeroloid. But the operator $I + Q$ shows that the converse is not true, where I is the identity operator and Q is the quasi-nilpotent operator defined on the Hilbert space $\ell^2(\mathbb{N})$ by $Q(x_1, x_2, \dots) = (0, x_1, \frac{x_2}{2}, \dots)$.
(iii) T is zeroloid if and only if T^n is zeroloid for every integer $n \geq 1$.
(iv) Let $(T, S) \in L(X) \times L(Y)$, then $T \oplus S$ is zeroloid if and only if T and S are zeroloid.
(v) Here and elsewhere denote by $comm(T) = \{S \in L(X) : TS = ST\}$. So if $Q \in comm(T)$ is a quasi-nilpotent or a power finite rank operator, then T is zeroloid if and only if $T + Q$ is zeroloid.

According to [4], the p-ascent $\tilde{p}(T)$ and the p-descent $\tilde{q}(T)$ of a pseudo-Fredholm operator $T \in L(X)$ are defined respectively, by $\tilde{p}(T) = p(T_M)$ and $\tilde{q}(T) = q(T_M)$, where M is any subspace which complemented by a subspace N such that $(M, N) \in GKD(T)$.

Proposition 2.3. *If $T \in L(X)$ is a pseudo-Fredholm operator, then the following statements are equivalent:*
(a) $\tilde{p}(T) < \infty$;
(b) T has the SVEP at 0;
(c) $\mathcal{H}_0(T) \cap \mathcal{K}(T) = \{0\}$;
(d) $\mathcal{H}_0(T)$ is closed.
dually, the following are equivalent:
(e) $\tilde{q}(T) < \infty$;

(f) T^* has the SVEP at 0;
(g) $\mathcal{H}_0(T) + \mathcal{K}(T) = X$.

Proof. (a) $\Longleftrightarrow$ (b) Let $(M, N) \in GKD(T)$, then T_M is semi-regular and T_N is quasi-nilpotent. As $p(T_M) = \tilde{p}(T)$ then by the implication (A) above, we deduce that $\tilde{p}(T) < \infty$ if and only if T_M has the SVEP at 0. Hence $\tilde{p}(T) < \infty$ if and only if T has the SVEP at 0. The equivalence (e) $\Longleftrightarrow$ (f) goes similarly. The equivalences (b) $\Longleftrightarrow$ (c), (c) $\Longleftrightarrow$ (d) and (f) $\Longleftrightarrow$ (g) are proved in [1, Theorem 2.79, Theorem 2.80]. □

Lemma 2.4. *For $T \in L(X)$, the following statements are equivalent:*
(i) T is zeroloid;
(ii) $\sigma_*(T) \subset \{0\}$, where $\sigma_* \in \{\sigma_{pf}, \sigma_{upbf}, \sigma_{lpbf}, \sigma_{upbw}, \sigma_{lpbw}, \sigma_{lgd}, \sigma_{rgd}, \sigma_{pbf}, \sigma_{pbw}\}$.

Proof. (i) $\Longrightarrow$ (ii) Obvious, since $\sigma_{gd}(T) = \operatorname{acc}\sigma(T)$.
(ii) $\Longrightarrow$ (i) If $\sigma_*(T) \subset \{0\}$, then $\mathbb{C} \setminus \{0\} \subset \Omega$, where Ω is the component of $(\sigma_{pf}(T))^C$. Suppose that there exists $\lambda \in \operatorname{acc}\sigma(T) \setminus \{0\}$, then $\lambda \notin \sigma_*(T)$ and hence $\tilde{p}(T - \lambda I) = \infty$ or $\tilde{q}(T - \lambda I) = \infty$, but this is impossible. Indeed, assume that $\tilde{p}(T - \lambda I) = \infty$, as $T - \lambda I$ is pseudo-Fredholm, from Proposition 2.3 we have $\mathcal{H}_0(T - \lambda I) \cap \mathcal{K}(T - \lambda I) \neq \{0\}$. And from [12, Corollary 4.3], we obtain $\overline{\mathcal{H}_0(T - \lambda I)} \cap \mathcal{K}(T - \lambda I) = \overline{\mathcal{H}_0(T - \mu I)} \cap \mathcal{K}(T - \mu I)$ for every $\mu \in \Omega$. This implies that $\tilde{p}(T - \mu I) = \infty$ for all $\mu \in \Omega \setminus \{0\}$ [otherwise $\mathcal{H}_0(T - \mu I)$ becomes closed for some $\mu \in \Omega \setminus \{0\}$ and then $\overline{\mathcal{H}_0(T - \lambda I)} \cap \mathcal{K}(T - \lambda I) = \{0\}$, which is impossible] and this is contradiction. Thus $\tilde{q}(T - \lambda I) = \infty$, but this leads (by the same argument) to a contradiction. Hence T is zeroloid. □

Proposition 2.5. *$T \in L(X)$ is zeroloid if and only if T_M and $T^*_{M^\perp}$ are zeroloid, where M is any closed T-invariant subspace.*

Proof. If T is zeroloid, then its resolvent $(\sigma(T))^C$ is connected. From [15, Proposition 2.10], we obtain that $\sigma(T) = \sigma(T_M) \cup \sigma(T^*_{M^\perp})$. Thus T_M and $T^*_{M^\perp}$ are zeroloid. Conversely, if T_M and $T^*_{M^\perp}$ are zeroloid, then T is zeroloid, since the inclusion $\sigma(T) \subset \sigma(T_M) \cup \sigma(T^*_{M^\perp})$ is always true. □

Definition 2.6. *Let $T \in L(X)$. A pair of subspaces $(M, N) \in Red(T)$ is a generalized Kato zeroloid decomposition associated to T [$(M, N) \in g_zKD(T)$ for brevity] if T_M is semi-regular and T_N is zeroloid. If such a pair exists, we say that T is a g_z-Kato operator.*

Example 2.7. *(i) Every zeroloid operator and every semi-regular operator are g_z-Kato.*
(ii) Every generalized Kato-meromorphic operator is g_z-Kato. But the converse is not true, see Example 4.13 below.

Our next result gives a punctured neighborhood theorem for g_z-Kato operators. Recall that the reduced minimal modulus $\gamma(T)$ of an operator T is defined by $\gamma(T) := \inf\limits_{x \notin \mathcal{N}(T)} \frac{\|Tx\|}{d(x, \mathcal{N}(T))}$, where $d(x, \mathcal{N}(T))$ is the distance between x and $\mathcal{N}(T)$.

Theorem 2.8. *Let $T \in L(X)$ be a g_z-Kato operator. For every $(M, N) \in g_zKD(T)$, there exists $\epsilon > 0$ such that for all $\lambda \in B(0, \epsilon) \setminus \{0\}$ we have*
(i) $T - \lambda I$ is pseudo-Fredholm.
(ii) $\alpha(T_M) = \dim \mathcal{N}(T - \lambda I) \cap \mathcal{K}(T - \lambda I) \leq \alpha(T - \lambda I)$.
(iii) $\beta(T_M) = \operatorname{codim}[\mathcal{R}(T - \lambda I) + \mathcal{H}_0(T - \lambda I)] \leq \beta(T - \lambda I)$.

Proof. Let $\epsilon = \gamma(T_M) > 0$ and let $\lambda \in B(0, \epsilon) \setminus \{0\}$. From [18, Theorem 4.7], $T_M - \lambda I$ is semi-regular, $\alpha(T_M) = \alpha(T_M - \lambda I)$ and $\beta(T_M) = \beta(T_M - \lambda I)$. As T_N is zeroloid then from [4], $T_N - \lambda I$ is pseudo-Fredholm with $\mathcal{N}(T_N - \lambda I) \cap \mathcal{K}(T_N - \lambda I) = \{0\}$ and $N = \mathcal{R}(T_N - \lambda I) + \mathcal{H}_0(T_N - \lambda I)$. Hence $T - \lambda I$ is pseudo-Fredholm, $\alpha(T_M) = \dim \mathcal{N}(T - \lambda I) \cap \mathcal{K}(T - \lambda I)$ and $\beta(T_M) = \operatorname{codim}[\mathcal{R}(T - \lambda I) + \mathcal{H}_0(T - \lambda I)]$. □

Since every pseudo-Fredholm operator is g_z-Kato, from Theorem 2.8 we immediately obtain the following corollary. Hereafter, we denote by $\sigma_{g_zK}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not } g_z\text{-Kato operator}\}$ the g_z-Kato spectrum.

