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Two-parameter conformable fractional semigroups and abstract Cauchy problems

Rachid Ameziane Hassani[a], Aziz Blali[b], Abdelkhalek El Amrani[a], Mouad El Beldi[a]

[a]*Department of Mathematics, Dhar El Mahraz Faculty of Sciences, University Sidi Mohammed Ben Abdellah, Morocco*
[b]*Department of Mathematics, École Normale Supérieure, University Sidi Mohammed Ben Abdellah, Morocco*

Abstract. The goal of this work is to introduce the two-parameter conformable fractional semigroups and provide a definition of its infinitesimal generator. For such generators, we develop multiple results. In addition, we show that the two-parameter conformable fractional semigroups provide a solution for two-parameter conformable fractional abstract Cauchy problems.

1. Introduction

Fractional differential equations are well known for their importance in the exploration of many phenomena and processes in various branches of science such as physics, chemistry, control systems, electrodynamics and aerodynamics (see [7],[8],[12],[13],[14],[17] and [20]). For more history on fractional calculus and recent developments we refer to [15], [16] and [18].

In [11], Khalil introduced a derivative called the conformable fractional derivative, which is a natural extension of the classical derivative. It is defined as follows:

Given a function $f : [0, +\infty[\to \mathbb{R}$. Then the conformable fractional derivative of order $\alpha \in]0, 1]$ at $t > 0$ (abbreviated α-derivative) is defined by

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

If this limit exists, then the function f is called α-differentiable at t. If f is α-differentiable in some $]0, b[$ where $b > 0$ and the limit $\lim_{t\to 0^+} T_\alpha (f)(t)$ exists, then the α-derivative at 0 is defined as $T_\alpha (f)(0) = \lim_{t\to 0^+} T_\alpha (f)(t)$.

This topic has sparked a lot of debate in the scientific community, and a lot of research papers (see [1],[16]).

In [2], Abdeljawad, Al Horani and Khalil introduced a one-parameter semigroup called the conformable fractional semigroup (abbreviated α-semigroup) associated with the α-derivative. They showed that this semigroup is a solution for the one-parameter conformable abstract Cauchy problems (abbreviated α-ACP).

Throughout this paper, we take $\alpha \in]0, 1]$.

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Email addresses: ra.ameziane@yahoo.fr (Rachid Ameziane Hassani), aziz.blali@usmba.ac.ma (Aziz Blali), abdelkhalek.elamrani@usmba.ac.ma (Abdelkhalek El Amrani), mouad.elbeldi@usmba.ac.ma (Mouad El Beldi)

Let X be a Banach space on a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$) with norm $\|.\|$, we will denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. A two-parameter family $(T(s,t))_{s,t\geq 0}$ of bounded linear operators in $\mathcal{L}(X)$ is called a two-parameter semigroup of bounded linear operators on X if it satisfies the following conditions:

1. $T(0,0) = I$ (I is the identity operator on X).
2. $T((s_1,t_1) + (s_2,t_2)) = T(s_1,t_1)\,T(s_2,t_2)$ for all $s_1,s_2,t_1,t_2 \geq 0$.

The theory of two-parameter semigroups was studied in [4]. The authors considered in [5] and [6] a special class of two-parameter semigroups. Two-parameter semigroups proved to be an effective tool to solve the two-parameter abstract Cauchy problems (see [10]).

In this paper, we introduce the two-parameter conformable fractional semigroups $(T_\alpha(s,t))_{s,t\geq 0}$. The problem is to define the infinitesimal generator for such semigroups and develop multiple proprieties for such generators, which will permit in the following to treat the two-parameter conformable fractional Cauchy problems.

To resolve this problem, we have organized our paper as follows:

In section 2, we present some preliminaries about the theory of the one-parameter conformable fractional semigroups of operators.

In section 3, we review the multi-variable conformable fractional calculus of vector-valued functions with values in Banach space. We also define the α-differentiability at 0 and present a relation between the α-derivative and the corresponding partial α-derivatives.

The two-parameter α-semigroup is defined in section 4, and multiple continuity relations are examined in this section.

In section 5, we define the α-infinitesimal generator of the two-parameter α-semigroups as the α-derivative at $(0,0)$ of $T_\alpha(.,.)\,x$ for a given $x \in X$. We use two methods to describe this generator, and we develop some essential properties regarding the α-generators.

In section 6, we apply the previous results to study the two-parameter α-ACP. We show that the two-parameter α-semigroup provides a solution for the two-parameter α-ACP.

2. Preliminaries

Definition 2.1 ([2]). *Let f be a vector-valued function defined by $f : [0,+\infty[\to X$ where X is a Banach space. Then the conformable fractional derivative of order $\alpha \in]0,1]$ at $t > 0$ is defined by*

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

If this limit exists then we say that f is α-differentiable at t, $D^\alpha f(t)$ is called the α-derivative of f at the point t.

If f is α-differentiable in some $]0,a[$ where $a > 0$, and $\lim_{t\to 0^+} D^\alpha f(t)$ exists, then $D^\alpha f(0) = \lim_{t\to 0^+} D^\alpha f(t)$.

For more details about the conformable fractional derivative see [3].

Now we give some reminders on the α-semigroups of one parameter (see [2] for more details).

Definition 2.2 ([2]). *Let $\alpha \in]0,a]$ for any $a > 0$. For a Banach space X, a family $(T_\alpha(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is called a one-parameter conformable fractional semigroup (or α-semigroup) of operators if*

1. $T_\alpha(0) = I$,
2. $T_\alpha\left((s+t)^{\frac{1}{\alpha}}\right) = T_\alpha\left(s^{\frac{1}{\alpha}}\right) T_\alpha\left(t^{\frac{1}{\alpha}}\right)$ *for all $t,s \in [0,\infty)$.*

If $\alpha = 1$, then 1-semigroups are just the usual semigroups.

