Filomat 37:7 (2023), 2105–2116
https://doi.org/10.2298/FIL2307105X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotic analysis for stock loans near maturity

Yongqing Xu[a]

[a]*College of Big data and Internet, Shenzhen Technology University,Shenzhen 518118*

Abstract. In this paper, we derive the asymptotic expressions of the scaled value function and the optimal redemption boundary of stock loan with dividend-paying near maturity. Using the equation satisfied by the derivative of the value function at the exercise boundary, we set up the asymptotic expression for the boundary. When the risk-free rate r is smaller than the loan rate β, i.e., $r < \beta$, the boundary tends to $Ke^{\beta T_0}$ in parabolic-logarithm form, this case is the main result. For the case $r \geq \beta$, the corresponding problem returns back to a usual American call option with interest-free rate $r - \beta$ and the existing results can be utilized to make proper adjustments for the stock loan. The matched expansion for the value function is performed with a small parameter. Numerical examples are provided to demonstrate the effectiveness of the proposed method.

1. Introduction

As a type of path-dependent options, the finite-time American options' valuation is an important and complicated problem in mathematical finance. When the time to expiry is short, or the time approaches maturity, the asymptotic problems of American options are of interest for theoretic and practical aims, which have been addressed in many literature during latest decades, herein we refer some papers [1]–[13]. Under short time framework, the main goals are to derive the asymptotic forms of the value functions of American put and call options and the optimal exercise boundaries of them. The existing studies make clear that the parameters including the interest rate, the constant dividend rate, the volatility and the time to maturity, influence the exercise boundaries and the value functions. In the finite-time American options' valuation, especially for call options, the dividend rate is a pivotal parameter compared to other ones. Some papers have disclosed that the optimal exercise boundaries are singular at expiry and they observe remarkably different asymptotic behaviors when the dividend rate varies from less than the interest rate to greater than it.

Stock loans are a kind of popular financial contract in the capital market, which are essentially an American call option with time-dependent strike price. As a formal literature record, stock loans were initially studied by Xia and Zhou [14], who investigated stock loans as a perpetual American call option with time-growth strike price and built the value function with a different structure relative to usual perpetual American call. After that, there are considerable studies about stock loans. Further works are

2020 *Mathematics Subject Classification*. 35K20, 91G20, 91G80
Keywords. stock loan, risk-free rate, dividend, loan rate, Black-Scholes equation, free boundary
Received: 08 March 2022; Revised: 15 May 2022; Accepted: 04 July 2022
Communicated by Miljana Jovanović
The work is supported in part by Shenzhen postdoctoral start-up fund(NO.202028555301050) and China Postdoctoral Science Foundation (NO.2015M572298).
Email address: xyq195752@sina.com; xyq195752@gmail.com (Yongqing Xu)

mainly extended along several directions, which include considering the contracts under other underlying processes, the ones with different terms/clauses, and the ones with finite maturity. For the extensions with the complicated underlying processes, we can refer some literature as follows: Zhang and Zhou [15] studied perpetual stock loans under regime-switching models; Prager and Zhang [17, 29] considered the finite-maturity stock loans under regime-switching and mean-reverting models and a Markov chain model; Wong and Wong [21] investigated perpetual stock loans with mean-reverting stochastic volatility models; Wong and Wong [22] studied the contracts with exponential phase-type Lévy models; Grasselli and Gómez [23] researched the contract in incomplete markets; Cai and Sun [25] studied the contracts under hyper-exponential jump diffusion models; Chen, Xu and Zhu [28] investigated the contracts under stochastic interest rate models; Fan, Xiang and Chen [30] studied the contracts based on the finite moment log-stable process; Fan and Zhou [31] investigated the contracts under the CGMY Models. For the second extensions with added clauses, we can refer several papers. Liu and Xu [16] considered the valuation of perpetual stock loans with cap limit of stock price; Liang, Wu and Jiang [18] studied the contracts with automatic termination clause, cap and margin; Lu and Putri [27] investigated stock loans with a margin call. For the extensions with finite maturity, we can refer Dai and Xu [20], Pascucci, Suárez-Taboada and Vázquez [24], Lu and Putri [26].

Under finite time framework, the free boundary of stock loans is treated as a function of time. Since stock loans possesses the early exercise feature of American options, it is necessary to ascertain the time-varying behavior of exercise boundary of the contract. In [20], the authors provided the limits of the redeeming boundaries under several different dividend distribution manners as the time tending to maturity. But these results looks a bit rough and not exquisite as taking into account short-time asymptotic form. To the best of our knowledge, there are no available literature concerning short-time asymptotics of stock loans. In this paper, we will investigate the problems of short-time approximation for stock loans, in which we derive asymptotic expansions of the free boundary and the scaled value function with respect to time-to-maturity. This paper is structured as follows: The second section provides a formulation of the model about the problem. By proper change of variable, we transform Black-Scholes equation into a second-order parabolic equation with constant coefficients. Next, we derive an expansion of the free boundary of stock loan about time-to-maturity. Then we derive the matched asymptotic expansion of the scaled value function. Some formulas needed in the computations are provided in the appendix.

2. The Model and Problem Formulation

The underlying asset (stock) price observes the geometric Brownian motion

$$dS_t = S_t[(r-q)dt + \sigma dW_t],$$

where $r > 0$ denotes the risk free interest rate, $q \geq 0$ is the dividend rate, $\sigma > 0$ is the volatility coefficient. $\{W_t\}_{t>0}$ is a standard Brownian motion defined on a filtered probability space $(\mathbb{S}, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. We denote the loan rate by the sign β. Assume that, during the mortgage period of the asset, the acquired dividend of the asset is attributed to the bank–the lender of the loan until the client (borrower) repays the loan. At initial time, the client obtains a loan of amount K from the bank with the stock as a collateral. When the borrower pays back the loan, the amount should be $Ke^{\beta t}$ at the repaying loan time $t \in [0, T_0]$ with T_0 being the expiry time of the loan. We write $P(S, t)$ as the value of the loan contract at time t with the current stock price $S_t := S$, which can be expressed as the value function of an optimal stopping problem

$$\begin{aligned} P(S,t) &= \sup_{\tau\in\mathcal{T}_{[t,T_0]}} \mathbb{E}_t[e^{-r(\tau-t)}(S_\tau - Ke^{\beta\tau})^+] \\ &= e^{\beta t} \sup_{\tau\in\mathcal{T}_{[t,T_0]}} \mathbb{E}_t[e^{-(r-\beta)(\tau-t)}(\hat{S}_\tau - K)^+] \\ &= e^{\beta t} \sup_{\tau\in\mathcal{T}_{[0,T_0-t]}} \mathbb{E}[e^{-(r-\beta)\tau}(\hat{S}e^{(r-\beta-q-\frac{1}{2}\sigma^2)\tau+\sigma W_\tau} - K)^+] \end{aligned} \tag{1}$$