Corollary 2.9. *The g_z-Kato spectrum $\sigma_{g_zK}(T)$ of an operator $T \in L(X)$ is compact.*

Proposition 2.10. *If $T \in L(X)$ is a g_z-Kato operator, then $\alpha(T_M)$, $\beta(T_M)$, $p(T_M)$ and $q(T_M)$ are independent of the choice of the generalized Kato zeroloid decomposition $(M, N) \in g_zKD(T)$.*

Proof. Let $(M_1, N_1), (M_2, N_2) \in g_zKD(T)$ and let $n \geq 1$. It is easily seen that T^n is also a g_z-Kato operator and $(M_1, N_1), (M_2, N_2) \in g_zKD(T^n)$. We put $\epsilon_n = \min\{\gamma(T^n_{M_1}), \gamma(T^n_{M_2})\}$. If $\lambda \in B(0, \epsilon_n) \setminus \{0\}$, then by Theorem 2.8 we obtain $\alpha(T^n_{M_1}) = \alpha(T^n_{M_2}) = \dim \mathcal{N}(T^n - \lambda I) \cap \mathcal{K}(T^n - \lambda I)$ and $\beta(T^n_{M_1}) = \beta(T^n_{M_2}) = \operatorname{codim} [\mathcal{R}(T^n - \lambda I) + \mathcal{H}_0(T^n - \lambda I)]$. Hence $p(T_{M_1}) = p(T_{M_2})$ and $q(T_{M_1}) = q(T_{M_2})$. □

Let $T \in L(X)$ be a g_z-Kato operator. Following Proposition 2.10, we denote by $\tilde{\alpha}(T) = \alpha(T_M)$, $\tilde{\beta}(T) = \beta(T_M)$, $\tilde{p}(T) = p(T_M)$ and $\tilde{q}(T) = q(T_M)$, where $(M, N) \in g_zKD(T)$ be arbitrary. If in addition, T_M is semi-Fredholm, then for every $(M', N') \in g_zKD(T)$ the operator $T_{M'}$ is also semi-Fredholm and $\text{ind}(T_M) = \text{ind}(T_{M'})$ (this result will be extended in Lemma 3.4).

The next lemma extends [30, Theorem A.16]. In the sequel, for $T \in L(X)$ and $(M, N) \in \text{Red}(T)$, we define the operator $T_{(M,N)} \in L(X)$ by $T_{(M,N)} = TP_M + P_N$, where P_M is the projection operator on X onto M.

Lemma 2.11. *Let $T \in L(X)$ and let $(M, N) \in Red(T)$. The following assertions are equivalent:*
(i) $\mathcal{R}(T_M)$ is closed;
(ii) $\mathcal{R}(T^*_{N^\perp})$ is closed;
(iii) $\mathcal{R}(T^*_{N^\perp}) \oplus M^\perp$ is closed in the weak--topology $\sigma(X^*, X)$ on X^*.*

Proof. As $(M, N) \in \text{Red}(T)$ then $(P_N)^* = P_{M^\perp}$ and $(TP_M)^* = T^* P_{N^\perp}$. So $(T_{(M,N)})^* = (TP_M + P_N)^* = T^* P_{N^\perp} + P_{M^\perp} = T^*_{(N^\perp, M^\perp)}$. Thus $\mathcal{R}(T_{(M,N)}) = \mathcal{R}(T_M) \oplus N$ and $\mathcal{R}((T_{(M,N)})^*) = \mathcal{R}(T^*_{N^\perp}) \oplus M^\perp$. Moreover, $\mathcal{R}(T_M)$ is closed if and only if $\mathcal{R}(T_{(M,N)})$ is closed. By applying [30, Theorem A.16] to the operator $T_{(M,N)}$, the proof is complete. □

From this Lemma and some known classical properties of pseudo-Fredholm and quasi-Fredholm operators, we immediately obtain:

Corollary 2.12. *Let $T \in L(X)$. The following statements hold:*
(i) If T is pseudo-Fredholm, then $\mathcal{R}(T^*) + \mathcal{H}_0(T^*)$ is closed in $\sigma(X^*, X)$.
(ii) If T is a Hilbert space quasi-Fredholm operator of degree d, then $\mathcal{R}(T^*) + \mathcal{N}(T^{d*})$ is closed in $\sigma(X^*, X)$.

The following lemma extends some well known results in spectral theory, as relation between nullity, deficiency and some other spectral quantities of a given operator T and its dual T^*.

Lemma 2.13. *Let $T \in L(X)$ and let $(M, N) \in Red(T)$. The following statements hold:*
(i) T_M is semi-regular if and only if $T^*_{N^\perp}$ is semi-regular.
(ii) If $\mathcal{R}(T_M)$ is closed, then $\alpha(T_M) = \beta(T^*_{N^\perp})$, $\beta(T_M) = \alpha(T^*_{N^\perp})$, $p(T_M) = q(T^*_{N^\perp})$ and $q(T_M) = p(T^*_{N^\perp})$.
(iii) $\sigma_a(T_M) = \sigma_s(T^*_{N^\perp})$, $\sigma_s(T_M) = \sigma_a(T^*_{N^\perp})$, $\sigma_*(T_M) = \sigma_*(T^*_{N^\perp})$ and $r(T_M) = r(T^*_{N^\perp})$, where $\sigma_* \in \{\sigma, \sigma_{se}, \sigma_e, \sigma_{sf}, \sigma_{bf}, \sigma_d, \sigma_b\}$. Moreover, if T_M is semi-Fredholm, then $ind(T_M) = -ind(T^*_{N^\perp})$.

Proof. (i) We have $\mathcal{N}(T_{(M,N)}) = \mathcal{N}(T_M)$ and $(T_{(M,N)})^n = T^n_{(M,N)}$ for every $n \in \mathbb{N}$. It is easy to see that T_M is semi-regular if and only if $T_{(M,N)}$ is semi-regular. As $(T_{(M,N)})^* = T^*_{(N^\perp, M^\perp)}$ then T_M is semi-regular if and only if $T^*_{N^\perp}$ is semi-regular.
(ii) We have $\mathcal{N}((T_{(M,N)})^n) = \mathcal{N}(T^n_M)$ and $\mathcal{R}((T_{(M,N)})^n) = \mathcal{R}(T^n_M) \oplus N$ for every $n \in \mathbb{N}$. As $\mathcal{R}(T_{(M,N)}) = \mathcal{R}(T_M) \oplus N$ is closed then $\alpha(T_M) = \alpha(T_{(M,N)}) = \beta(T^*_{(N^\perp, M^\perp)}) = \beta(T^*_{N^\perp})$. The other equalities go similarly.
(iii) As $(T_M \oplus 0_N)^* = (TP_M)^* = T^* P_{N^\perp} = T^*_{N^\perp} \oplus 0_{M^\perp}$, then $\sigma_*(T_M) \cup \sigma_*(0_N) = \sigma_*(T_M \oplus 0_N) = \sigma_*(T^*_{N^\perp} \oplus 0_{M^\perp}) = \sigma_*(T^*_{N^\perp}) \cup \sigma_*(0_{M^\perp})$. We know that $\sigma_*(S) = \emptyset$ for every nilpotent operator S with $\sigma_* \in \{\sigma_{bf}, \sigma_d\}$. Furthermore, the first and the second points imply that $0 \in \sigma_*(T_M)$ if and only if $0 \in \sigma_*(T^*_{N^\perp})$, where $\sigma_* \in \{\sigma, \sigma_{se}, \sigma_e, \sigma_{sf}, \sigma_b\}$. So $\sigma_*(T_M) = \sigma_*(T^*_{N^\perp})$ and $r(T_M) = r(T^*_{N^\perp})$. The proof of the other equalities spectra is obvious, see Lemma 2.11. Moreover, if T_M is semi-Fredholm, then $T^*_{N^\perp}$ is also semi-Fredholm and $\text{ind}(T_M) = -\text{ind}(T^*_{N^\perp})$. □

Corollary 2.14. *Let $T \in L(X)$ and let $(M, N) \in Red(T)$. Then $(M, N) \in g_zKD(T)$ if and only if $(N^\perp, M^\perp) \in g_zKD(T^*)$. In particular, if T is g_z-Kato, then T^* is g_z-Kato.*

Proposition 2.15. *If $T \in L(X)$ is g_z-Kato, then*
(a) There exist $S, R \in L(X)$ such that:
(i) $T = S + R$, $RT = TR = 0$, S is quasi-Fredholm of degree $d \leq 1$ and R is zeroloid.
(ii) $\mathcal{N}(S) + \mathcal{N}(R) = X$ and $\mathcal{R}(S) \oplus \overline{\mathcal{R}(R)}$ is closed.
(b) There exist $S, R \in L(X)$ such that $SR = RS = (S + R) - I = T$, S is semi-regular and R is zeroloid.

Proof. (a) Let $(M, N) \in g_zKD(T)$. The operators $S = TP_M$ and $R = TP_N$ respond to the statement (a). Indeed, as T_N is zeroloid and $\operatorname{acc} \sigma(R) = \operatorname{acc} \sigma(T_N)$ then R is zeroloid. Suppose that $M \notin \{\{0\}, X\}$ (the other case is trivial) and let $n \in \mathbb{N} \geq 1$, then $\mathcal{N}(S^n) = N \oplus \mathcal{N}(T_M^n)$ and $\mathcal{R}(S) = \mathcal{R}(T_M)$ is closed. As T_M is semi-regular, it follows that $\mathcal{N}(S^n) + \mathcal{R}(S) = N + \mathcal{N}(T_M^n) + \mathcal{R}(T_M) = N + \mathcal{N}(T_M) + \mathcal{R}(T_M) = \mathcal{N}(S) + \mathcal{R}(S)$. Consequently, S is quasi-Fredholm of degree $d \leq 1$. Moreover, $\mathcal{N}(S) + \mathcal{N}(R) = X$ and $\mathcal{R}(S) \oplus \overline{\mathcal{R}(R)} = \mathcal{R}(T_M) \oplus \overline{\mathcal{R}(T_N)}$ is closed.
(b) Let $(M, N) \in g_zKD(T)$. If we take $S = T_{(M,N)}$ and $R = T_{(N,M)}$, then $SR = RS = (S + R) - I = T$, $S = T_M \oplus I_N$ is semi-regular and $R = I_M \oplus T_N$ is zeroloid. □

In the case of Hilbert space operator T, the next proposition shows that the statement (a) of Proposition 2.15 is equivalent to say that T is g_z-Kato.