Definition 2.3 ([2]). *A α-semigroup $(T_\alpha(t))_{t\geq 0}$ is called a C_0-α-semigroup, if for each $x \in X$, $T_\alpha(t)\,x \to x$ as $t \to 0^+$.*

Proposition 2.4. 1. *Let* $(T_\alpha(t))_{t\geq 0}$ *be a* C_0-α*-semigroup. For any* $t \geq 0$, *we set*

$$S(t) = T_\alpha\left(t^{\frac{1}{\alpha}}\right),$$

then $(S(t))_{s\geq 0}$ *is a one-parameter* C_0*-semigroup.*

2. *Let* $(T(t))_{t\geq 0}$ *be a one-parameter* C_0*-semigroup. For any* $t \geq 0$, *we set*

$$T_\alpha(t) = T(t^\alpha),$$

then $(T_\alpha(t))_{t\geq 0}$ *is a one-parameter* C_0-α*-semigroup.*

3. *Let* $(T_\alpha(t))_{t\geq 0}$ *be a* C_0-α*-semigroup. Then there exists constants* $\omega \geq 0$ *and* $M \geq 1$ *such that for all* $t \geq 0$

$$\|T_\alpha(t)\| \leq Me^{\omega t^\alpha}.$$

Proof. 1. and 2. are easily verified.
For 3. we notice that for all $t \geq 0$

$$\|T_\alpha(t)\| = \left\|T_\alpha\left((t^\alpha)^{\frac{1}{\alpha}}\right)\right\| = \|S(t^\alpha)\|,$$

but from 1. we have that $(S(t))_{t\geq 0}$ is a one-parameter C_0-semigroup, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that for all $t \geq 0$ $\|S(t^\alpha)\| \leq Me^{\omega t^\alpha}$. Hence

$$\|T_\alpha(t)\| \leq Me^{\omega t^\alpha}.$$

□

Using 3. of the previous Proposition, we get the following result.

Corollary 2.5. *Let* $(T_\alpha(t))_{t\geq 0}$ *be a* C_0-α*-semigroup. Then for any* $x \in X$, *the map* $t \mapsto T_\alpha(t)x$ *is continuous, that is* $(T_\alpha(t))_{t\geq 0}$ *is strongly continuous.*

Definition 2.6 ([2]). *Let* $(T_\alpha(t))_{t\geq 0}$ *be a* α*-semigroup. The* α*-infinitesimal generator of* $(T_\alpha(t))_{t\geq 0}$ *is defined on*

$$D(A) = \left\{x \in X : \lim_{t\to 0^+} D^\alpha(T_\alpha(t)x) \ \text{exists}\right\},$$

by setting

$$Ax = \lim_{t\to 0^+} D^\alpha(T_\alpha(t)x),$$

for all $x \in D(A)$.

Theorem 2.7 ([2]). *Let* $(T_\alpha(t))_{t\geq 0}$ *be a* C_0-α*-semigroup, where* $\alpha \in]0,1]$ *and let* A *be its infinitesimal generator. Then for* $x \in \mathcal{D}(A)$, $T_\alpha(t)x \in \mathcal{D}(A)$ *and*

$$D^\alpha(T_\alpha(t)x) = AT_\alpha(t)x = T_\alpha(t)Ax.$$

3. Multivariable conformable fractional calculus

Definition 3.1 ([9]). *Let* f *be a vector-valued function defined by* $f : \mathbb{R}^{+^2} \to X$ *where* X *is a Banach space and let* $\alpha \in]0,1]$. *We say that* f *is* α*-differentiable at* (s,t), $s,t > 0$ *if there is a linear transformation* $L : \mathbb{R}^2 \to X$ *such that*

$$\lim_{(h,k)\to(0,0)} \frac{\left\|f\left(s + hs^{1-\alpha}, t + kt^{1-\alpha}\right) - f(s,t) - L(h,k)\right\|}{\|(h,k)\|} = 0.$$

The linear transformation L *if it exists, is unique and we shall denote it by* $D^\alpha f(s,t)$ *and called the conformable fractional derivative (or* α*-derivative) of* f *of order* $\alpha \in]0,1]$ *at* (s,t).

Definition 3.2. *Let f be a vector-valued function defined by $f : \mathbb{R}^{+^2} \to X$ where X is a Banach space and let $\alpha \in]0,1]$. We say that f is α-differentiable at $(0,0)$ if the following assertions are satisfied*

1. *$D^\alpha f(s,t)$ exists in an open of the form $]0,a[\times]0,b[$, $a,b>0$ and $\lim_{(s,t)\to(0^+,0^+)} D^\alpha f(s,t)$ exists.*
2. *The one-parameter vector valued functions defined by $s \mapsto f(s,0)$ and $t \mapsto f(0,t)$ are α-differentiable in $]0,a[$ and $]0,b[$ respectively.*
   In this case, we will take

$$D^\alpha f(0,0) = \lim_{(s,t)\to(0^+,0^+)} D^\alpha f(s,t).$$

The following two theorems are proved with the same method as theorems 3.8 and 3.9 in [9].