where $\mathbb{E}_t$ is the expectation operator under $\mathcal{F}_t$ and $\mathcal{T}_{[t,T_0]}$ denotes the set of all stopping times with values in $[t, T_0]$. The process $\hat{S}_t := e^{-\beta t}S_t$ is the discounted process of stock price, which follows a stochastic differential equation $d\hat{S}_t = \hat{S}_t[(r-\beta-q)dt + \sigma dW_t]$. The simplified notation $\hat{S}$ is the discounted stock price at time t, namely, $\hat{S}_t := \hat{S}$. According to above expressions, we can observe the following facts: (1) As $r \geq \beta$, the value function of stock loan $P(S,t)$ can be denoted as $e^{\beta t}P_1(\hat{S},t)$, where $P_1(\hat{S},t)$ is a usual American call option with the underlying price $\hat{S}$ and non-negative interest rate $r-\beta$ (≥ 0). (2) As $r < \beta$, the value function of stock loan $P(S,t)$ can be denoted as $e^{\beta t}P_2(\hat{S},t)$, where $P_2(\hat{S},t)$ is an unusual American call option with the underlying price $\hat{S}$ and negative interest rate $r-\beta$ (< 0). From another perspective, the value function satisfies the following variational inequality problem

$$P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + (r-q)SP_S - rP = 0, \qquad as \quad 0 < S < B(t), \quad 0 < t < T_0. \tag{2}$$

$$P(B(t),t) = B(t) - Ke^{\beta t}, \quad P_S(B(t),t) = 1. \tag{3}$$

$$\lim_{S\to 0} P(S,t) = 0 \tag{4}$$

$$P(S,T_0) = \max\{S - Ke^{\beta T_0}, 0\}, \quad B(T_0) = \begin{cases} e^{\beta T_0}K, & as \quad r < \beta; \\ e^{\beta T_0}K\max\{1, \frac{r-\beta}{q}\}, & as \quad r \geq \beta, \ q > 0. \end{cases} \tag{5}$$

where $B(t)$ is the exercise boundary of the loan contract, defined as $B(t) \triangleq \inf\{S > 0 | P(S,t) = S - Ke^{\beta t}\}$, $\forall t \in [0, T_0]$. The exercise time of the contract is a stopping time defined by $T^* = \inf\{t \in [0,T_0] | S_t \geq B(t)\}$. Valuation of the stock loan is different from usual American call option in which the strike price is placed by the time-growth one. The strike price with a fast growing factor has caused a trouble of deriving the value function of stock loan under perpetual time framework (see [14]). In finite time setting, we overcome the problem by introducing the following changes of variables:

Let $x = \ln\frac{e^{-\beta t}S}{K}$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$, $b(\tau) = \ln\frac{e^{-\beta t}B(t)}{K}$, $\alpha = \frac{2(r-\beta)}{\sigma^2}$, $\ell = \frac{2q}{\sigma^2}$, $p(x,\tau) = e^{-\beta t}P(S,t)/K$. Under above new variables, the function $p(x,\tau)$ satisfies the following variational inequalities

$$\max\{\mathcal{L}p - p_\tau, p_0 - p\} = 0, \qquad \text{in } \ R \times (0,\infty) \tag{6}$$

$$p(x,0) \triangleq p_0(x) = (e^x - 1)_+, \qquad \text{at } \ \tau = 0. \tag{7}$$

$$b(\tau) = \inf\{x | p(x,\tau) = p_0(x)\} = \sup\{x | p(x,\tau) > p_0(x)\}, \forall \tau > 0. \tag{8}$$

$$b(0) = \begin{cases} 0, & if \quad \alpha < 0; \\ \max\{0, \ln\frac{\alpha}{\ell}\}, & if \quad \alpha \geq 0, \ \ell > 0. \end{cases} \tag{9}$$

where the operator $\mathcal{L}$ is defined by $\mathcal{L}p = \partial_{xx}p + (\alpha - \ell - 1)\partial_x p - \alpha p$. The above variational inequalities (6-9) can be also formulated as a free boundary value problem

$$\begin{cases} p_\tau = \mathcal{L}p, & x < b(\tau), \quad \tau > 0 \\ p(x,\tau) = e^x - 1, & x \geq b(\tau) \\ p_x(b(\tau),\tau) = e^{b(\tau)}, & \tau > 0 \\ p(-\infty,\tau) = 0, & \tau > 0 \end{cases} \tag{10}$$

Our goal is to derive the asymptotic forms of the exercise boundary $b(\tau)$ and the value function $p(x,\tau)$ of stock loan when the time τ approaches zero. Next sections will provide detailed derivation procedures for them.

3. Short Time Asymptotics for the Exercise Boundary

In this section, we will deal with the asymptotic expansion for the exercise boundary. In [20], the limiting behavior of the boundary was provided, but the result looks rough and not accurate. To improve the limiting result, we provide a more accurate form for the boundary. To this end, we need to make the following asymptotic analysis. For the free boundary problem in (10), the corresponding Green's function is

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x+(\alpha-\ell-1)\tau)^2}{4\tau} - \alpha\tau}.$$

By Green's theorem, we can write the function $p(x,\tau)$ in integral of G as follows:

$$\begin{aligned} p(x,\tau) &= \int_0^\infty (e^y - 1)G(x-y,\tau)dy + \int_0^\tau \int_{b(u)}^\infty (\ell e^y - \alpha)G(x-y,\tau-u)dydu. \\ &:= I_1(x,\tau) + I_2(x,\tau). \end{aligned}$$

By direct computations, it produces the explicit expressions as follows

$$\begin{aligned} I_1(x,\tau) &= e^{x-\ell\tau} N(\tfrac{x+(\alpha-\ell+1)\tau}{\sqrt{2\tau}}) - e^{-\alpha\tau} N(\tfrac{x+(\alpha-\ell-1)\tau}{\sqrt{2\tau}}); \\ I_2(x,\tau) &= \int_0^\tau \ell e^{x-\ell(\tau-u)} N(\tfrac{x-b(u)+(\alpha-\ell+1)(\tau-u)}{\sqrt{2(\tau-u)}})du - \int_0^\tau \alpha e^{-\alpha(\tau-u)} N(\tfrac{x-b(u)+(\alpha-\ell-1)(\tau-u)}{\sqrt{2(\tau-u)}})du, \end{aligned}$$

where the sign $N(\cdot)$ is the standard normal distribution function, i.e., $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$. To analyze the asymptotic form of p, we need to introduce the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \sim \frac{e^{-x^2}}{\sqrt{\pi} x}, \quad \text{as} \quad x \to +\infty. \tag{11}$$

It is not hard to derive an equality of the relationship between $N(x)$ and $\operatorname{erfc}(x)$

$$N(x) = 1 - \frac{1}{2}\operatorname{erfc}(\frac{x}{\sqrt{2}}). \tag{12}$$

By virtue of equation(12), we can rewrite the formulations of two above functions $I_1(x,\tau)$ and $I_2(x,\tau)$ involving in the error function and obtain the following results

$$I_1(x,\tau) = e^{x-\ell\tau} \left(1 - \tfrac{1}{2}\operatorname{erfc}(\tfrac{x+(\alpha-\ell+1)\tau}{2\sqrt{\tau}})\right) - e^{-\alpha\tau} \left(1 - \tfrac{1}{2}\operatorname{erfc}(\tfrac{x+(\alpha-\ell-1)\tau}{2\sqrt{\tau}})\right) \tag{13}$$

$$\begin{aligned} I_2(x,\tau) &= \int_0^\tau \ell e^{x-\ell(\tau-u)} (1 - \tfrac{1}{2}\operatorname{erfc}(\tfrac{x-b(u)+(\alpha-\ell+1)(\tau-u)}{2\sqrt{(\tau-u)}}))du \\ &\quad - \int_0^\tau \alpha e^{-\alpha(\tau-u)} (1 - \tfrac{1}{2}\operatorname{erfc}(\tfrac{x-b(u)+(\alpha-\ell-1)(\tau-u)}{2\sqrt{(\tau-u)}}))du. \end{aligned} \tag{14}$$