Proposition 2.16. *If H is a Hilbert space, then $T \in L(H)$ is g_z-Kato if and only if there exist $S, R \in L(H)$ such that $T = S + R$ and*
(i) $RT = TR = 0$, S is quasi-Fredholm of degree $dis(S) \leq 1$, R is a zeroloid operator;
(ii) $\mathcal{N}(S) + \mathcal{N}(R) = H$ and $\mathcal{R}(S) \oplus \overline{\mathcal{R}(R)}$ is closed.

Proof. Assume that S is quasi-Fredholm of degree 1 (the case of S semi-regular is obvious), then from the proof of [27, Theorem 2.2], there exists $(M, N) \in GKD(S)$ such that $T_M = S_M$ and $T_N = R_N$. As R is zeroloid then Proposition 2.5 entails that T_N is zeroloid. Thus T is g_z-Kato. For the converse, see Proposition 2.15. □

3. g_z-Fredholm operators

Definition 3.1. *$T \in L(X)$ is said to be an upper semi-g_z-Fredholm (resp., lower semi-g_z-Fredholm, g_z-Fredholm) operator if there exists $(M, N) \in Red(T)$ such that T_M is an upper semi-Fredholm (resp., lower semi-Fredholm, Fredholm) operator and T_N is zeroloid. T is said a semi-g_z-Fredholm if it is an upper or a lower semi-g_z-Fredholm.*

Every zeroloid operator is g_z-Fredholm. Every generalized Drazin-meromorphic semi-Fredholm is a semi-g_z-Fredholm, and we show by Example 4.13 that the converse is generally not true.

The next proposition gives some relations between semi-g_z-Fredholm and g_z-Kato operators.

Proposition 3.2. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is semi-g_z-Fredholm [resp., upper semi-g_z-Fredholm, lower semi-g_z-Fredholm, g_z-Fredholm];
(ii) T is g_z-Kato and $\min\{\tilde{\alpha}(T), \tilde{\beta}(T)\} < \infty$ [resp., T is g_z-Kato and $\tilde{\alpha}(T) < \infty$, T is g_z-Kato and $\tilde{\beta}(T) < \infty$, T is g_z-Kato and $\max\{\tilde{\alpha}(T), \tilde{\beta}(T)\} < \infty$];
(iii) T is g_z-Kato and $0 \notin \operatorname{acc} \sigma_{spbf}(T)$ [resp., T is g_z-Kato and $0 \notin \operatorname{acc} \sigma_{upbf}(T)$, T is g_z-Kato and $0 \notin \operatorname{acc} \sigma_{lpbf}(T)$, T is g_z-Kato and $0 \notin \operatorname{acc} \sigma_{pbf}(T)$], where $\sigma_{spbf}(T) := \sigma_{upbf}(T) \cup \sigma_{lpbf}(T)$.

Proof. $(i) \Longleftrightarrow (ii)$ Assume that T is semi-g_z-Fredholm, then there exists $(A, B) \in \operatorname{Red}(T)$ such that T_A is semi-Fredholm and T_B is zeroloid. From [5, Corollary 3.7], there exists $(M, N) \in g_zKD(T)$ such that T_M is semi-Fredholm. Thus T is g_z-Kato operator and $\min\{\tilde{\alpha}(T), \tilde{\beta}(T)\} = \min\{\alpha(T_M), \beta(T_M)\} < \infty$. The converse is obvious. The other equivalence cases go similarly.
$(ii) \Longleftrightarrow (iii)$ Is a consequence of Theorem 2.8. □

Corollary 3.3. *$T \in L(X)$ is g_z-Fredholm if and only if T is an upper and a lower semi-g_z-Fredholm.*

The following lemma will allow us to define the index for semi-g_z-Fredholm operators.

Lemma 3.4. *Let $T \in L(X)$. If there exist two pair of closed T-invariant subspaces (M, N) and (M', N') such that $M \oplus N = M' \oplus N'$ is closed, T_M and $T_{M'}$ are semi-Fredholm, T_N and $T_{N'}$ are zeroloid, then $ind(T_M) = ind(T_{M'})$.*

Proof. As T_M and $T_{M'}$ are semi-Fredholm operators then from the punctured neighborhood theorem for semi-Fredholm operators, there exists $\epsilon > 0$ such that $B(0,\epsilon) \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C$, $\text{ind}(T_M - \lambda I) = \text{ind}(T_M)$ and $\text{ind}(T_{M'} - \lambda I) = \text{ind}(T_{M'})$ for every $\lambda \in B(0,\epsilon)$. From [4, Remark 2.4] and the fact that T_N and $T_{N'}$ are zeroloid, we conclude that $B_0 := B(0,\epsilon) \setminus \{0\} \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C \cap \sigma_{gd}(T_N)^C \cap \sigma_{gd}(T_{N'})^C \subset \sigma_{spbf}(T_{M\oplus N})^C$. Let $\lambda \in B_0$, then $(T - \lambda I)_{M\oplus N}$ is pseudo semi-B-Fredholm and $\text{ind}((T - \lambda I)_{M\oplus N}) = \text{ind}(T_M - \lambda I) + \text{ind}(T_N - \lambda I) = \text{ind}(T_{M'} - \lambda I) + \text{ind}(T_{N'} - \lambda I)$. Thus $\text{ind}(T_M) = \text{ind}(T_{M'})$. □

Definition 3.5. *Let $T \in L(X)$ be a semi-g_z-Fredholm. We define its index ind(T) as the index of T_M, where M is a closed T-invariant subspace which has a complementary closed T-invariant subspace N such that T_M is semi-Fredholm and T_N is zeroloid. From Lemma 3.4, the index of T is independent of the choice of the pair (M,N) appearing in Definition 3.1 of T as a semi-g_z-Fredholm. In addition, we have from Proposition 3.2, ind$(T) = \tilde{\alpha}(T) - \tilde{\beta}(T)$.*

We say that $T \in L(X)$ is an upper semi-g_z-Weyl (resp., lower semi-g_z-Weyl, g_z-Weyl) operator if T is an upper semi-g_z-Fredholm (resp., lower semi-g_z-Fredholm, g_z-Fredholm) with $\text{ind}(T) \le 0$ (resp., $\text{ind}(T) \ge 0$, $\text{ind}(T) = 0$).

Remark 3.6. *(i) Every zeroloid operator T is g_z-Fredholm with $\tilde{\alpha}(T) = \tilde{\beta}(T) = ind(T) = 0$. A pseudo semi-B-Fredholm is semi-g_z-Fredholm and its usual index coincides with its index as a semi-g_z-Fredholm.*
(ii) T is g_z-Fredholm if and only if T is semi-g_z-Fredholm with an integer index. And T is g_z-Weyl if and only if T is upper and lower semi-g_z-Weyl.

Proposition 3.7. *If $T \in L(X)$ and $S \in L(Y)$ are semi-g_z-Fredholm, then*
(i) T^n is semi-g_z-Fredholm and $ind(T^n) = n.ind(T)$ for every integer $n \ge 1$.
(ii) $T \oplus S$ is semi-g_z-Fredholm and $ind(T \oplus S) = ind(T) + ind(S)$.

Proof. (i) As T is semi-g_z-Fredholm, then there exists $(M,N) \in \text{Red}(T)$ such that T_M is semi-Fredholm and T_N is zeroloid. So $(M,N) \in Red(T^n)$, T^n_M is semi-Fredholm and T^n_N is zeroloid. Thus $\text{ind}(T^n) = \text{ind}(T^n_M) = n.\text{ind}(T_M) = n.\text{ind}(T)$.
(ii) Since $T \in L(X)$ and $S \in L(Y)$ are semi-g_z-Fredholm, then there exist $(M_1,N_1) \in \text{Red}(T)$ and $(M_2,N_2) \in Red(S)$ such that T_{M_1} and T_{M_2} are semi-Fredholm, T_{N_1} and T_{N_2} are zeroloid. Hence $T_{M_1\oplus M_2}$ is semi-Fredholm and $T_{N_1\oplus N_2}$ is zeroloid. Moreover, $(M_1 \oplus M_2, N_1 \oplus N_2) \in Red(T \oplus S)$. Hence $\text{ind}(T \oplus S) = \text{ind}((T \oplus S)_{M_1\oplus M_2}) = \text{ind}(T_{M_1}) + \text{ind}(S_{M_2}) = \text{ind}(T) + \text{ind}(S)$. □

Denote by $\sigma_{ug_zf}(T)$, $\sigma_{lg_zf}(T)$, $\sigma_{sg_zf}(T)$, $\sigma_{g_zf}(T)$, $\sigma_{ug_zw}(T)$, $\sigma_{lg_zw}(T)$, $\sigma_{sg_zw}(T)$ and $\sigma_{g_zw}(T)$ respectively, the upper semi-g_z-Fredholm spectrum, the lower semi-g_z-Fredholm spectrum, the semi-g_z-Fredholm, the g_z-Fredholm spectrum, the upper semi-g_z-Weyl spectrum, the lower semi-g_z-Weyl spectrum, the semi-g_z-Weyl spectrum and the g_z-Weyl spectrum of T.