Theorem 3.3. *If a vector valued function $f : \mathbb{R}^{+^2} \to X$ is α-differentiable at (s,t) with $s,t>0$ then f is continuous at (s,t).*

Theorem 3.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}^{+^2}$ be a vector valued function defined by $f(t) = (f_1(t), f_2(t))$ and let $g : \mathbb{R}^{+^2} \to X$ be a vector valued function. If f is α-differentiable at $a>0$ and if g is α-differentiable at $f(a)$ with $f_i(a)>0$, $i=1,2$. Then the composition $g\circ f$ is α-differentiable at a and*

$$D^\alpha g\circ f(a) = D^\alpha g(f(a)) \circ f(a)^{\alpha-1} \circ D^\alpha f(a)$$

where $f(a)^{\alpha-1}$ is the linear transformation defined by

$$f(a)^{\alpha-1}(x,y) = \left(x\,[f_1(a)]^{\alpha-1}, y\,[f_2(a)]^{\alpha-1}\right).$$

Definition 3.5 ([9]). *Let $f : \mathbb{R}^{+^n} \to X$ be a vector valued fuction with n variables and $a=(a_1,..,a_n)$ be a point whose i^{th} component $a_i>0$, then the limit*

$$\lim_{\varepsilon\to 0} \frac{f\left(a_1,...,a_{i-1}, a_i+\varepsilon(a_i)^{1-\alpha},...,a_n\right) - f(a)}{\varepsilon}$$

if it exists, is denoted by $\frac{\partial^\alpha}{\partial t_i^\alpha} f(a)$ and called the i^{th} conformable partial derivative (partial α-derivative) of f of order $\alpha \in]0,1]$ at a.

Theorem 3.6 ([9]). *Let $f : \mathbb{R}^{+^2} \to X$ be a vector valued function. If f is α-differentiable at (a,b) where $a,b>0$ then $\frac{\partial^\alpha}{\partial t_i^\alpha} f(a,b)$ exist for $i=1,2$ and*

$$D^\alpha f(a,b) = \left(\frac{\partial^\alpha}{\partial t_1^\alpha} f(a,b), \frac{\partial^\alpha}{\partial t_2^\alpha} f(a,b)\right).$$

4. Continuity of the two-parameter conformable fractional semigroups

Definition 4.1. *Let X be a Banach space, and let $\alpha \in]0,a]$ for any $a>0$. A family $(T_\alpha(s,t))_{s,t\geq 0} \subset \mathcal{L}(X)$ is called a two-parameter conformable fractional semigroup or simply a two-parameter α-semigroup on the Banach space X if the following conditions are satisfied.*

1. *$T_\alpha(0,0) = I$ with I is the identity operator in $\mathcal{L}(X)$.*
2. *$T_\alpha\left((s_1+s_2)^{\frac{1}{\alpha}}, (t_1+t_2)^{\frac{1}{\alpha}}\right) = T_\alpha\left((s_1)^{\frac{1}{\alpha}}, (t_1)^{\frac{1}{\alpha}}\right) T_\alpha\left((s_2)^{\frac{1}{\alpha}}, (t_2)^{\frac{1}{\alpha}}\right)$ for all $s_1,s_2,t_1,t_2 \geq 0$.*

Example 4.2. *Let $(F_\alpha(s))_{s\geq 0}$ and $(G_\alpha(t))_{t\geq 0}$ be two commuting one-parameter α-semigroups, we easily verify that the family $(T_\alpha(s,t))_{s,t\geq 0} \subset \mathcal{L}(X)$ defined by*

$$T_\alpha(s,t) = F_\alpha(s)\, G_\alpha(t) \;\; , s,t \geq 0$$

is a two-parameter α-semigroup. Indeed we have

1. $T_\alpha(0,0) = F_\alpha(0)\, G_\alpha(0) = I \circ I = I.$
2. *We have for all* $s_1, s_2, t_1, t_2 \geq 0$.

$$\begin{aligned} T_\alpha\left((s_1+s_2)^{\frac{1}{\alpha}}, (t_1+t_2)^{\frac{1}{\alpha}}\right) &= F_\alpha\left((s_1+s_2)^{\frac{1}{\alpha}}\right) G_\alpha\left((t_1+t_2)^{\frac{1}{\alpha}}\right) \\ &= F_\alpha\left((s_1)^{\frac{1}{\alpha}}\right) F_\alpha\left((s_2)^{\frac{1}{\alpha}}\right) G_\alpha\left((t_1)^{\frac{1}{\alpha}}\right) G_\alpha\left((t_2)^{\frac{1}{\alpha}}\right) \\ &= F_\alpha\left((s_1)^{\frac{1}{\alpha}}\right) G_\alpha\left((t_1)^{\frac{1}{\alpha}}\right) F_\alpha\left((s_2)^{\frac{1}{\alpha}}\right) G_\alpha\left((t_2)^{\frac{1}{\alpha}}\right) \\ &= T_\alpha\left((s_1)^{\frac{1}{\alpha}}, (t_1)^{\frac{1}{\alpha}}\right) T_\alpha\left((s_2)^{\frac{1}{\alpha}}, (t_2)^{\frac{1}{\alpha}}\right). \end{aligned}$$

Example 4.3. *Let A and B be two bounded commuting linear operators on X, $a, b \in \mathbb{R}\setminus\{0\}$ and define for any $s, t \geq 0$ $T(s,t) = e^{a\sqrt{s}A + b\sqrt{t}B}$. Then $(T(s,t))_{s,t\geq 0}$ is a $\frac{1}{2}$-semigroup with two parameters. In fact*