On the optimal exercise boundary, we have $p_\tau(b(\tau),\tau) = 0$, which has appeared as a useful tool to discuss the analysis of American option near expiry. Thus we can obtain an equation for the boundary $b(\tau)$:

$$\partial_\tau I_1(b(\tau),\tau) + \partial_\tau I_2(b(\tau),\tau) = 0. \tag{15}$$

To solve equation (15), we write $b(\tau)$ with respect to b_0 and a new unknown function $a(\tau) : b(\tau) = b_0 + 2\sqrt{\tau} a(\tau)$. Since $\alpha \geq 0$, the loan contract can reduce to a usual American call option with non-negative interest rate, which case can be adapted to the existing results of American call option (see [1]). We mainly treat the case $\alpha < 0$ and $b_0 = 0$ for this case. In the following, we will show that $\lim_{\tau \to 0^+} a(\tau) = +\infty$ for the case $\alpha < 0$.

Actually, since $\frac{\partial I_2}{\partial x}$ is an integral from 0 to τ, it holds that $\lim_{\tau\to 0^+}\frac{\partial I_2}{\partial x}=0$. By the equation $\frac{\partial p}{\partial x}=\frac{\partial I_1}{\partial x}+\frac{\partial I_2}{\partial x}$, we thus obtain that

$$\lim_{\tau\to 0^+}\frac{\partial p}{\partial x}=\lim_{\tau\to 0^+}\frac{\partial I_1}{\partial x}.$$

Using direct computations, we get that

$$\frac{\partial I_1}{\partial x}=e^{x-\ell\tau}\left(1-\tfrac{1}{2}\mathrm{erfc}(\tfrac{x+(\alpha-\ell+1)\tau}{2\sqrt{\tau}})+\tfrac{1}{2\sqrt{\pi}}\tfrac{1}{\sqrt{\tau}}e^{-(\frac{x+(\alpha-\ell+1)\tau}{2\sqrt{\tau}})^2}\right)-e^{-\alpha\tau}\tfrac{1}{2\sqrt{\pi}}\tfrac{1}{\sqrt{\tau}}e^{-(\frac{x+(\alpha-\ell-1)\tau}{2\sqrt{\tau}})^2}.$$

Using the equation (10) and $b(0)=0$ for $\alpha<0$, it holds that $p_x(b(0),0)=e^{b(0)}=1$. We deduce that

$$\lim_{\tau\to 0^+}\frac{\partial I_1}{\partial x}(b(\tau),\tau)=\lim_{\tau\to 0^+}(1-\frac{1}{2}\mathrm{erfc}(a(\tau))=1.$$

Thus we have that $\lim_{\tau\to 0^+}\mathrm{erfc}(a(\tau))=0$. According to the property of the error function $\mathrm{erfc}(\cdot)$, we can deduce that $\lim_{\tau\to 0^+}a(\tau)=+\infty$.

To obtain the asymptotic expression of $b(\tau)$ as τ tends to zero, we need to derive the asymptotic forms of $\partial_\tau I_1(b(\tau),\tau)$ and $\partial_\tau I_2(b(\tau),\tau)$. Firstly, we compute the derivative $\partial_\tau I_1(x,\tau)$ as follows:

$$\begin{aligned}\partial_\tau I_1(x,\tau) &= -\ell e^{x-\ell\tau}(1-\tfrac{1}{2}\mathrm{erfc}(\tfrac{x+(\alpha-\ell+1)\tau}{2\sqrt{\tau}}))+e^{x-\ell\tau}\tfrac{1}{\sqrt{\pi}}e^{-(\frac{x+(\alpha-\ell+1)\tau}{2\sqrt{\tau}})^2}(\tfrac{-x}{4\tau^{3/2}}+\tfrac{\alpha-\ell+1}{4\sqrt{\tau}})\\ &\quad+\alpha e^{-\alpha\tau}(1-\tfrac{1}{2}\mathrm{erfc}(\tfrac{x+(\alpha-\ell-1)\tau}{2\sqrt{\tau}}))-e^{-\alpha\tau}\tfrac{1}{\sqrt{\pi}}e^{-(\frac{x+(\alpha-\ell-1)\tau}{2\sqrt{\tau}})^2}(\tfrac{-x}{4\tau^{3/2}}+\tfrac{\alpha-\ell-1}{4\sqrt{\tau}}).\end{aligned}$$

At $x=b(\tau)$, as $\tau\to 0^+$, it holds that $b(\tau)\to 0, e^{b(0)-\ell\tau}\to 1, e^{-\alpha\tau}\to 1, \frac{x+(\alpha-\ell\pm 1)\tau}{2\sqrt{\tau}})\to+\infty$, and $\mathrm{erfc}(\frac{x+(\alpha-\ell\pm 1)\tau}{2\sqrt{\tau}})\to 0$, $\frac{x+(\alpha-\ell\pm 1)\tau}{2\sqrt{\tau}}\sim\frac{x}{2\sqrt{\tau}}$. Thus, we combine these results and obtain that, as $\tau\to 0^+$,

$$\partial_\tau I_1(b(\tau),\tau)\sim(\alpha-\ell)+\tfrac{1}{2\sqrt{\pi\tau}}e^{-b^2(\tau)/4\tau}. \tag{16}$$

Next, we will derive the asymptotic expression of $\partial_\tau I_2(b(\tau),\tau)$. To this end, we set change of variable $u=\tau z$ and define $C(x,z,\tau):=\frac{x-b(u)}{2\sqrt{\tau-u}}=\frac{x/\tau^{1/2}-2z^{1/2}a(\tau z)}{2\sqrt{1-z}}$, then the variables in erfc of I_2 become

$$\frac{x-b(u)+(\alpha-\ell\pm 1)(\tau-u)}{2\sqrt{(\tau-u)}}=C(x,z,\tau)+\frac{(\alpha-\ell\pm 1)\tau^{1/2}\sqrt{1-z}}{2}.$$

Then, in the expression of I_2, we make variable change $u=\tau z$ and use $C(x,\tau,z)$. By Taylor's expansion, we evaluate I_2 up to terms of order $\tau^{3/2}$. Letting $\tau\to 0$, we have that

$$I_2(x,\tau)\sim(\ell e^x-\alpha)\tau[1-\tfrac{1}{2}\int_0^1\mathrm{erfc}(C(x,z,\tau)dz]+[\ell e^x\tfrac{(\alpha-\ell+1)}{2}-\tfrac{\alpha(\alpha-\ell-1)}{2}]\tfrac{\tau^{3/2}}{2}\int_0^1\tfrac{2\sqrt{1-z}}{\sqrt{\pi}}e^{-C^2(x,z,\tau)}dz \tag{17}$$