Corollary 3.8. *For every $T \in L(X)$, we have $\sigma_{g_zf}(T) = \sigma_{ug_zf}(T) \cup \sigma_{lg_zf}(T)$ and $\sigma_{g_zw}(T) = \sigma_{ug_zw}(T) \cup \sigma_{lg_zw}(T)$.*

Proposition 3.9. *Let $T \in L(X)$ be a semi-B-Fredholm operator which is semi-g_z-Fredholm. Then T is quasi semi-B-Fredholm and its index as a semi-B-Fredholm coincides with its index as a semi-g_z-Fredholm.*

Proof. Let $(M,N) \in \text{Red}(T)$ such that T_M is semi-Fredholm and T_N is zeroloid. Since T is semi-B-Fredholm then T_N is Drazin invertible. So there exists $(A,B) \in Red(T_N)$ such that T_A is invertible and T_B is nilpotent. It is easy to get that $M \oplus A$ is closed, so that $T_{M\oplus A}$ is semi-Fredholm. Consequently, $T = T_{M\oplus A} \oplus T_B$ is quasi semi-B-Fredholm. Furthermore, the punctured neighborhood theorem for semi-Fredholm operators implies that $\text{ind}(T_M) = \text{ind}(T_{[m_T]})$. □

From [29, Theorem 7] and the previous proposition, we obtain the following corollary.

Corollary 3.10. *Every B-Fredholm operator $T \in L(X)$ is g_z-Fredholm and its usual index coincides with its index as a g_z-Fredholm operator.*

Proposition 3.11. *If $T \in L(X)$ is a semi-g_z-Fredholm operator, then T^* is semi-g_z-Fredholm, $\tilde{\alpha}(T) = \tilde{\beta}(T^*)$, $\tilde{\beta}(T) = \tilde{\alpha}(T^*)$ and $ind(T) = -ind(T^*)$.*

Proof. See Lemma 2.13. □

Our next definition gives a new class of operators that extends the class of semi-Browder operators.

Definition 3.12. *We say that $T \in L(X)$ is an upper semi-g_z-Browder (resp., lower semi-g_z-Browder, g_z-Browder) if T is a direct sum of an upper semi-Browder (resp., lower semi-Browder, Browder) operator and a zeroloid operator.*

Proposition 3.13. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is an upper semi-g_z-Browder [resp., lower semi-g_z-Browder, g_z-Browder];
(ii) T is an upper g_z-Weyl and T has the SVEP at 0 [resp., T is a lower semi-g_z-Weyl and T^* has the SVEP at 0, T is g_z-Weyl and T or T^* has the SVEP at 0];
(iii) T is an upper semi-g_z-Fredholm and T has the SVEP at 0 [resp., T is a lower semi-g_z-Fredholm and T^* has the SVEP at 0, T is g_z-Fredholm and $T \oplus T^*$ has the SVEP at 0].

Proof. (i) $\Longleftrightarrow$ (ii) Suppose that T is g_z-Browder, then there exists $(M, N) \in g_zKD(T)$ such that T_M is Browder. So T_M, $(T_M)^*$, T_N and $(T_N)^*$ have the SVEP at 0. Thus T and T^* have the SVEP at 0. Conversely, if T is g_z-Weyl and T or T^* has the SVEP at 0, then there exists $(M, N) \in g_zKD(T)$ such that T_M is Weyl and T_M or $(T_M)^*$ has the SVEP at 0. So $\max\{\tilde{\alpha}(T), \tilde{\beta}(T)\} < \infty$ and $\min\{\tilde{p}(T), \tilde{q}(T)\} < \infty$. This implies from [1, Lemma 1.22] that $\max\{\tilde{p}(T), \tilde{q}(T)\} < \infty$ and then T_M is Browder. Therefore T is g_z-Browder. The other equivalence cases go similarly.
(i) $\Longleftrightarrow$ (iii) Suppose that T is g_z-Fredholm and $T \oplus T^*$ has the SVEP at 0. Let $(M, N) \in g_zKD(T)$ such that T_M is Fredholm and T_N is zeroloid. Hence $T_M \oplus (T_M)^*$ has the SVEP at 0. From the implications (A) and (B) mentioned in the introduction, we deduce that T_M is Browder and then T is g_z-Browder. The converse is clear and the other equivalence cases go similarly. □

The proofs of the following results are obvious and are left to the reader.

Proposition 3.14. *If $T \in L(X)$ is semi-g_z-Fredholm, then there exists $\epsilon > 0$ such that $B_0 := B(0, \epsilon) \setminus \{0\} \subset (\sigma_{spbf}(T))^C$ and $ind(T) = ind(T - \lambda I)$ for every $\lambda \in B_0$.*

Corollary 3.15. *For every $T \in L(X)$, the following assertions hold:*
(i) $\sigma_{ug_zf}(T)$, $\sigma_{lg_zf}(T)$, $\sigma_{sg_zf}(T)$, $\sigma_{g_zf}(T)$, $\sigma_{ug_zw}(T)$, $\sigma_{lg_zw}(T)$, $\sigma_{sg_zw}(T)$ and $\sigma_{g_zw}(T)$ are compact.
(ii) If Ω is a component of $(\sigma_{ug_zf}(T))^C$ or $(\sigma_{lg_zf}(T))^C$, then the index $ind(T - \lambda I)$ is constant as λ ranges over Ω.

Corollary 3.16. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is semi-g_z-Weyl [resp., upper semi-g_z-Weyl, lower semi-g_z-Weyl, g_z-Weyl];
(ii) T is g_z-Kato and $0 \notin acc\, \sigma_{spbw}(T)$ [resp., T is g_z-Kato and $0 \notin acc\, \sigma_{upbw}(T)$, T is g_z-Kato and $0 \notin acc\, \sigma_{lpbw}(T)$, T is g_z-Kato and $0 \notin acc\, \sigma_{pbw}(T)$], where $\sigma_{spbw}(T) := \sigma_{upbw}(T) \cup \sigma_{lpbw}(T)$.

4. g_z-invertible operators

Recall [1] that $T \in L(X)$ is said to be Drazin invertible if there exists an operator $S \in L(X)$ which commutes with T with $STS = S$ and $T^nST = T^n$ for some integer $n \in \mathbb{N}$. The index of a Drazin invertible operator T is defined by $i(T) = \min\{n \in \mathbb{N} : \exists S \in L(X)$ such that $ST = TS, STS = S$ and $T^nST = T^n\}$.

Proposition 4.1. *Let $T \in L(X)$. If $p(T) < \infty$ (resp., $q(T) < \infty$) then $p(T) = dis(T)$ (resp., $q(T) = dis(T)$). Moreover, if T is Drazin invertible, then $i(T) = dis(T)$.*

Proof. Suppose that $p(T) < \infty$, then $\mathcal{N}(T_{[n]}) = \{0\}$ for every $n \geq p(T)$. This implies that $\mathcal{N}(T_{[d]}) = \{0\}$, where $d := \text{dis}(T)$. Thus $p(T) \leq d$, and as we always have $d \leq \min\{p(T), q(T)\}$ then $p(T) = d$. If $q(T) < \infty$, then $X = \mathcal{R}(T) + \mathcal{N}(T^n)$ for every $n \geq q(T)$. Since $\mathcal{R}(T) + \mathcal{N}(T^d) = \mathcal{R}(T) + \mathcal{N}(T^m)$ for every integer $m \geq d$, then $X = \mathcal{R}(T) + \mathcal{N}(T^d)$. Hence $T_{[d]}$ is surjective and consequently $q(T) = d$. If in addition T is Drazin invertible, then the proof of the equality desired is an immediate consequence of [1, Theorem 1.134]. □

Definition 4.2. *We say that T is quasi left Drazin invertible (resp., quasi right Drazin invertible) if there exists $(M, N) \in KD(T)$ such that T_M is bounded below (resp., surjective).*

Proposition 4.3. *Let $T \in L(X)$. The following hold:*
(i) T is Drazin invertible if and only if T is quasi left and quasi right Drazin invertible.
(ii) If T is quasi left Drazin invertible, then T is left Drazin invertible.
(iii) If T is quasi right Drazin invertible, then T is right Drazin invertible.
Furthermore, the converses of (ii) and (iii) are true in the case of Hilbert space.