1. $T(0,0) = e^{a\sqrt{0}A + b\sqrt{0}B} = I.$
2. *For all* $s_1, s_2, t_1, t_2 \geq 0$ *and* $a, b \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned} T\left((s_1+s_2)^2, (t_1+t_2)^2\right) &= e^{a\sqrt{(s_1+s_2)^2}A + b\sqrt{(t_1+t_2)^2}B} \\ &= e^{as_1A + bt_1B + as_2A + bt_2B} = e^{as_1A + bt_1B} e^{as_2A + bt_2B} \\ &= T\left((s_1)^2, (t_1)^2\right) T\left((s_2)^2, (t_2)^2\right). \end{aligned}$$

Remark 4.4.
1. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup. For any $s, t \geq 0$ we set*

$$S(s,t) = T_\alpha\left(s^{\frac{1}{\alpha}}, t^{\frac{1}{\alpha}}\right),$$

then $(S(s,t))_{s,t\geq 0}$ is a two-parameter semigroup.

2. *Let $(T(s,t))_{s,t\geq 0}$ be a two-parameter semigroup. For any $s, t \geq 0$ we set*

$$T_\alpha(s,t) = T(s^\alpha, t^\alpha)$$

then $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter α-semigroup.

3. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup. Then $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ are one-parameter α-semigroups.*

Definition 4.5. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup on a Banach space X, then*

1. *We say that $(T_\alpha(s,t))_{s,t\geq 0}$ is uniformly continuous if we have*

$$\lim_{(s,t)\to(0^+,0^+)} \|T_\alpha(s,t) - I\| = 0.$$

2. *We say that $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup if for all $x \in X$ we have*

$$\lim_{(s,t)\to(0^+,0^+)} \|T_\alpha(s,t)x - x\| = 0$$

Proposition 4.6.
1. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup. Then $(T_\alpha(s,t))_{s,t\geq 0}$ is a C_0-α-semigroup if and only if $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{s,t\geq 0}$ are one-parameter C_0-α-semigroups.*
2. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup. For any $s, t \geq 0$ we set*

$$S(s,t) = T_\alpha\left(s^{\frac{1}{\alpha}}, t^{\frac{1}{\alpha}}\right),$$

then $(S(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup.

3. *Let $(T(s,t))_{s,t\geq 0}$ be a two-parameter C_0-semigroup. For any $s,t \geq 0$ we set*

$$T_\alpha(s,t) = T(s^\alpha, t^\alpha)$$

then $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup.

Proof. 1. If $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup, then in particular if $s = 0$ and $t = 0$ we get that $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ are one-parameter C_0-α-semigroups.
For the converse, we observe that for any $s,t \geq 0$

$$\begin{aligned} T_\alpha(s,t) &= T_\alpha\left((s^\alpha+0)^{\frac{1}{\alpha}}, (0+t^\alpha)^{\frac{1}{\alpha}}\right) \\ &= T_\alpha\left((s^\alpha)^{\frac{1}{\alpha}}, 0^{\frac{1}{\alpha}}\right) T_\alpha\left(0^{\frac{1}{\alpha}}, (t^\alpha)^{\frac{1}{\alpha}}\right) = T_\alpha\left(0^{\frac{1}{\alpha}}, (t^\alpha)^{\frac{1}{\alpha}}\right) T_\alpha\left((s^\alpha)^{\frac{1}{\alpha}}, 0^{\frac{1}{\alpha}}\right) \\ &= T_\alpha(s,0)\, T_\alpha(0,t) = T_\alpha(0,t)\, T_\alpha(s,0). \end{aligned}$$

If we suppose that $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ are one-parameter C_0-α-semigroups, then for any $s,t \geq 0$ and $x \in X$, we have

$$\begin{aligned} \|T_\alpha(s,t)x - x\| &= \|T_\alpha(s,0)\, T_\alpha(0,t)x - T_\alpha(s,0)x + T_\alpha(s,0)x - x\| \\ &= \|T_\alpha(s,0)(T_\alpha(0,t)x - x) + T_\alpha(s,0)x - x\| \\ &\leq \|T_\alpha(s,0)\|\,\|T_\alpha(0,t)x - x\| + \|T_\alpha(s,0)x - x\|. \end{aligned}$$

We apply the uniform boundedness principle, and we get that there exist $a > 0$ and $M > 0$ such that $\|T_\alpha(s,0)\| \leq M$ for all $s \in]0,a[$. Then for any $t \geq 0$, $s \in]0,a[$ and $x \in X$, we have

$$\|T_\alpha(s,t)x - x\| \leq M\,\|T_\alpha(0,t)x - x\| + \|T_\alpha(s,0)x - x\|$$

which tends towards zero when $(s,t) \to (0^+,0^+)$, and this shows that $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup.

2. It is clear that $(S(s,t))_{s,t\geq 0}$ is a two-parameter semigroup. Let $\alpha \in]0,a]$ for any $a > 0$ and let $(h,k) = \left(s^{\frac{1}{\alpha}}, t^{\frac{1}{\alpha}}\right)$ with $s,t > 0$, we have $(h,k) \to (0^+,0^+)$ as $(s,t) \to (0^+,0^+)$. Let $x \in X$, from the preceding disscussion we obtain

$$\begin{aligned} \lim_{(s,t)\to(0^+,0^+)} S(s,t)x &= \lim_{(s,t)\to(0^+,0^+)} T_\alpha\left(s^{\frac{1}{\alpha}}, t^{\frac{1}{\alpha}}\right)x \\ &= \lim_{(h,k)\to(0^+,0^+)} T_\alpha(h,k)x = x \end{aligned}$$