Letting $x=b(\tau)$ and $\tau\to 0$, we take the τ derivative of (17) to obtain $\partial_\tau I_2(b(\tau),\tau)$. When $\beta>r$, it implies $b_0=0$ and $b(\tau)=2\tau^{1/2}a(\tau)$. Equation (15) multiplies by factor $2\tau^{1/2}$ and then become

$$\begin{aligned}\frac{e^{-a^2(\tau)}}{\sqrt{\pi}} &\sim \lim_{x\to b(\tau)}\Bigg[-\tau^{1/2}(\alpha-\ell-2\ell\tau^{1/2}a(\tau))\int_0^1\mathrm{erfc}(C(x,z,\tau))dz\\ &\quad-\tau^{3/2}(\alpha-\ell-2\ell\tau^{1/2}a(\tau))\Big(-\int_0^1\frac{2}{\sqrt{\pi}}e^{-C^2(x,z,\tau)}C_\tau(x,z,\tau))dz\Big)\\ &\quad-\frac{3\tau}{2}\frac{1}{2}[\frac{\alpha+\ell-(\alpha+\ell)^2}{2}]\int_0^1\frac{2\sqrt{1-z}}{\sqrt{\pi}}e^{-C^2(x,z,\tau))}dz\\ &\quad-\tau^2[\frac{\alpha+\ell-(\alpha+\ell)^2}{2}]\int_0^1\frac{2\sqrt{1-z}}{\sqrt{\pi}}(-2C(x,z,\tau))C_\tau(x,z,\tau)))e^{-C^2(x,z,\tau))}dz\Bigg]\end{aligned} \tag{18}$$

By virtue of the similar discussion in ([1]), we obtain that $\int_0^1 \operatorname{erfc}(C(b(\tau), z, \tau))dz \sim \lim_{z\to 1^-} -\frac{C_{zz}}{2|C_z|^3}$, as $\tau \to 0$. Since $C(b(\tau), z, \tau) = \frac{a(\tau)-\sqrt{z}a(\tau z)}{\sqrt{1-z}}$, the asymptotic form of the first- and second-order derivatives of C w.r.t τ are given as follows. Direct computations show that

$$C_z(b(\tau), z, \tau)) = -\frac{a(\tau z) + 2\tau z a'(\tau z)}{2\sqrt{z}(1-z)^{1/2}} + \frac{a(\tau) - \sqrt{z}a(\tau z)}{2(1-z)^{3/2}}.$$

For fixed $\tau > 0$, we can deduce that $\lim_{z\to 1^-} \frac{a(\tau)-\sqrt{z}a(\tau z)}{1-z} = \frac{a(\tau)+2\tau a'(\tau)}{2}$ by L'Hospital rule. Thus we have that, as $z \to 1^-$,

$$C_z(b(\tau), z, \tau)) \sim -\frac{a(\tau) + 2\tau a'(\tau)}{4(1-z)^{1/2}}.$$

Furthermore, we can obtain that, as $z \to 1^-$,

$$C_{zz}(b(\tau), z, \tau)) \sim -\frac{a(\tau) + 2\tau a'(\tau)}{8(1-z)^{3/2}}.$$

Combining above results on two derivatives of C, we will obtain the following expression, as $\tau \to 0^+$,

$$\int_0^1 \operatorname{erfc}(C(b(\tau), z, \tau))dz \sim \lim_{z\to 1^-} -\frac{C_{zz}}{2|C_z|^3} = O\big(a(\tau)^{-2}\big).$$

For the second integral in (18), we have that $\lim_{\tau\to 0^+} \frac{-2\tau}{\sqrt{\pi}} \int_0^1 e^{-C^2(b(\tau),z,\tau)} C_\tau(b(\tau), z, \tau))dz = 4$. Actually, using the discussions of asymptotic evaluation of Gaussian integrals in ([1]), we have that, as $\tau \to 0^+$,

$$\frac{-2\tau}{\sqrt{\pi}} \int_0^1 e^{-C^2(b(\tau),z,\tau)} C_\tau(b(\tau), z, \tau))dz \sim \frac{-2\tau}{\sqrt{\pi}} \lim_{z\to 1^-} C_\tau(b(\tau), z, \tau)) \frac{\sqrt{\pi}}{|C_z(b(\tau), z, \tau))|}.$$

Because of the derivative $C_\tau(x, z, \tau) = \frac{-\frac{x}{2\tau^{3/2}} - 2z^{3/2}a'(\tau z)}{2\sqrt{1-z}}$, we can obtain that

$$\lim_{\tau\to 0^+} \frac{-2\tau}{\sqrt{\pi}} \lim_{z\to 1^-} C_\tau(b(\tau), z, \tau)) \frac{\sqrt{\pi}}{|C_z(b(\tau), z, \tau))|} = 4 \lim_{\tau\to 0^+} \lim_{z\to 1^-} \frac{a(\tau) + 2z^{3/2}\tau a'(\tau)}{a(\tau) + 2\tau a'(\tau)} = 4.$$

Thus we keep the term of order $\tau^{1/2}$ in the right side of (18), and obtain the following main result

$$\frac{e^{-a^2(\tau)}}{\sqrt{\pi}} \sim 4(\ell - \alpha)\sqrt{\tau}, \quad \text{for } \ell \geq 0 \text{ and } \alpha < 0, \tag{19}$$

which implies that, as $\tau \to 0^+$,

$$a^2(\tau) \sim \ln \frac{1}{4(\ell - \alpha)\sqrt{\pi\tau}}, \quad \text{for } \ell \geq 0 \text{ and } \alpha < 0, \tag{20}$$

from which we can also see that $\lim_{\tau\to 0^+} a(\tau) = +\infty$. Via the equality $B(t) = Ke^{\beta t}e^{b(\tau)}$, we can acquire that the free boundary of stock loan near expiry observes the asymptotic behavior with $\bar{r} = r - \beta < 0$, as $t \to T_0^-$,

$$B(t) \sim Ke^{\beta t}\left[1 + \sigma\sqrt{(T_0 - t)\ln[\sigma^2/(32\pi(q - \bar{r})^2(T_0 - t))]}\right]. \tag{21}$$

Besides, when the interest rate r is larger than the loan rate β, i.e., $r > \beta$, this case returns back to the usual American call option, according to the available results in [1], we can write the boundary of stock loan with dividends as follows: as $t \to T_0^-$

$$\begin{aligned} &B(t) \sim Ke^{\beta t}\Big[1 + \sigma\sqrt{(T_0 - t)\ln[\sigma^2/(8\pi(T_0 - t)(q - \bar{r})^2)]}\Big]; && q > \bar{r} \\ &B(t) \sim Ke^{\beta t}\Big[1 + \sigma\sqrt{2(T_0 - t)\ln[1/(4\sqrt{\pi}q(T_0 - t))]}\Big]; && q = \bar{r} \\ &B(t) \sim \tfrac{\bar{r}}{q}Ke^{\beta t}[1 + \sigma\gamma_0\sqrt{2(T_0 - t)}]; && 0 \le q < \bar{r}. \end{aligned} \tag{22}$$

where γ_0 is some constant.