Proof. (i) Assume that T is Drazin invertible, then $n := p(T) = q(T) < \infty$. It is well known that $(\mathcal{R}(T^n), \mathcal{N}(T^n)) \in \text{Red}(T)$, $T_{\mathcal{R}(T^n)}$ is invertible and $T_{\mathcal{N}(T^n)}$ is nilpotent. So T is quasi left and quasi right Drazin invertible. Conversely, if T is quasi left and quasi right Drazin invertible, then $\tilde{\alpha}(T) = \tilde{\beta}(T) = 0$. Therefore $\alpha(T_M) = \tilde{\alpha}(T) = \tilde{\beta}(T) = \beta(T_M) = 0$ for every $(M,N) \in KD(T)$. Thus T is Drazin invertible.
(ii) Let $(M,N) \in \text{Red}(T)$ such that T_M is bounded below and T_N is nilpotent of degree d. As a bounded below operator is semi-regular, we deduce from [5, Theorem 2.21] that $d = \text{dis}(T)$. Clearly, $\mathcal{R}(T^n)$ is closed and $T_{[n]} = (T_M)_{[n]}$ is bounded below for every integer $n \geq d$. Hence T is left Drazin invertible. Conversely, assume that T is left Drazin invertible Hilbert space operator. Then T is upper semi-B-Fredholm, which entails from [10, Theorem 2.6] and [5, Corollary 3.7] that there exists $(M,N) \in KD(T)$ such that T_M is upper semi-Browder. Using [4, Lemma 2.17], we conclude that T_M is bounded below and then T is quasi left Drazin invertible.
(iii) Goes similarly with (ii). □

Proposition 4.4. *$T \in L(X)$ is an upper semi-Browder [resp., lower semi-Browder] if and only if T is a quasi left Drazin invertible [resp., quasi right Drazin invertible] and $\dim N < \infty$ for every (or for some) $(M,N) \in KD(T)$.*

Proof. If T is an upper semi-Browder, then T is upper semi-Fredholm. From [5, Corollary 3.7], there exists $(M,N) \in KD(T)$ with T_M is upper semi-Browder. It follows from [4, Lemma 2.17] that T_M is bounded below. Let $(A,B) \in KD(T)$ be arbitrary. Since a nilpotent operator $S \in L(Y)$ is semi-Fredholm iff $\dim Y < \infty$, then $\dim B < \infty$. The converse is obvious and the other case goes similarly. □

Definition 4.5. *$T \in L(X)$ is said to be left g_z-invertible (resp., right g_z-invertible) if there exists $(M,N) \in g_zKD(T)$ such that T_M is bounded below (resp., surjective). T is called g_z-invertible if it is left and right g_z-invertible.*

Remark 4.6. *(i) It is clear that T is g_z-invertible if and only if there exists $(M,N) \in g_zKD(T)$ such that T_M is invertible.*
(ii) Every generalized Drazin-meromorphic invertible operator is g_z-invertible.

We prove in the following result that the class of g_z-invertible operators preserves some properties of Drazin invertibility [16, 24].

Theorem 4.7. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is g_z-invertible;
(ii) T is g_z-Browder;
(iii) There exists $(M,N) \in g_zKD(T)$ such that T_M is Drazin invertible;
(iv) There exists a Drazin invertible operator $S \in L(X)$ such that $TS = ST$, $STS = S$ and $T^2S - T$ is zeroloid. A such S is called a g_z-inverse of T;
(v) There exists a bounded projection P on X which commutes with T, $T + P$ is generalized Drazin invertible and TP is zeroloid;
(vi) There exists a bounded projection P on X commuting with T such that there exist $U, V \in L(X)$ which satisfy $P = TU = VT$ and $T(I - P)$ is zeroloid;
(vii) T is g_z-Kato and $\tilde{p}(T) = \tilde{q}(T) < \infty$.

Proof. The equivalences (i) $\Longleftrightarrow$ (ii) and (i) $\Longleftrightarrow$ (iii) are immediate consequences of Propositions 4.3 and 4.4.
(i) $\Longleftrightarrow$ (iv) Assume that T is g_z-invertible and let $(M,N) \in g_zKD(T)$ such that T_M is invertible. The operator $S = (T_M)^{-1} \oplus 0_N$ is Drazin invertible. Moreover, $TS = ST = I_M \oplus 0_N$, $STS = S$ and $T^2S - T = 0_M \oplus (-T_N)$. As T_N is zeroloid then $T^2S - T$ is also zeroloid. Conversely, suppose that there exists a Drazin invertible operator S such that $TS = ST$, $STS = S$ and $T^2S - T$ is zeroloid. Then TS is a projection. If we take $M = \mathcal{R}(TS)$ and $N = \mathcal{N}(TS)$, then $(M,N) \in \text{Red}(T) \cap Red(S)$. We have T_M is one-to-one. Indeed, $x \in \mathcal{N}(T_M)$ implies

that $x = TSy$ and $Tx = 0$, so $x = (TS)^2y = STx = 0$. Since $\mathcal{R}(T_M) = M$ then T_M is invertible. Let us to show that $S = (T_M)^{-1} \oplus 0_N$. We have $S_N = 0_N$, since $S = STS$. Let $x = TSy \in M$, as $Sy = STSy \in M$ then $Sx = Sy = (T_M)^{-1}T_MSy = (T_M)^{-1}x$. Hence $S = (T_M)^{-1} \oplus 0_N$ and $T^2S - T = 0_M \oplus (-T_N)$. Thus T_N is zeroloid and then T is g_z-invertible.
(i) $\Longleftrightarrow$ (v) Suppose that there exists a bounded projection P on X which commutes with T, $T + P$ is generalized Drazin invertible and TP is zeroloid. Then $(A, B) := (\mathcal{N}(P), \mathcal{R}(P)) \in \text{Red}(T)$, $T_A = (T + P)_A$ is generalized Drazin invertible and $T_B = (TP)_B$ is zeroloid. Thus there exists $(C, D) \in Red(T_A)$ such that T_C is invertible and T_D is quasi-nilpotent. Hence $(C, D \oplus B) \in g_zKD(T)$ and then T is g_z-invertible. Conversely, let $(M, N) \in g_zKD(T)$ such that T_M is invertible. Clearly, $P := 0_M \oplus I_N$ is a projection and $TP = PT$. Furthermore, $TP = 0_M \oplus T_N$ is zeroloid and $T + P = T_M \oplus (T + I)_N$ is generalized Drazin invertible, since $-1 \notin \text{acc}\,\sigma(T_N) = \sigma_{gd}(T_N)$.
(vi) $\Longrightarrow$ (i) Suppose that there exists a bounded projection P on X commuting with T such that there exist $U, V \in L(X)$ which satisfy $P = TU = VT$ and $T(I - P)$ is zeroloid. In addition, we assume that $U, V \in \text{comm}(T)$ (for the general case, one can see the proof of the implication (v) $\Longrightarrow$ (vi) of [35, Theorem 2.4]). Then $I_M \oplus 0_N = T_MU_M \oplus T_NU_N = V_MT_M \oplus V_NT_N$, where $(M, N) := (\mathcal{R}(P), \mathcal{N}(P)) \in \text{Red}(T)$, and thus $T_MU_M = V_MT_M = I_M$ and $T_NU_N = V_NT_N = 0_N$. Hence T_M is invertible. Moreover, T_N is zeroloid, since $T(I - P) = 0_M \oplus T_N$ is zeroloid. Consequently, T is g_z-invertible.
(iv) $\Longrightarrow$ (vi) and (i) $\Longleftrightarrow$ (vii) are clear. □

The next two theorems are analogous to the previous one.

Theorem 4.8. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is left g_z-invertible;
(ii) T is upper semi-g_z-Browder;
(iii) There exists $(M, N) \in g_zKD(T)$ such that T_M is quasi left Drazin invertible;
(iv) T is g_z-Kato and $\tilde{p}(T) = 0$;
(v) T is g_z-Kato and $0 \notin acc\,\sigma_{lgd}(T)$.

Theorem 4.9. *Let $T \in L(X)$. The following statements are equivalent:*
(i) T is right g_z-invertible;
(ii) T is lower semi-g_z-Browder;
(iii) There exists $(M, N) \in g_zKD(T)$ such that T_M is quasi right Drazin invertible;
(iv) T is g_z-Kato and $\tilde{q}(T) = 0$;
(v) T is g_z-Kato and $0 \notin acc\,\sigma_{rgd}(T)$.

Corollary 4.10. *If $T \in L(X)$ is g_z-invertible and S is a g_z-inverse of T, then TST is the Drazin inverse of S and $p(S) = q(S) = dis(S) \leq 1$.*

Proof. Obvious. □

Hereafter, $\sigma_{lg_zd}(T)$, $\sigma_{rg_zd}(T)$ and $\sigma_{g_zd}(T)$ are respectively, the left g_z-invertible spectrum, the right g_z-invertible spectrum and the g_z-invertible spectrum of T.

Theorem 4.11. *For every $T \in L(X)$ we have $\sigma_{g_zd}(T) = acc\,(acc\,\sigma(T))$.*

Proof. Let $\mu \notin \text{acc}\,(\text{acc}\,\sigma(T))$. Without loss of generality we assume that $\mu = 0$ [note that $\text{acc}\,\text{acc}\,\sigma(T - \alpha I) = \text{acc}\,(\text{acc}\,\sigma(T)) - \alpha$, for every complex α]. If $0 \notin \text{acc}\,\sigma(T)$, then T is generalized Drazin invertible and in particular g_z-invertible. If $0 \in \text{acc}\,\sigma(T)$ then $0 \in \text{acc}\,(\text{iso}\,\sigma(T))$. We distinguish two cases:
Case 1: $\text{acc}\,(\text{iso}\,\sigma(T)) \neq \{0\}$. It follows that $\epsilon := \inf\limits_{\lambda \in acc\,(iso\,\sigma(T))\setminus\{0\}} |\lambda| > 0$. Moreover, the sets $F_2 := D(0, \frac{\epsilon}{2}) \cap \overline{\text{iso}\,\sigma(T)}$ and $F_1 := ((\text{acc}\,\sigma(T)) \setminus \{0\}) \cup (\overline{\text{iso}\,\sigma(T)} \setminus F_2)$ are closed and disjoint. Indeed, $F_1 \cap F_2 = F_2 \cap [(\text{acc}\,\sigma(T)) \setminus \{0\}] \subset [\text{acc}\,(\text{iso}\,\sigma(T)) \setminus \{0\}] \cap D(0, \frac{\epsilon}{2}) = \emptyset$. As $0 \notin \text{acc}\,(\text{acc}\,\sigma(T))$ then $(\text{acc}\,\sigma(T)) \setminus \{0\}$ is closed. Let us to show that $C := (\overline{\text{iso}\,\sigma(T)} \setminus F_2)$ is closed. If $\lambda \in \text{acc}\,C$ (the case of $\text{acc}\,C = \emptyset$ is obvious), then $\lambda \in \overline{\text{iso}\,\sigma(T)}$. Let $(\lambda_n)_n \subset C$ be a non stationary sequence that converges to λ, it follows that $\lambda \neq 0$. We have $\lambda \notin F_2$. Otherwise, $\lambda \in D(0, \frac{\epsilon}{2})$ and then $\lambda \notin \text{acc}\,(\text{iso}\,\sigma(T)$. So $\lambda \in \text{iso}\,\sigma(T)$ and this is a contradiction. Therefore C is closed and then F_1 is