3. Similar to 2.
□

Proposition 4.7. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup on a Banach space X. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T_\alpha(s,t)\| \leq Me^{\omega(s^\alpha+t^\alpha)}.$$

Proof. Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup on a Banach space X, then by the previous result $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ are one-parameter C_0-α-semigroups, so there exist constants $\omega_1, \omega_2 \geq 0$ and $M_1, M_2 \geq 1$ such that $\|T_\alpha(s,0)\| \leq M_1 e^{\omega_1 s^\alpha}$ and $\|T_\alpha(0,t)\| \leq Me^{\omega_2 t^\alpha}$. Let $\omega = \max(\omega_1,\omega_2)$ and $M = M_1 M_2$. Thus

$$\begin{aligned} \|T_\alpha(s,t)\| &= \|T_\alpha(s,0)\, T_\alpha(0,t)\| \\ &\leq Me^{\omega(s^\alpha+t^\alpha)}. \end{aligned}$$

□

Definition 4.8. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup on a Banach space X. We say that $(T_\alpha(s,t))_{s,t\geq 0}$ is strongly continuous if*

$$\lim_{(s,t)\to(s_0,t_0)} \|T_\alpha(s,t)x - T_\alpha(s_0,t_0)x\| = 0$$

for all $x \in X$ and $s_0, t_0 > 0$ with $(s,t) \to (0^+, 0^+)$ if $(s_0,t_0) = (0,0)$.

Corollary 4.9. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup on a Banach space X, then $(T_\alpha(s,t))_{s,t\geq 0}$ is strongly continuous if and only if $(T_\alpha(s,t))_{s,t\geq 0}$ is a C_0-α-semigroup.*

Proof. If $(T_\alpha(s,t))_{s,t\geq 0}$ is strongly continuous, then it is clear that $(T_\alpha(s,t))_{s,t\geq 0}$ is a C_0-α-semigroup.
Conversely, let $s_0, t_0 \geq 0$, we have to show that

$$\forall x \in X, \lim_{(s,t)\to(s_0,t_0)} \|T_\alpha(s,t)x - T_\alpha(t_0,s_0)x\| = 0.$$

Let $(s,t) \in [0,s_0[\times [0,t_0[$ and $x \in X$,

$$\begin{aligned}\|T_\alpha(s,t)x - T_\alpha(t_0,s_0)x\| &\leq \|T_\alpha(s,t)\| \|x - T_\alpha(t_0-t, s_0-s)x\| \\ &\leq Me^{\omega(s^\alpha+t^\alpha)} \|x - T_\alpha(t_0-t, s_0-s)x\|,\end{aligned}$$

but

$$\lim_{(s,t)\to(s_0,t_0)} e^{\omega(s^\alpha+t^\alpha)} \|x - T_\alpha(t_0-t, s_0-s)x\| = 0.$$

Hence $\lim_{(s,t)\to(s_0,t_0)} \|T_\alpha(s,t)x - T_\alpha(t_0,s_0)x\| = 0.$
Now, let $(s,t) \in]s_0, s_0+1] \times]t_0, t_0+1]$ and $x \in X$,

$$\begin{aligned}\|T_\alpha(s,t)x - T_\alpha(t_0,s_0)x\| &\leq \|T_\alpha(s_0,t_0)\| \|T_\alpha(t-t_0, s-s_0)x - x\| \\ &\leq Me^{\omega\left((s_0)^\alpha+(t_0)^\alpha\right)} \|T_\alpha(t-t_0, s-s_0)x - x\|,\end{aligned}$$

and

$$\lim_{(s,t)\to(s_0,t_0)} \|T_\alpha(t-t_0, s-s_0)x - x\| = 0.$$

Hence $\lim_{(s,t)\to(s_0,t_0)} \|T_\alpha(s,t)x - T_\alpha(t_0,s_0)x\| = 0.$ □

5. The α-infinitesimal generator of a two-parameter C_0-α-semigroup

Definition 5.1. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter α-semigroup on the Banach space X. The α-infinitesimal generator of $(T_\alpha(s,t))_{s,t\geq 0}$ is defined on*

$$D(A) = \{x \in X : T_\alpha(.,.)x \text{ is } \alpha\text{-differentiable at } (0,0)\}$$

by setting for all $x \in D(A)$

$$Ax = D^\alpha(T_\alpha(0,0)x).$$

Lemma 5.2. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup on the Banach space X and let $x \in D(A)$, then we have*

$$Ax = \left(\lim_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} T_\alpha(s,0)x, \lim_{t\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} T_\alpha(0,t)x\right).$$

Proof. Let $x \in D(A)$, then $T_\alpha(.,.)\,x$ is α-differentiable at $(0,0)$

1. $D^\alpha(T_\alpha(s,t)\,x)$ exists in some $]0,a[\times]0,b[$, $a,b>0$ and

$$D^\alpha(T_\alpha(0,0)\,x) = \lim_{(s,t)\to(0^+,0^+)} D^\alpha(T_\alpha(s,t)\,x)$$

exists.

2. $\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\,x)$ and $\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)\,x)$ exist in $]0,a[$ and $]0,b[$ respectively.