4. Matched Asymptotic Expansions of Stock Loan

In this section, we shall derive matched asymptotic expansions to the scaled value function with respect to short time. For studying short time behavior of the scaled value function $p(x, \tau)$, we define a new function $h(x, \tau) := e^{\alpha\tau}(p(x, \tau) - e^x + 1)$. Then in continuation region of the contract, $h(x, \tau)$ satisfies the equation

$$h_\tau = h_{xx} + (\alpha - \ell - 1)h_x + e^{\alpha\tau}(\alpha - \ell e^x).$$

Let $\tau = \delta T$ with $T = O(1)$ and δ is a small parameter. Then the function h satisfies the equation for x and T as follows:

$$h_T = \delta\Big[h_{xx} + (\alpha - \ell - 1)h_x + e^{\delta\alpha T}(\alpha - \ell e^x)\Big] \tag{23}$$

and with the boundary and initial conditions

$$h(b(\tau), T) = h_x(b(\tau), T) = 0;\ h(x, T) = e^{\delta\alpha T}(1 - e^x),\ \text{as} x \to -\infty; h(x, 0) = (1 - e^x)_+. \tag{24}$$

For $x < 0$ and $x = O(1)$, we employ a Taylor's expansion in the powers of δ to obtain the outer expansion as follows:

$$h = 1 - e^x + \delta\alpha(1 - e^x)T + O(\delta^2), \quad x < 0. \tag{25}$$

A local expansion in the region $x = b(\tau) + \delta z$ with $z = O(1)$ is proposed to cater for the boundary conditions. Thus the equation for $h(z, T)$ is given by

$$\begin{aligned} &\delta h_T - b'(\tau)h_z = h_{zz} + \delta(\alpha - \ell - 1)h_z + \delta^2 e^{\delta\alpha T}(\alpha - \ell e^{b(\tau)+\delta z}) \\ &h(b, T) = h_z(b, T) = 0, \end{aligned}$$

this will lead to $h = O(\delta^2)$. Since the outer expansion becomes invalid for $x = O(\delta^{1/2})$, we need an inner expansion with a scaling $x = \delta^{1/2}X$ and $X = O(1)$ which links between the outer region and a region near $b(\tau)$. We define the following expansion

$$h(x, \tau) = \delta^{1/2}h_0(X, T) + \delta h_1(X, T) + \delta^{3/2}h_2(X, T) + O(\delta^2). \tag{26}$$

Using three equalities $h_T = \delta^{1/2}h_{0T} + \delta h_{1T} + \delta^{3/2}h_{2T}$, $h_x = h_{0X} + \delta^{1/2}h_{1X} + \delta h_{2X}$ and $h_{xx} = \delta^{-1/2}h_{0XX} + h_{1XX} + \delta^{1/2}h_{2XX}$, we combine them with (23) and match the coefficients of the powers of δ of two sides. We can derive the sequence of three subproblems:

Subproblem 1. For h_0, it satisfies an equation with conditions

$$h_{0T} = h_{0XX}, \quad \text{in} -\infty < X < \infty, T > 0,$$

$$h_0(X, 0) = \max(-X, 0), \quad \text{as } X \to +\infty, h_0 \to 0, \ \text{as } X \to -\infty,\ h_0 \sim -X.$$

By Green's theorem, we can derive the solution

$$h_0(X,T) = \int_{-\infty}^{0} (-y)\frac{1}{\sqrt{4\pi T}}e^{-\frac{(X-y)^2}{4T}}dy = \frac{\sqrt{T}}{\sqrt{\pi}}e^{-\frac{X^2}{4T}} - \frac{X}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}}).$$

Furthermore, we can write $h_0(X,T) = \sqrt{T}f_0(\xi)$ with $\xi = \frac{X}{2\sqrt{T}}$ and $f_0(\xi) = \frac{1}{\sqrt{\pi}}e^{-\xi^2} - \xi\mathrm{erfc}(\xi)$.

Subproblem 2. For h_1, it satisfies an equation with conditions

$$h_{1T} = h_{1XX} + (\alpha-\ell-1)h_{0X} + (\alpha-\ell), \quad \text{in} -\infty < X < \infty,\ T > 0,$$

$$h_1(X,0) = -\frac{X^2}{2}I(X<0); \quad \text{as } X \to +\infty,\ h_{1X} \to 0;\ \text{as } X \to -\infty,\ h_1 \sim -\frac{X^2}{2}.$$

The solution to h_1 can be expressed as

$$h_1(X,T) = \int_{-\infty}^{0} \frac{-\frac{1}{2}y^2}{\sqrt{4\pi T}}e^{-\frac{(X-y)^2}{4T}}dy + \int_0^T\int_{-\infty}^{\infty}\Big(\frac{-(\alpha-\ell-1)}{2}\mathrm{erfc}(\frac{y}{2\sqrt{s}}) + (\alpha-\ell)\Big)\frac{1}{\sqrt{4\pi(T-s)}}e^{-\frac{(X-y)^2}{4(T-s)}}dyds$$

since $h_{0X} = -\frac{1}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}})$. The first integral in h_1 may be formulated as $-\frac{X^2}{4}\mathrm{erfc}(\frac{X}{2\sqrt{T}}) + \frac{T}{\sqrt{\pi}}\frac{X}{2\sqrt{T}}e^{-\frac{X^2}{4T}} - \frac{T}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}})$. To simplify the second integral in h_1, we need to utilize an important equality $\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)e^{-u^2}du = \mathrm{erfc}(\dfrac{a}{\sqrt{b^2+1}})$, which will be proved in Appendix 1. By the equality, we can compute the second integral in h_1 as follows

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}\Big(\frac{-(\alpha-\ell-1)}{2}\mathrm{erfc}(\frac{y}{2\sqrt{s}}) + (\alpha-\ell)\Big)\frac{1}{\sqrt{4\pi(T-s)}}e^{-\frac{(X-y)^2}{4(T-s)}}dyds \\
&= \int_0^T\int_{-\infty}^{\infty}\Big(\frac{-(\alpha-\ell-1)}{2}\mathrm{erfc}(\frac{X-2\sqrt{T-s}u}{2\sqrt{s}}) + (\alpha-\ell)\Big)\frac{1}{\sqrt{\pi}}e^{-u^2}duds;\ \ (u = \frac{X-y}{2\sqrt{T-s}}) \\
&= \frac{-(\alpha-\ell-1)}{2}\int_0^T \mathrm{erfc}(\frac{\frac{X}{2\sqrt{s}}}{\sqrt{\frac{T-s}{s}+1}})ds + (\alpha-\ell)T \\
&= \frac{-(\alpha-\ell-1)}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}})T + (\alpha-\ell)T.
\end{aligned}$$