closed. As $\sigma(T) = F_1 \cup F_2$ then there exists $(M, N) \in \mathrm{Red}(T)$ such that $\sigma(T_M) = F_1$ and $\sigma(T_N) = F_2$. So T_M is invertible and $0 \in \operatorname{acc}\sigma(T_N)$. Let $v \in F_2$, then $v \notin \operatorname{acc}\sigma(T_N) \setminus \{0\}$, since $F_1 \cap F_2 = F_2 \cap (\operatorname{acc}\sigma(T) \setminus \{0\}) = \emptyset$. Hence $\operatorname{acc}\sigma(T_N) = \{0\}$ and T is g_z-invertible.
Case 2: $\operatorname{acc}(\operatorname{iso}\sigma(T)) = \{0\}$. Then $F_2 := D(0,1) \cap \overline{\operatorname{iso}\sigma(T)}$ and $F_1 := ((\operatorname{acc}\sigma(T)) \setminus \{0\}) \cup (\overline{\operatorname{iso}\sigma(T)} \setminus F_2)$ are closed disjoint subsets and give the desired result. For this, if $\lambda \in \overline{C}$, where $C := \overline{\operatorname{iso}\sigma(T)} \setminus F_2$, then there exists a sequence $(\lambda_n) \subset C$ that converges to λ. As $\operatorname{acc}(\operatorname{iso}\sigma(T)) = \{0\}$ and $\lambda(\neq 0) \in \overline{\operatorname{iso}\sigma(T)}$ then $\lambda \in \operatorname{iso}\sigma(T)$. Therefore $(\lambda_n)_n$ is stationary and so $\lambda \in C$. Thus F_1 is closed and hence there exists $(M, N) \in \mathrm{Red}(T)$ such that $\sigma(T_M) = F_1$ and $\sigma(T_N) = F_2$. Conclusion, T is g_z-invertible.
Conversely, if T is g_z-invertible, then $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is zeroloid. And then there exists $\epsilon > 0$ such that $B(0, \epsilon) \setminus \{0\} \subset (\sigma(T_1))^C \cap (\operatorname{acc}\sigma(T_2))^C \subset (\operatorname{acc}\sigma(T))^C$. Thus $0 \notin \operatorname{acc}(\operatorname{acc}\sigma(T))$. □

From the previous theorem and some well known results in perturbation theory, we obtain the following corollary.

Corollary 4.12. *Let $T \in L(X)$. The following statements hold:*
(i) $\sigma_{lg_zd}(T)$, $\sigma_{rg_zf}(T)$ and $\sigma_{g_zd}(T)$ are compact.
(ii) $\sigma_{g_zd}(T) = \sigma_{g_zd}(T^*)$.
(iii) If $S \in L(Y)$, then $T \oplus S$ is g_z-invertible if and only if T and S are g_z-invertible.
(iv) T is g_z-invertible if and only if T^n is g_z-invertible for some (equivalently for every) integer $n \geq 1$.
(v) If $Q \in comm(T)$ is quasi-nilpotent, then $\sigma_{g_zd}(T) = \sigma_{g_zd}(T + Q)$.
(vi) If $F \in \mathcal{F}_0(X) \cap comm(T)$, then $\sigma_{g_zd}(T) = \sigma_{g_zd}(T + F)$, where $\mathcal{F}_0(X)$ is the set of all power finite rank operators.

Example 4.13. *Let $T \in L(X)$ be the operator such that $\sigma(T) = \sigma_d(T) = \overline{\{\frac{1}{n}\}}$. Then T is g_z-invertible and not generalized Drazin-meromorphic invertible, since $0 \in acc\,\sigma_d(T)$ (see [35, Theorem 5]). Note also that T is not generalized Kato-meromorphic. Otherwise, we get $\tilde{\alpha}(T) = \tilde{\beta}(T) = 0$, since T is g_z-invertible. Hence T is generalized Drazin-meromorphic invertible and this is a contradiction.*

Proposition 4.14. *Let $T \in L(X)$. The following statements are equivalent:*
(i) $0 \in iso\,(acc\,\sigma(T))$ (i.e. T is g_z-invertible and not generalized Drazin invertible);
(ii) $T = T_1 \oplus T_2$, where T_1 is invertible and $acc\,\sigma(T_2) = \{0\}$;
(iii) T is g_z-Kato and there exists a non stationary sequence of isolated points of $\sigma(T)$ that converges to 0.

Proof. (i) $\Longrightarrow$ (ii) Follows directly from the proof of Theorem 4.11. Note here that $\operatorname{acc}\sigma(T_N) = \{0\}$ for every $(M, N) \in g_zKD(T)$.
(ii) $\Longrightarrow$ (iii) As $T = T_1 \oplus T_2$, T_1 is invertible and $\operatorname{acc}\sigma(T_2) = \{0\}$, then $0 \in \operatorname{iso}(\operatorname{acc}\sigma(T))$ and there exists a non stationary sequence $(\lambda_n)_n \subset \operatorname{iso}\sigma(T_2)$ that converges to 0. Thus T is g_z-invertible and there exists $N \in \mathbb{N}$ such that $\lambda_n \in \sigma(T) \setminus \operatorname{acc}\sigma(T) = \operatorname{iso}\sigma(T)$ for all $n \geq N$.
(iii) $\Longrightarrow$ (i) Assume that $T = T_1 \oplus T_2$, T_1 is semi-regular, T_2 is zeroloid and there exists a non stationary sequence $(\lambda_n)_n$ of isolated point of $\sigma(T)$ that converges to 0. Hence $0 \in \operatorname{acc}\sigma(T)$ and $T \oplus T^*$ has the SVEP at 0. This entails that T is g_z-invertible and then $0 \in \operatorname{iso}(\operatorname{acc}\sigma(T))$. □

Recall that $\sigma \subset \sigma(T)$ is called a spectral set (called also isolated part) of T if σ and $\sigma(T) \setminus \sigma$ are closed, see [17]. Let T be a g_z-invertible operator which is not generalized Drazin invertible. From Proposition 4.14, we conclude that there exists a non-zero strictly decreasing sequence $(\lambda_n)_n \subset \operatorname{iso}\sigma(T)$ that converges to 0 such that $\sigma := \overline{\{\lambda_n : n \in \mathbb{N}\}}$ is a spectral set of T. If P_σ is the spectral projection associated to σ, then $(M_\sigma, N_\sigma) := (\mathcal{N}(P_\sigma), \mathcal{R}(P_\sigma)) \in g_zKD(T)$, $\sigma(T_{N_\sigma}) = \sigma$ and $\sigma(T_{M_\sigma}) = \sigma(T) \setminus \sigma$. Thus $T + rP_\sigma = T_{M_\sigma} \oplus (T + rI)_{N_\sigma}$ is invertible for every $|r| > |\lambda_0|$ and then the operator $T_\sigma^D := (T + rP_\sigma)^{-1}(I - P_\sigma) = (T_{M_\sigma})^{-1} \oplus 0_{N_\sigma}$ is a g_z-inverse of T and depends only on σ. Note that $P_\sigma = I - TT_\sigma^D \in \operatorname{comm}^2(T) := \{S \in \operatorname{comm}(L) : L \in \operatorname{comm}(T)\}$, so that $(M_\sigma, N_\sigma) \in \mathrm{Red}(S)$ for every operator $S \in \operatorname{comm}(T)$ and $T_\sigma^D \in \operatorname{comm}^2(T)$. Note also that $T + P_\sigma$ is generalized Drazin invertible and TP_σ is zeroloid.

Lemma 4.15. *Let $T \in L(X)$ be a g_z-invertible operator and $(M, N) \in g_zKD(T)$ such that T_M invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$. Then $\sigma(T_N) \setminus \{0\} \subset iso\,\sigma(T)$ and for every $S \in comm(T)$ we have $(M, N) \in Red(S)$.*

Proof. If T is generalized Drazin invertible, then $0 \notin \operatorname{acc}\sigma(T)$ and so $\operatorname{acc}\sigma(T_N) = \emptyset$, hence $\sigma(T_N)$ is a finite set of isolated points of $\sigma(T)$. Let P_σ be the spectral projection associated to $\sigma = \sigma(T_N)$. From [17, Proposition 2.4] and the fact that $P_\sigma \in \operatorname{comm}^2(T)$ we deuce that $(M, N) = (\mathcal{N}(P_\sigma), \mathcal{R}(P_\sigma)) \in \operatorname{Red}(S)$ for every $S \in \operatorname{comm}(T)$. If T is not generalized Drazin invertible, then there exists a strictly decreasing sequence $(\lambda_n)_n$ of isolated point of $\sigma(T)$ that converges to 0 and such that $\sigma(T_N) = \overline{\{\lambda_n : n \in \mathbb{N}\}}$. Thus $\sigma(T_N) \setminus \{0\} \subset \operatorname{iso}\sigma(T)$. Let P be the spectral projection associated to the spectral set $\sigma(T_N)$, then $(M, N) = (\mathcal{N}(P), \mathcal{R}(P))$ and so $(M, N) \in \operatorname{Red}(S)$ for every $S \in \operatorname{comm}(T)$. □

Remark 4.16. *It is not difficult to see that the following assertions are aquivalent:*
(i) $\exists (M, N) \in Red(S)$ *such that* T_M *is invertible for every* $S \in comm(T)$;
(ii) $\exists L \in comm^2(T)$ *such that* $L = L^2 T$.