Let $(s,t)\in]0,a[\times]0,b[$, then $\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\,x)$ and $\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)\,x)$ exist, and we have

$$\begin{aligned} D^\alpha(T_\alpha(0,0)\,x) &= \lim_{(s,t)\to(0^+,0^+)} D^\alpha(T_\alpha(s,t)\,x) \\ &= \lim_{(s,t)\to(0^+,0^+)} \left(\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\,x), \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)\,x)\right) = (l_1,l_2)\in X\times X \end{aligned}$$

with

$$\lim_{(s,t)\to(0^+,0^+)} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\,x) = l_1 \text{ and } \lim_{(s,t)\to(0^+,0^+)} \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)\,x) = l_2.$$

We have to show that

$$\lim_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\,x) = l_1 \text{ and } \lim_{t\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)\,x) = l_2.$$

First, we remark that for any $(s,t)\in]0,a[\times]0,b[$,

$$\begin{aligned} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\,x) &= \lim_{\varepsilon\to 0} \frac{T_\alpha\left(s+\varepsilon s^{1-\alpha},t\right)x - T_\alpha(s,t)\,x}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{T_\alpha(0,t)\left(T_\alpha\left(s+\varepsilon s^{1-\alpha},0\right)x - T_\alpha(s,0)\,x\right)}{\varepsilon} \\ &= T_\alpha(0,t)\left[\lim_{\varepsilon\to 0} \frac{T_\alpha\left(s+\varepsilon s^{1-\alpha},0\right)x - T_\alpha(s,0)\,x}{\varepsilon}\right] \\ &= T_\alpha(0,t)\left[\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\,x)\right]. \end{aligned}$$

Similarly

$$\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)\,x) = T_\alpha(s,0)\left[\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)\,x)\right].$$

We have $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup, then Proposition 4.6 gives that $(T_\alpha(0,t))_{t\geq 0}$ is a one-parameter C_0-α-semigroup. Thus for any $s\in]0,a[$

$$\begin{aligned} \lim_{t\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\,x) &= \lim_{t\to 0^+} T_\alpha(0,t)\left[\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\,x)\right] \\ &= \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\,x). \end{aligned}$$

Similarly, we obtain for all $t\in]0,b[$

$$\lim_{s\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)\,x) = \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)\,x).$$

Let $\varepsilon > 0$, there exist $0 < \delta_1 \leq a$ and $0 < \delta_2 \leq b$ such that if $(s,t) \in]0,\delta_1[\times]0,\delta_2[$ then

$$\left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) - l_1 \right\| \leq \frac{\varepsilon}{2}$$

and there exist $0 < \eta_1 \leq b$ such that if $t \in]0,\eta_1[$ we have for all $s \in]0,a[$

$$\left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) - \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (s,0) x \right) \right\| \leq \frac{\varepsilon}{2}.$$

Let $\gamma_1 = \inf (\delta_1, \eta_1)$ and let $t \in]0,\gamma_1[$, then we have for all $s \in]0,\delta_1[$

$$\begin{aligned} \left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right) - l_1 \right\| &\leq \left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right) - \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) \right\| + \left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) - l_1 \right\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

then

$$\lim_{s \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right) = l_1.$$

Hence

$$\begin{aligned} \lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) &= \lim_{s \to 0^+} \lim_{t \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,t) x \right) \\ &= \lim_{s \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (s,t) x \right) &= \lim_{t \to 0^+} \lim_{s \to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (s,t) x \right) \\ &= \lim_{t \to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (0,t) x \right). \end{aligned}$$

□

Let $(T_\alpha (s,t))_{s,t \geq 0}$ be a two-parameter C_0-α-semigroup, then $(T_\alpha (s,0))_{s \geq 0}$ and $(T_\alpha (0,t))_{t \geq 0}$ are one-parameter C_0-α-semigroups.

Let A_1 and A_2 the linear operators defined by

$$D(A_1) = \left\{ x \in X : \lim_{s \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right) \text{ exists} \right\},$$

$$D(A_2) = \left\{ x \in X : \lim_{t \to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (0,t) x \right) \text{ exists} \right\}$$

and

$$A_1 x = \lim_{s \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha (s,0) x \right) \text{ for all } x \in D(A_1),$$

$$A_2 x = \lim_{t \to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha (0,t) x \right) \text{ for all } x \in D(A_2).$$

It is clear that A_1 and A_2 are the α-infinitesimal generators of the one-parameter C_0-α-semigroups $(T_\alpha (s,0))_{s \geq 0}$ and $(T_\alpha (0,t))_{t \geq 0}$ respectively.

Theorem 5.3. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup and let A be its α-infinitesimal generator, then we have*

$$D(A) = D(A_1) \cap D(A_2),$$

and we can consider the α-infinitesimal generator as a linear operator $A : D(A) \subset X \to X \times X$ defined by

$$\forall x \in D(A),\, Ax = (A_1 x, A_2 x).$$

Proof. Let $x \in D(A)$, then by the previous lemma $\lim\limits_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)x)$ and $\lim\limits_{t\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x)$ exist and

$$\begin{aligned} Ax &= \left(\lim_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} T_\alpha(s,0)x, \lim_{t\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha} T_\alpha(0,t)x\right) \\ &= (A_1 x, A_2 x). \end{aligned}$$

Therefore, $D(A) \subset D(A_1) \cap D(A_2)$ and $Ax = (A_1x, A_2x)$ for all $x \in D(A)$.

Now, let $x \in D(A_1) \cap D(A_2)$ then $\lim\limits_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)x)$ and $\lim\limits_{t\to 0^+} \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x)$ exist, then $\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)x)$ and $\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x)$ exist on an open of the form $]0,a[$, $a > 0$ and $]0,b[$, $b > 0$ respectively, but we have for any $(s,t) \in]0,a[\times]0,b[$,

$$\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)x) = T_\alpha(0,t)\left[\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)x)\right]$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)x) = T_\alpha(s,0)\left[\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x)\right],$$

then $\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)x)$ exists for all $s \in]0,a[$ and $t \geq 0$ and $\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)x)$ exists for all $t \in]0,a[$ and $s \geq 0$.