Thus we write the expression of h_1 as follows:

$$h_1(X,T) = T\Big(-\xi^2\mathrm{erfc}(\xi) - \frac{(\alpha-\ell)}{2}\mathrm{erfc}(\xi) + \frac{1}{\sqrt{\pi}}\xi e^{-\xi^2} + (\alpha-\ell)\Big) := Tf_1(\xi),$$

where $\xi = \frac{X}{2\sqrt{T}}$ and $f_1(\xi)$ satisfies

$$\begin{aligned}
&f_1'' + 2\xi f_1' - 4f_1 = -2(\alpha-\ell-1)f_0' - 4(\alpha-\ell),\ \text{in } -\infty < \xi < \infty; \\
&\text{as } \xi \to +\infty,\ f_1 \to (\alpha-\ell),\ f_1' \to 0; \quad \text{as } \xi \to -\infty,\ f_1 \sim -2\xi^2.
\end{aligned}$$

Subproblem 3. For h_2, it satisfies an equation with conditions

$$h_{2T} = h_{2XX} + (\alpha-\ell-1)h_{1X} - \ell X \quad \text{in} -\infty < X < \infty,\ T > 0,$$

$$h_2(X,0) = \max(-\frac{X^3}{6}, 0); \quad \text{as } X \to +\infty,\ h_{2X} \to -\ell T;\ \text{as } X \to -\infty,\ h_2 \sim -\frac{X^3}{6} - \alpha XT.$$

By Green's theorem, the solution of h_2 can be denoted as

$$h_2(X,T) = \int_{-\infty}^{0}\frac{-\frac{1}{6}y^3}{\sqrt{4\pi T}}e^{-\frac{(X-y)^2}{4T}}dy + \int_0^T\int_{-\infty}^{\infty}\Big((\alpha-\ell-1)h_{1X}(y,s) - \ell y\Big)\frac{1}{\sqrt{4\pi(T-s)}}e^{-\frac{(X-y)^2}{4(T-s)}}dyds.$$

First integral in h_2 can be computed as $-\frac{X^3}{12}\mathrm{erfc}(\frac{X}{2\sqrt{T}}) - \frac{XT}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}}) + \frac{2}{3}\frac{T\sqrt{T}}{\sqrt{\pi}}(\frac{X}{2\sqrt{T}})^2 e^{-(\frac{X}{2\sqrt{T}})^2} + \frac{2}{3}\frac{T\sqrt{T}}{\sqrt{\pi}}e^{-(\frac{X}{2\sqrt{T}})^2}$. Because of $h_{1X} = \sqrt{T}\Big(-\frac{X}{2\sqrt{T}}\mathrm{erfc}(\frac{X}{2\sqrt{T}}) + \frac{\alpha-\ell+1}{2\sqrt{\pi}}e^{-(\frac{X}{2\sqrt{T}})^2}\Big)$, and using the above integral equality involved error function, we can compute the second integral in h_2 as follows:

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}\Big((\alpha-\ell-1)h_{1X}(y,s)-\ell y\Big)\frac{1}{\sqrt{4\pi(T-s)}}e^{-\frac{(X-y)^2}{4(T-s)}}dyds\\
&=\int_0^T\int_{-\infty}^{\infty}\Big[(\alpha-\ell-1)\sqrt{s}\Big(-\frac{y}{2\sqrt{s}}\mathrm{erfc}(\frac{y}{2\sqrt{s}})+\frac{\alpha-\ell+1}{2\sqrt{\pi}}e^{-(\frac{y}{2\sqrt{s}})^2}\Big)-\ell y\Big]\frac{1}{\sqrt{4\pi(T-s)}}e^{-\frac{(X-y)^2}{4(T-s)}}dyds\\
&=\int_0^T\int_{-\infty}^{\infty}(\alpha-\ell-1)\sqrt{s}\Big(-(\frac{X}{2\sqrt{s}}-\sqrt{\frac{T-s}{s}}u)\mathrm{erfc}(\frac{X}{2\sqrt{s}}-\sqrt{\frac{T-s}{s}}u)\Big)\frac{1}{\sqrt{\pi}}e^{-u^2}du\\
&\quad+\int_0^T\int_{-\infty}^{\infty}\frac{(\alpha-\ell)^2-1}{2\sqrt{\pi}}\sqrt{s}e^{(\frac{X}{2\sqrt{s}}-\sqrt{\frac{T-s}{s}}u)^2}e^{-u^2}du-\int_0^T\int_{-\infty}^{\infty}\ell(X-2\sqrt{T-s}u)\frac{e^{-u^2}}{\sqrt{\pi}}du\\
&=-(\alpha-\ell-1)\int_0^T\sqrt{s}\Big(\frac{X}{2\sqrt{s}}\mathrm{erfc}(\frac{\frac{X}{2\sqrt{s}}}{\sqrt{\frac{T-s}{s}+1}})-\frac{1}{\sqrt{\pi}}(\frac{T-s}{s})\sqrt{\frac{s}{T}}e^{-(\frac{X}{2\sqrt{T}})^2}\Big)ds\\
&\quad+\frac{(\alpha-\ell)^2-1}{2\sqrt{\pi}}\int_0^T\sqrt{s}\sqrt{\frac{s}{T}}e^{-\frac{X^2}{4T}}ds-\ell XT\\
&=-(\alpha-\ell-1)\frac{XT}{2}\mathrm{erfc}(\frac{X}{2\sqrt{T}})+(\alpha-\ell-1)\frac{T\sqrt{T}}{2\sqrt{\pi}}e^{-\frac{X^2}{4T}}+\frac{(\alpha-\ell)^2-1}{2\sqrt{\pi}}\frac{T\sqrt{T}}{2}e^{-\frac{X^2}{4T}}-\ell XT
\end{aligned}$$

where the second equality is true by variable change $u=\frac{X-y}{2\sqrt{T-s}}$. The third equality comes from two important equalities

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)e^{-u^2}du=\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}})$$

and

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(a-bu)\mathrm{erfc}(a-bu)e^{-u^2}du=a\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}})-\frac{b^2}{\sqrt{\pi}}\frac{e^{-\frac{a^2}{b^2+1}}}{\sqrt{b^2+1}}.$$

which are proven in the Appendix 1. We also write the expression of h_2 in the variables $\xi=\frac{X}{2\sqrt{T}}$ and T as follows: $h_2(X,T)=T^{\frac{3}{2}}f_2(\xi)$, and

$$f_2(\xi)=-\frac{2}{3}\xi^3\mathrm{erfc}(\xi)-(\alpha-\ell)\xi\mathrm{erfc}(\xi)+\frac{2}{3\sqrt{\pi}}(\xi^2+1)e^{-\xi^2}+\Big(\frac{\alpha-\ell-1}{2\sqrt{\pi}}+\frac{(\alpha-\ell)^2-1}{4\sqrt{\pi}}\Big)e^{-\xi^2}-2\ell\xi,$$

which satisfies the following equation with conditions

$$\begin{aligned}
&f_2''+2\xi f_2'-6f_2=-2(\alpha-\ell-1)f_1'+8\ell\xi,\ \text{ in } -\infty<\xi<\infty;\\
&\text{as } \xi\to+\infty,\ f_2\to-\infty,\ f_2'\to-2\ell;\quad \text{as } \xi\to-\infty,\ f_2\sim-\frac{4}{3}\xi^3-2\alpha\xi.
\end{aligned}$$