Theorem 4.17. *Let* $T \in L(X)$. *The following statements are equivalent:*
(i) T *is* g_z*-invertible;*
(ii) $0 \notin acc(acc\,\sigma(T))$;
(iii) There exists $(M, N) \in g_zKD(T)$ *such that* T_M *invertible and* $\sigma(T_M) \cap \sigma(T_N) = \emptyset$;
(iv) There exists a spectral set σ *of* T *such that* $0 \notin \sigma(T) \setminus \sigma$ *and* $\sigma \setminus \{0\} \subset iso\,\sigma(T)$;
(v) There exists a bounded projection $P \in comm^2(T)$ *such that* $T + P$ *is generalized Drazin invertible and* TP *is zeroloid.*

Proof. For the equivalence (i) $\Longleftrightarrow$ (ii), see Theorem 4.11. For the equivalences (i) $\Longleftrightarrow$ (iii) and (i) $\Longleftrightarrow$ (v), see Theorem 4.7 and the paragraph preceding Lemma 4.15 (the case of T is generalized Drazin invertible is clear). The proof of the equivalence (iii) $\Longleftrightarrow$ (iv) is a consequence of Lemma 4.15 and the spectral decomposition theorem. □

Proposition 4.18. *For every* g_z*-invertible operator* $T \in L(X)$, *the following statements hold:*
(i) Let $(M, N), (M', N') \in g_zKD(T)$ *such that* T_M, $T_{M'}$ *are invertible and* $\sigma(T_M) \cap \sigma(T_N) = \sigma(T_{M'}) \cap \sigma(T_{N'}) = \emptyset$. *If* $(T_M)^{-1} \oplus 0_N = (T_{M'})^{-1} \oplus 0_{N'}$, *then* $(M, N) = (M', N')$.
(ii) Let σ, σ' *two spectral sets of* T *such that* $0 \notin \sigma(T) \setminus (\sigma \cap \sigma')$ *and* $(\sigma \cup \sigma') \setminus \{0\} \subset iso\,\sigma(T)$. *If* $(T + rP_\sigma)^{-1}(I - P_\sigma) = (T + r'P_{\sigma'})^{-1}(I - P_{\sigma'})$, *where* P_σ *is the spectral projection of* T *associated to* σ, $|r| > \max\limits_{\lambda\in\sigma} |\lambda|$ *and* $|r'| > \max\limits_{\lambda\in\sigma'} |\lambda|$, *then* $\sigma = \sigma'$.

Proof. (i) From the proof of Lemma 4.15, we have $(M, N) = (\mathcal{N}(P_\sigma), \mathcal{R}(P_\sigma))$ and $(M', N') = (\mathcal{N}(P_{\sigma'}), \mathcal{R}(P_{\sigma'}))$, where $\sigma = \sigma(T_N)$ and $\sigma' = \sigma(T_{N'})$. As $(T_M)^{-1} \oplus 0_N = (T_{M'})^{-1} \oplus 0_{N'}$ then $\sigma(T_M) = \sigma(T_{M'})$ and thus $\sigma(T_N) = \sigma(T_{N'})$. This proves that $(M, N) = (M', N')$.
(ii) Follows from (i). □

The previous Proposition 4.18 gives a sense to the next remark.

Remark 4.19. *If* $T \in L(X)$ *is* g_z*-invertible, then*
(i) For every $(M, N) \in g_zKD(T)$ *such that* T_M *is invertible and* $\sigma(T_M) \cap \sigma(T_N) = \emptyset$, *the* g_z*-inverse operator* $T^D_{(M,N)} := (T_M)^{-1} \oplus 0_N \in comm^2(T)$, *and we call* $T^D_{(M,N)}$ *the* g_z*-inverse of* T *associated to* (M, N).
(ii) If σ *is a spectral set of* T *such that* $0 \notin \sigma(T) \setminus \sigma$ *and* $\sigma \setminus \{0\} \subset iso\,\sigma(T)$, *then the operator* $T^D_\sigma := (T + rP_\sigma)^{-1}(I - P_\sigma) \in comm^2(T)$ *is a* g_z*-inverse of* T, *where* $|r| > \max\limits_{\lambda\in\sigma} |\lambda|$, *and we call* T^D_σ *the* g_z*-inverse of* T *associated to* σ.

Note that if $T \in L(X)$ is generalized Drazin invertible which is not invertible, then by [24, Lemma 2.4] and Proposition 4.18 we conclude that the Drazin inverse of T is exactly the g_z-inverse of T associated to $\sigma = \{0\}$, in other words $T^D = T^D_{\{0\}}$.

Proposition 4.20. *Let* $T, S \in L(X)$ *two commuting* g_z*-invertible. If* σ *and* σ' *are spectral sets of* T *and* S, *respectively such that* $0 \notin (\sigma(T) \setminus \sigma) \cup (\sigma(S) \setminus \sigma')$, $\sigma \setminus \{0\} \subset iso\,\sigma(T)$ *and* $\sigma' \setminus \{0\} \subset iso\,\sigma(S)$, *then* $T, S, T^D_\sigma, S^D_{\sigma'}$ *are mutually commutative.*

Proof. As $TS = ST$ then the previous remark entails that $T_{\sigma}^{D} = (T + rP_{\sigma})^{-1}(I - P_{\sigma}) \in \text{comm}(S_{\sigma'}^{D})$, and analogously for other operators. $\square$

The following proposition describe the relation between the g_z-inverse of a g_z-invertible operator T associated to (M, N) and the g_z-inverse of T associated to a spectral set σ. It's proof is clear.

Proposition 4.21. *If $T \in L(X)$ is g_z-invertible and $(M, N) \in g_zKD(T)$ such that T_M is invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$, then $T_{(M,N)}^{D} = T_{\sigma}^{D}$, where $\sigma = \sigma(T_N)$. In other words $T_{\sigma(T_N)}^{D} = (T_M)^{-1} \oplus 0_N$.*

Our next theorem gives a generalization of [24, Theorem 4.4] in the case of the complex Banach algebra $L(X)$. Denote by $\text{Hol}(T)$ the set of all analytic functions defined on an open neighborhood of $\sigma(T)$.

Theorem 4.22. *If $0 \in \sigma(T) \setminus acc(acc\,\sigma(T))$, then for every spectral set σ such that $0 \in \sigma$ and $\sigma \setminus \{0\} \subset iso\,\sigma(T)$ we have*

$$T_{\sigma}^{D} = f_{\sigma}(T),$$

where $f_{\sigma} \in Hol(T)$ defined by $f_{\sigma} = 0$ in a neighborhood of σ and $f_{\sigma}(\lambda) = \lambda^{-1}$ in a neighborhood of $\sigma(T) \setminus \sigma$. Moreover $\sigma(T_{\sigma}^{D}) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(T) \setminus \sigma\}$.

Proof. Let Ω_1 and Ω_2 two disjoint open sets such that $\sigma \subset \Omega_1$ and $\sigma(T) \setminus \sigma \subset \Omega_2$ (for the construction of Ω_1 and Ω_2, see the paragraph below) and let $g \in \text{Hol}(T)$ be the function defined by

$$g(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \Omega_1 \\ 0 & \text{if } \lambda \in \Omega_2 \end{cases}$$

It is clear that $P_{\sigma} = g(T)$ and as $T_{\sigma}^{D} = (T + rP_{\sigma})^{-1}(I - P_{\sigma})$ (where $|r| > \max\limits_{\lambda \in \sigma} |\lambda|$ be arbitrary), then the function $f_{\sigma}(\lambda) = (\lambda + rg(\lambda))^{-1}(1 - g(\lambda))$ has the required property. Moreover, we have $\sigma(T_{\sigma}^{D}) = f_{\sigma}(\sigma(T)) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(T) \setminus \sigma\}$. $\square$

According to [17], if σ is a spectral set of T then there exist two disjoint open sets Ω_1 and Ω_2 such that $\sigma \subset \Omega_1$ and $\sigma(T) \setminus \sigma \subset \Omega_2$. Choose a Cauchy domains S_1 and S_2 such that $\sigma \subset S_1$, $\sigma(T) \setminus \sigma \subset S_2$, $\overline{S_1} \subset \Omega_1$ and $\overline{S_2} \subset \Omega_2$. It follows that the spectral projection corresponding to σ is

$$P_{\sigma} = \frac{1}{2i\pi} \int_{\partial S_1} (\lambda I - T)^{-1} d\lambda.$$

Moreover, if $0 \in \sigma$ and $\sigma \setminus \{0\} \subset \text{iso}\,\sigma(T)$, then from Theorem 4.22 we conclude that

$$T_{\sigma}^{D} = \frac{1}{2i\pi} \int_{\partial S_2} \lambda^{-1} (\lambda I - T)^{-1} d\lambda.$$

5. Weak SVEP and applications

As a continuation of some results proved in [19, 22], we begain this part by the next theorem which gives a new characterization of some Browder's type theorems in terms of spectra introduced and studied in the preceding parts.