Let $(s,t) \in]0,a[\times]0,b[$ and $h,k > 0$, we set

$$J(h,k) = T_\alpha\left(s + hs^{1-\alpha}, t + kt^{1-\alpha}\right)x - T_\alpha(s,t)x - \left(\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)x), \frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(s,t)x)\right)\begin{pmatrix} h \\ k \end{pmatrix}.$$

We have

$$\begin{aligned} J(h,k) = T_\alpha(0,t)&\left(T_\alpha\left(s + hs^{1-\alpha}, 0\right)x - T_\alpha(s,0)x - h\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)x)\right) \\ &+ T_\alpha\left(s + hs^{1-\alpha}, 0\right)\left(T_\alpha\left(0, t + kt^{1-\alpha}\right)x - T_\alpha(0,t)x - k\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x)\right) \\ &+ k\left(T_\alpha\left(s + hs^{1-\alpha}, 0\right) - T_\alpha(s,0)\right)\frac{\partial^\alpha}{\partial t^\alpha}(T_\alpha(0,t)x). \end{aligned}$$

Since $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup. Then Proposition 4.6 gives that $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ are one-parameter C_0-α-semigroups, then there exist constants $\omega_1, \omega_2 \geq 0$ and $M_1, M_2 \geq 1$ such that for any $s,t \geq 0$ we have $\|T_\alpha(s,0)\| \leq M_1 e^{\omega_1 s^\alpha}$ and $\|T_\alpha(0,t)\| \leq M_2 e^{\omega_2 t^\alpha}$.

Given the fact that we have $\frac{h}{\|(h,k)\|} \leq 1$ and $\frac{k}{\|(h,k)\|} \leq 1$, we obtain the following inequalities:

$$\begin{aligned}
\frac{\|J(h,k)\|}{\|(h,k)\|} \leq{}& \frac{h}{\|(h,k)\|}\|T_\alpha(0,t)\|\left\|\frac{T_\alpha\left(s+hs^{1-\alpha},0\right)x-T_\alpha(s,0)x}{h}-\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,0)x\right)\right\| \\
&+\frac{k}{\|(h,k)\|}\left\|T_\alpha\left(s+hs^{1-\alpha},0\right)\right\|\left\|\frac{T_\alpha\left(0,t+kt^{1-\alpha}\right)x-T_\alpha(0,t)x}{k}-\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\| \\
&+\frac{k}{\|(h,k)\|}\left\|\left(T_\alpha\left(s+hs^{1-\alpha},0\right)-T_\alpha(s,0)\right)\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\| \\
\leq{}& M_1e^{\omega_1 t^\alpha}\left\|\frac{T_\alpha\left(s+hs^{1-\alpha},0\right)x-T_\alpha(s,0)x}{h}-\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,0)x\right)\right\| \\
&+M_2e^{\omega_2\left(s+hs^{1-\alpha}\right)^\alpha}\left\|\frac{T_\alpha\left(0,t+kt^{1-\alpha}\right)x-T_\alpha(0,t)x}{k}-\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\| \\
&+\left\|\left(T_\alpha\left(s+hs^{1-\alpha},0\right)-T_\alpha(s,0)\right)\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\|.
\end{aligned}$$

We have for all $(s,t) \in]0,a[\times]0,b[$

$$\lim_{h\to 0}\left\|\frac{T_\alpha\left(s+hs^{1-\alpha},0\right)x-T_\alpha(s,0)x}{h}-\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,0)x\right)\right\|=0$$

and

$$\lim_{k\to 0}\left\|\frac{T_\alpha\left(0,t+kt^{1-\alpha}\right)x-T_\alpha(0,t)x}{k}-\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\|=0.$$

If we put $\varepsilon = hs^{1-\alpha}$ then $\varepsilon \to 0$ as $h \to 0$, and we have $(T_\alpha(s,0))_{s\geq 0}$ is strongly continuous, so we get

$$\begin{aligned}
&\lim_{h\to 0}\left\|\left(T_\alpha\left(s+hs^{1-\alpha},0\right)-T_\alpha(s,0)\right)\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\| \\
&\quad=\lim_{\varepsilon\to 0}\left\|\left(T_\alpha(s+\varepsilon,0)-T_\alpha(s,0)\right)\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(0,t)x\right)\right\|=0.
\end{aligned}$$

Finally

$$\lim_{(h,k)\to(0,0)}\frac{\|J(h,k)\|}{\|(h,k)\|}=0,$$

which means that $D^\alpha(T_\alpha(s,t)x)$ exists for all $x \in D(A_1)\cap D(A_2)$ and $(s,t) \in]0,a[\times]0,a[$ and

$$D^\alpha\left(T_\alpha(s,t)x\right)=\left(\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,t)x\right),\frac{\partial^\alpha}{\partial t^\alpha}\left(T_\alpha(s,t)x\right)\right).$$

We have for any $x \in D(A_1)\cap D(A_2)$

$$\lim_{(s,t)\to(0^+,0^+)}\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,t)x\right)=A_1x$$

Indeed, we have for all $s \in]0,a[$

$$\begin{aligned}
\lim_{t\to 0^+}\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,t)x\right)&=\lim_{t\to 0^+}T_\alpha(0,t)\left[\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,0)x\right)\right] \\
&=\frac{\partial^\alpha}{\partial s^\alpha}\left(T_\alpha(s,0)x\right)\ \text{exist,}
\end{aligned}$$

and we have for any $x \in D(A_1) \cap D(A_2)$

$$\lim_{s \to 0^+} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,0)\,x\right) = A_1 x.$$