Besides, as $\xi\to+\infty$, the asymptotic forms of f_0, f_1 and f_2, which are needed for matching aims, are given by

$$\begin{aligned}
&f_0(\xi)\sim\frac{1}{\sqrt{\pi}}(\frac{1}{2\xi^2}-\frac{3}{4\xi^4})e^{-\xi^2},\quad \text{as } \xi\to+\infty;\\
&f_1(\xi)\sim-\frac{1}{2\sqrt{\pi}}(\alpha-\ell)\frac{e^{-\xi^2}}{\xi}+(\alpha-\ell),\quad \text{as } \xi\to+\infty;\\
&f_2(\xi)\sim\frac{1}{\sqrt{\pi}}\Big(\frac{2}{3}-\frac{\alpha-\ell+1}{2}+\frac{(\alpha-\ell)^2-1}{4}\Big)e^{-\xi^2}-2\ell\xi,\quad \text{as } \xi\to+\infty.
\end{aligned}$$

Thus we have acquired the inner expansion for $h(x,\tau)$ as follows:

$$h(x,\tau) = \tau^{\frac{1}{2}} f_0(\frac{x}{2\sqrt{\tau}}) + \tau f_1(\frac{x}{2\sqrt{\tau}}) + \tau^{\frac{3}{2}} f_2(\frac{x}{2\sqrt{\tau}}) + O(\tau^2), \quad x = O(\sqrt{\tau}). \tag{27}$$

When approaching the free boundary, $x = b(\tau) + O(\tau)$, the function h takes on the asymptotic behavior $h(x,\tau) \sim O(\tau^2), \quad x = b(\tau) + O(\tau)$.

5. Numerical Examples

In this section, we study numerical examples with the parameters $K = 50, r = 0.08, \beta = 0.1, q = 0.02, \sigma = 0.2, \tau = 0.05, 0.1$. and the values of the underlying asset in the interval $[40, 60]$. Using finite difference method, we compute the exact value of stock loan and get the exact boundaries 55.99 for $\tau = 0.1$. By the expression (20)or (21), we can obtain the asymptotic boundary 53.66 for $\tau = 0.1$. The exact and asymptotic boundaries for $\tau = 0.05$ are 54.43 and 52.83, respectively. The percentages of error of asymptotic boundary are about $2.9\% - 5\%$. With the shorter time-to-maturity, the error is smaller. We also compute asymptotic value function of the contract based on the asymptotic boundary. From figure 1, we see that the asymptotic value functions can offer efficient substitutions for the true ones when the time-to-maturity is short. The results are displayed in the following figure.

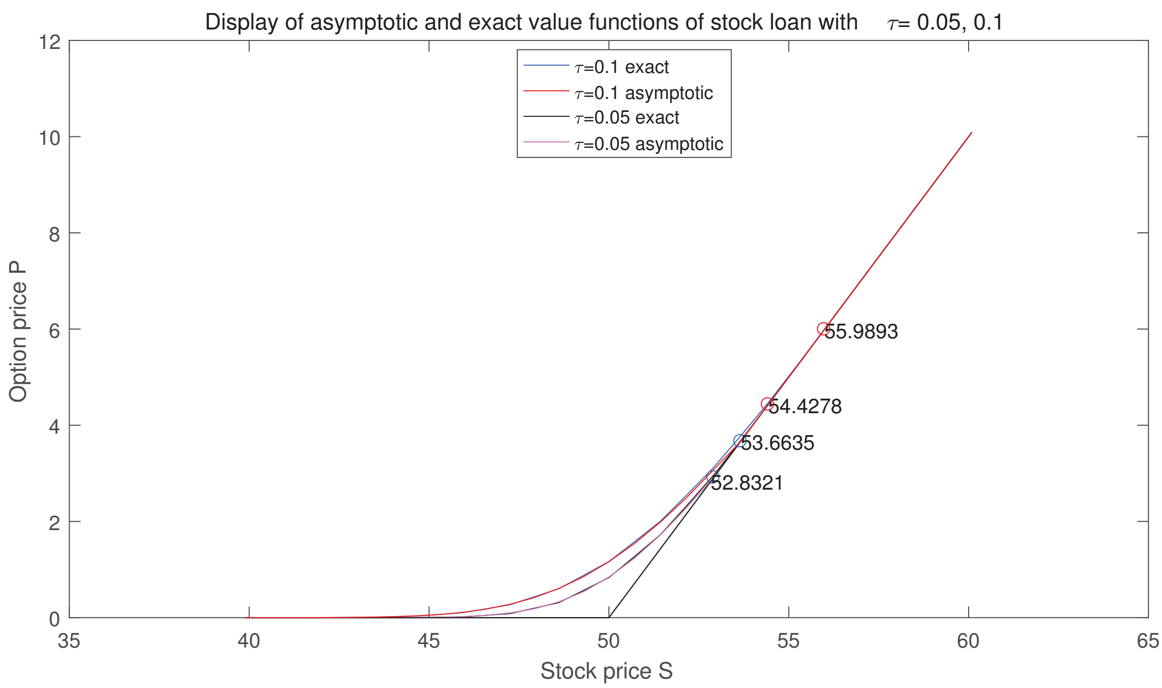

Figure 1: Comparisons of the exact and asymptotic value functions of Stock loan

6. Conclusion

In above contents, we have derived asymptotic expressions of the redemption boundary and value function of stock loan with short time to maturity. Using the equation that the option has no time value at the redemption boundary, we set up the asymptotic expression for the boundary. When the risk-free rate r is smaller than the loan rate β, i.e., $r < \beta$, the result shows that the boundary tends to $Ke^{\beta T_0}$ in parabolic-logarithm form, this case is the main result. For the case $r \geq \beta$, the corresponding problem returns back to a usual American call option with interest-free rate $r-\beta$ and the existing results can be utilized to make proper adjustments for the stock loan. The matched expansion for the value function is derived with respect to a small parameter. Numerical examples are provided to show that the proposed method is effective.

Acknowledgement: The author thanks the reviewers for the constructive advice, which have improved the paper sufficiently.

Appendix 1

In this appendix, we prove the equality $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)e^{-u^2}du = \mathrm{erfc}(\frac{a}{\sqrt{b^2+1}})$, which is very important and used in former computation of related integral. To this end, we define a function $g(a) := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)e^{-u^2}du$, which can be treated as an integral with parameter a. Since $\mathrm{erfc}'(x) = -\frac{2}{\sqrt{\pi}}e^{-x^2}$, we can derive that

$$\begin{aligned}
g'(a) &= -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{\pi}}e^{-(a-bu)^2}e^{-u^2}du;\\
&= -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{\pi}}e^{-(b^2+1)\left(u-\frac{ab}{b^2+1}\right)^2}e^{-\frac{a^2}{b^2+1}}du;\\
&= -\frac{2}{\sqrt{\pi}\sqrt{b^2+1}}e^{-\frac{a^2}{b^2+1}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-v^2}dv; \quad \left(v = \sqrt{b^2+1}(u-\tfrac{ab}{b^2+1})\right)\\
&= -\frac{2}{\sqrt{\pi}\sqrt{b^2+1}}e^{-\frac{a^2}{b^2+1}}.
\end{aligned}$$