Theorem 5.1. *For $T \in L(X)$, we have*
(i) $T \in (B)$ if and only if $\sigma_{g_zw}(T) = \sigma_{g_zd}(T)$.
(ii) $T \in (B_e)$ if and only if $\sigma_{g_zf}(T) = \sigma_{g_zd}(T)$.
(iii) $T \in (aB)$ if and only if $\sigma_{ug_zw}(T) = \sigma_{lg_zd}(T)$.

Proof. (i) If$\lambda \notin \sigma_{g_zw}(T)$, then from Corollary 3.16 we have $\lambda \notin \operatorname{acc}\sigma_{pbw}(T)$ [note that $\operatorname{acc}\sigma_{pbw}(T-\lambda I) = \operatorname{acc}(\sigma_{pbw}(T)) - \lambda$]. Since $T \in (B)$ then [22, Theorem 2.6] or [19, Theorem 2.8] implies that $\lambda \notin \operatorname{acc}\sigma_{gd}(T)$, and this implies from Theorem 4.11 that $\lambda \notin \sigma_{g_zd}(T)$. As the inclusion $\sigma_{g_zw}(T) \subset \sigma_{g_zd}(T)$ is always true, it follows that $\sigma_{g_zw}(T) = \sigma_{g_zd}(T)$. Conversely, let $\lambda \notin \sigma_w(T)$, then $\lambda \notin \sigma_{g_zw}(T) = \sigma_{g_zd}(T)$. On the other hand, [5, Corollary 3.7] implies that there exists $(M, N) \in \operatorname{Red}(T)$ such that $T_M - \lambda I$ is semi-regular and $T_N - \lambda I$ is nilpotent. Since $T - \lambda I$ is g_z-invertible then $p(T_M - \lambda I) = \tilde{p}(T - \lambda I) = \tilde{q}(T - \lambda I) = q(T_M - \lambda I) = 0$, and so $T_M - \lambda I$ is invertible. Hence $T - \lambda I$ is Browder and consequently $T \in (B)$. Using [22, Corollary 2.10] or [19, Corollary 2.14], the point (ii) goes similarly with (i). And Using [22, Theorem 2.7], we obtain analogously the point (iii). □

Definition 5.2. *Let A be a subset of $\mathbb{C}$. We say that $T \in L(X)$ has the Weak SVEP on A (T has the W_A-SVEP for brevity) if there exists a subset $B \subset A$ such that T has the SVEP on B and T^* has the SVEP on $A \backslash B$. If T has the $W_{\mathbb{C}}$-SVEP, then T is said to have the Weak SVEP (T has the W-SVEP for brevity).*

Remark 5.3. *(i) Let A be a subset of $\mathbb{C}$. Then $T \in L(X)$ has the W_A-SVEP if and only if for every $\lambda \in A$, at least T or T^* has the SVEP at λ.*
(ii) If T or T^* has the SVEP, then T has the W-SVEP. But the converse is not generally true. For this, the left shift operator $L \in L(\ell^2(\mathbb{N}))$ defined by $L(x_1, x_2, \dots) = (x_2, x_3, \dots)$ has the W-SVEP, but it does not have the SVEP.
(iii) The operator $L \oplus L^*$ does not have the W-SVEP.

The next theorem gives a sufficient condition for an operator $T \in L(X)$ to have the W-SVEP.

Theorem 5.4. *Let $T \in L(X)$. If*

$$X_T(\emptyset) \times X_{T^*}(\emptyset) \subset \{(x, 0) : x \in X\} \bigcup \{(0, f) : f \in X^*\},$$

then T has the W-SVEP.

Proof. Let $\lambda \in \mathbb{C}$ and let $V, W \subset \mathbb{C}$ two open neighborhood of λ. Let $f : V \longrightarrow X$ and $g : W \longrightarrow X^*$ two analytic functions such that $(T - \mu I)f(\mu) = 0$ and $(T^* - vI)g(v) = 0$ for every $(\mu, v) \in V \times W$. If we take $U = V \cap W$, then [1, Theorem 2.9] implies that $\sigma_T(f(\mu)) = \sigma_T(0) = \emptyset = \sigma_{T^*}(0) = \sigma_{T^*}(g(\mu))$ for every $\mu \in U$. Hence $(f(\mu), g(v)) \in X_T(\emptyset) \times X_{T^*}(\emptyset)$ for every $\mu, v \in U$. We discuss two cases. The first, there exists $\mu \in U$ such that $g(\mu) \neq 0$. As $(f(v), g(\mu)) \in X_T(\emptyset) \times X_{T^*}(\emptyset)$ for every $v \in U$ then by hypotheses $f \equiv 0$ on U. The identity theorem for analytic functions entails that T has the SVEP at λ. The second, $g(\mu) = 0$ for every $\mu \in U$. In the same way, we prove that T^* has the SVEP at λ. Hence T has the W-SVEP. □

Question: Similarly to [1, Theorem 2.14] which characterizes the SVEP of $T \in L(X)$ in terms of its local spectral subspace $X_T(\emptyset)$, we ask if the converse of Theorem 5.4 is true?

The next proposition characterizes the classes (B) and (aB) in terms of the Weak SVEP.

Proposition 5.5. *If $T \in L(X)$, then*
(a) For $\sigma_* \in \{\sigma_w, \sigma_{bw}, \sigma_{g_zw}\}$, the following statements are equivalent:
(i) $T \in (B)$;
(ii) T has the Weak SVEP on $\sigma_*(T)^C$;
(iii) For all $\lambda \notin \sigma_*(T)$, $T \oplus T^*$ has the SVEP at λ;
(iv) For all $\lambda \notin \sigma_*(T)$, T has the SVEP at λ;
(v) For all $\lambda \notin \sigma_*(T)$, T^* has the SVEP at λ.
(b) For $\sigma_* \in \{\sigma_e, \sigma_{bf}, \sigma_{g_zf}\}$, the following statements are equivalent:
(i) $T \in (B_e)$;
(ii) For all $\lambda \notin \sigma_*(T)$, $T \oplus T^*$ has the SVEP at λ.
(c) For $\sigma_* \in \{\sigma_{uw}, \sigma_{ubw}, \sigma_{ug_zw}\}$, the following statements are equivalent:
(i) $T \in (aB)$;
(ii)] T has the Weak SVEP on $\sigma_*(T)^C$;
(iii) For all $\lambda \notin \sigma_*(T)$, T has the SVEP at λ.

Proof. (a) For $\sigma_* = \sigma_{g_zw}$, we have only to show (ii) $\Longrightarrow$ (i), and the other implications are clair. Let $\lambda \notin \sigma_{g_zw}(T)$, then there exists $(M, N) \in \mathrm{Red}(T)$ such that $T_M - \lambda I$ is Weyl and $T_N - \lambda I$ is zeroloid. Hence T or T^* has the SVEP at λ is equivalent to say that T_M or $(T_M)^*$ has the SVEP at λ, and this is equivalent to $\min\{p(T_M - \lambda I), q(T_M - \lambda I)\} < \infty$. Therefore $T_M - \lambda I$ is Browder and then $\lambda \notin \sigma_{g_zd}(T)$. From Theorem 5.1, it follows that $T \in (B)$. For $\sigma_* \in \{\sigma_w, \sigma_{bw}\}$, the proof of (ii) $\Longrightarrow$ (i) is similar, and the other implications are already done in [1]. The assertions (b) and (c) go similarly with (a). Note that some implications of assertions (b) and (c) are already done in [1, 6, 19, 22]. □

We end this part by the next result which extends [1, Theorem 5.6].

Theorem 5.6. *If the g_z-Weyl spectrum of $T \in L(X)$ has empty interior that is, $\mathrm{int}\,\sigma_{g_zw}(T) = \emptyset$, then the following statements are equivalent:*
(i) $T \in (B)$;
(ii) $T \in (B_e)$;
(iii) $T \in (aB)$;
(iv) T has the SVEP;
(v) T^* has the SVEP;
(vi) $T \oplus T^*$ has the SVEP;
(vii) T has the W-SVEP.

Proof. (i) $\Longrightarrow$ (vi) As $T \in (B)$ then by Proposition 5.5, $T \oplus T^*$ has the SVEP on $\sigma_{g_zw}(T)^C$. Let $\lambda \in \sigma_{g_zw}(T)$, $U \subset \mathbb{C}$ be an open neighborhood of λ and $f : U \longrightarrow X$ be an analytic function which satisfies $(\mu I - T)f(\mu) = 0$, for every $\mu \in U$. The hypothesis $\mathrm{int}\,\sigma_{g_zw}(T) = \emptyset$ implies that there exists $\gamma \in U \cap (\sigma_{g_zw}(T))^C$. Hence $f \equiv 0$ on U, since T has the SVEP at γ. It then follows that T has the SVEP at λ. Analogously we prove that T^* has the SVEP at λ, and consequently $T \oplus T^*$ has the SVEP. It is clear that the statement (vi) implies without condition on T all other statements. Furthermore, all statements imply (i). This completes the proof. □

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