Since it is easy to compute $g(0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(-bu)e^{-u^2}du = 1$ by method of polar coordinates on half-plane and $g(a)-g(0) = \int_0^a g'(t)dt$, we can obtain that

$$\begin{aligned}
g(a) &= 1-\frac{2}{\sqrt{\pi}}\int_0^a\frac{e^{-\frac{v^2}{b^2+1}}}{\sqrt{b^2+1}}dv = 1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{\sqrt{b^2+1}}}e^{-z^2}dz\\
&= \frac{2}{\sqrt{\pi}}\int_{\frac{a}{\sqrt{b^2+1}}}^{\infty}e^{-z^2}dz = \mathrm{erfc}(\frac{a}{\sqrt{b^2+1}}).
\end{aligned}$$

Next, we prove the second equality

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(a-bu)\mathrm{erfc}(a-bu)e^{-u^2}du = a\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}}) - \frac{b^2}{\sqrt{\pi}}\frac{e^{-\frac{a^2}{b^2+1}}}{\sqrt{b^2+1}}.$$

Actually, we can deduce that

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(a-bu)\mathrm{erfc}(a-bu)e^{-u^2}du\\
&= \frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)e^{-u^2}du + \frac{b}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{erfc}(a-bu)de^{-u^2}
\end{aligned}$$

$$= a\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}}) + \frac{b}{2\sqrt{\pi}}\Big[\mathrm{erfc}(a-bu)e^{-u^2}\Big|_{-\infty}^{\infty} - \frac{2b}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2-(a-bu)^2}du\Big]$$
$$= a\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}}) - \frac{b^2}{\sqrt{\pi}}\frac{e^{-\frac{a^2}{b^2+1}}}{\sqrt{b^2+1}}\int_{-\infty}^{\infty}\frac{e^{-v^2}}{\sqrt{\pi}}dv;\ \Big(v = \sqrt{b^2+1}(u - \tfrac{ab}{b^2+1})\Big)$$
$$= a\mathrm{erfc}(\frac{a}{\sqrt{b^2+1}}) - \frac{b^2}{\sqrt{\pi}}\frac{e^{-\frac{a^2}{b^2+1}}}{\sqrt{b^2+1}}.$$

References

[1] J. D. Evans, R. Kuske, and J. B. Keller, American Options on Assets with dividends near expiry, Mathematical Finance, 12 (2002) 219–237.
[2] J. Goodman and N. D. Ostrov, On the early exercise boundary of the American put option, SIAM J. Appl. Math. 62 (2002) 1823–1835.
[3] G. Barles, J. Burdeau, M. Romano, and N. Samsoen, Critical stock price near expiration, Mathematical Finance 5 (1995) 77–95.
[4] R. A. Kuske and J. B. Keller, Optimal exercise boundary for an American put option, Applied Mathematical Finance, 5 (1998) 107–116.
[5] C. Knessl, A note on a moving boundary problem arising in the American put option, Stud. Appl. Math., 107 (2001) 157–183.
[6] D. Lamberton and S. Villeneuve, Critical price near maturity for an American option on a dividend-paying stock, Ann. Appl. Probab. 13 (2003) 800–815.
[7] R. Stamicar, D. Sevcovic, and J. Chadam, The early exercise boundary for the American put near expiry: numerical approximation, Canad. Appl. Math. Quart. 7 (1999) 427–444.
[8] E. Chevalier, Critical price near maturity for an American option on a dividend-paying stock in a local volatility model, Mathematical Finance 15 (2005) 439–463.
[9] X. F. Chen and J. Chadam, A mathematical analysis of the optimal exercise boundary for American put options, SIAM J. Math. Anal. 38 (2006), 1613–1641.
[10] K. Nyström, On the behaviour near expiry for multi-dimensional American options, J. Math. Anal. Appl. 339 (2008) 644–654.
[11] S. Levendorskii, American and European options in multi-factor jump-diffusion models, near expiry, Finance Stoch. 12 (2008) 541–560.
[12] P. Laurence and S. Salsa, Regularity of the free boundary of an American option on several assets, Comm. Pure Appl. Math. 62 (2009) 969–994.
[13] F.N. Hu and C. Knessl, Asymptotics of barrier option pricing under the CEV process, Appl. Math. Finance 17 (2010) 261–300.
[14] J. M. Xia and X. Y. Zhou, Stock loans, Mathematical Finance 17 (2007) 307–317.
[15] Q. Zhang and X. Y. Zhou, Valuation of stock loans with regime switching, SIAM J. Control Optim. 48 (2009) 1229–1250.
[16] G. Y. Liu and Y. Q. Xu, Capped stock loans, Comput. Math. Appl. 59 (2010) 3548–3558.
[17] D. Prager and Q. Zhang, Stock loan valuation under a regime-switching model with mean-reverting and finite maturity, J. Syst. Sci. Complex. 23 (2010) 572-583.
[18] Z. X. Liang, W. M. Wu and S. Q. Jiang, Stock loan with automatic termination clause, cap and margin, Comput. Math. Appl. 60 (2010) 3160–3176.
[19] Z. X. Liang and W. M. Wu, Variational inequalities in stock loan models, Optim. Eng. 13 (2012) 459–470.
[20] M. Dai and Z. Q. Xu, Optimal redeeming strategy of stock loans with finite maturity, Mathematical Finance 21 (2011) 775–793.
[21] T. W. Wong and H. Y. Wong, Stochastic volatility asymptotics of stock loans: valuation and optimal stopping, J. Math. Anal. Appl. 394 (2012) 337–346.
[22] T. W. Wong and H. Y. Wong, Valuation of stock loans using exponential phase-type Lévy models, Appl. Math. Comput. 222 (2013) 275–289.
[23] M. Grasselli and C. Gómez, Stock loans in incomplete markets, Appl. Math. Finance 20 (2013) 118–136.
[24] A. Pascucci, M. Suárez-Taboada, and C. Vázquez, Mathematical analysis and numerical methods for a PDE model of a stock loan pricing problem, J. Math. Anal. Appl. 403 (2013) 38–53.
[25] N. Cai and L. H. Sun, Valuation of stock loans with jump risk, J. Econom. Dynam. Control 40 (2014) 213–241.
[26] X. P. Lu and R. M. Putri, Semi-analytic valuation of stock loans with finite maturity, Commun. Nonlinear Sci. Numer. Simul. 27 (2015) 206–215.
[27] X. P. Lu and R. M. Putri, Finite maturity margin call stock loans, Oper. Res. Lett. 44 (2016) 12–18.
[28] W. T. Chen, L. B. Xu, and S. P. Zhu, Stock loan valuation under a stochastic interest rate model, Comput. Math. Appl. 70 (2015) 1757–1771.
[29] D. Prager and Q. Zhang, Valuation of stock loans under a Markov chain model. J. Syst. Sci. Complex. 29 (2016) 171–186.
[30] C. Y. Fan, K. L. Xiang, and S. Z. Chen, Stock loan valuation based on the finite moment log-stable process, Comput. Math. Appl. 75 (2018) 374–387.
[31] C. Y. Fan and C. H. Zhou, Pricing stock loans with the CGMY model, Discrete Dyn. Nat. Soc. 2019, Art. ID 6903019, 11 pp.