Let $\varepsilon > 0$, there exists $0 < \delta_1 \leq a$ such that if $s \in]0, \delta_1[$ then

$$\left\| \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,0)\,x\right) - A_1 x \right\| \leq \frac{\varepsilon}{2},$$

and there exists $0 < \delta_2 \leq b$ such that if $t \in]0, \delta_2[$ we have for all $s \in]0, a[$

$$\left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right) - \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,0)\,x\right) \right\| \leq \frac{\varepsilon}{2}$$

Let $(s,t) \in]0, \delta_1[\times]0, \delta_2[$ then

$$\begin{aligned} \left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right) - A_1 x \right\| &\leq \left\| \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right) - \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,0)\,x\right) \right\| + \left\| \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,0)\,x\right) - A_1 x \right\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq \varepsilon. \end{aligned}$$

Thus,

$$\lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right) = A_1 x.$$

Similarly, we show that for any $x \in D(A_1) \cap D(A_2)$

$$\lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,t)\,x\right) = A_2 x.$$

Hence, we have

$$\begin{aligned} \lim_{(s,t) \to (0^+,0^+)} D^\alpha \left(T_\alpha(s,t)\,x\right) &= \lim_{(s,t) \to (0^+,0^+)} \left(\frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right), \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,t)\,x\right) \right) \\ &= \left(\lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,t)\,x\right), \lim_{(s,t) \to (0^+,0^+)} \frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(s,t)\,x\right) \right) \\ &= \left(A_1 x, A_2 x\right). \end{aligned}$$

Finally, we have shown that

1. $D^\alpha \left(T_\alpha(s,t)\,x\right)$ exists in some $]0, a[\times]0, b[$, $a, b > 0$ and

$$D^\alpha \left(T_\alpha(0,0)\,x\right) = \lim_{(s,t) \to (0^+,0^+)} D^\alpha \left(T_\alpha(s,t)\,x\right)$$

   exists.
2. $\frac{\partial^\alpha}{\partial s^\alpha} \left(T_\alpha(s,0)\,x\right)$ and $\frac{\partial^\alpha}{\partial t^\alpha} \left(T_\alpha(0,t)\,x\right)$ exist in $]0, a[$ and $]0, b[$ respectively.

Therefore, $T_\alpha(s,t)\,x$ is α-differentiable at $(0,0)$, then $x \in D(A)$.
Hence $D(A_1) \cap D(A_2) \subset D(A)$ and $\forall x \in D(A_1) \cap D(A_2)$, $Ax = (A_1 x, A_2 x)$. □

Theorem 5.4. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup, then we can consider the α-infinitesimal generator of $(T_\alpha(s,t))_{s,t\geq 0}$ as a linear transformation*
$A : \mathbb{R}^{+^2} \to L(D(A_1) \cap D(A_2), X)$ defined by

$$A(h,k) = hA_1 + kA_2,$$

where A_1 and A_2 are the α-infinitesimal generators of the one-parameter C_0-α-semigroups $(T_\alpha(s,0))_{s\geq 0}$ and $(T_\alpha(0,t))_{t\geq 0}$ respectively.

Proof. Let $x \in D(A) = D(A_1) \cap D(A_2)$ then $D^\alpha(T_\alpha(0,0)x)$ exists as a linear transformation $L(.,.) : \mathbb{R}^{+^2} \to X$ defined by

$$L(h,k) = D^\alpha(T_\alpha(0,0)x)\begin{pmatrix} h \\ k \end{pmatrix} = hA_1x + kA_2x.$$

Let $\hat{L}(.,.) : \mathbb{R}^{+^2} \to L(D(A_1) \cap D(A_2), X)$ defined by

$$\hat{L}(h,k) = hA_1 + kA_2,$$

then $\hat{L}(.,.)$ is a linear transformation and we have for any $(h,k) \in \mathbb{R}^{+^2}$ and $x \in D(A_1) \cap D(A_2)$

$$L(h,k) = \hat{L}(h,k)x,$$

then we have for all $x \in D(A_1) \cap D(A_2)$

$$L(.,.) = \hat{L}(.,.)x$$

therefore, for any $x \in D(A_1) \cap D(A_2)$

$$\begin{aligned} Ax &= D^\alpha(T_\alpha(0,0)x) \\ &= L(.,.) \\ &= \hat{L}(.,.)x. \end{aligned}$$

Thus,

$$A = \hat{L}(.,.).$$

Therefore, we can consider the α-infinitesimal generator A as a linear transformation as follows

$$A : \mathbb{R}^{+^2} \to L(D(A_1) \cap D(A_2), X)$$

defined by

$$A(h,k) = hA_1 + kA_2.$$

□

Remark 5.5. *If $(T_\alpha(s,t))_{s,t\geq 0}$ is a two-parameter C_0-α-semigroup and A is its α-infinitesimal generator, then in the preceding results, we have seen two approaches to define A.*

The first approach is to consider A as a linear operator $A : D(A) \subset X \to X \times X$ defined by

$$Ax = (A_1x, A_2x) \text{ for all } x \in D(A).$$

The second approach is to consider A as a linear transformation $A : \mathbb{R}^{+^2} \to L(D(A_1) \cap D(A_2), X)$ defined by

$$A(h,k) = hA_1 + kA_2.$$

Next, we will denote the α-infinitesimal generator of $(T_\alpha(s,t))_{s,t\geq 0}$ by (A_1, A_2), and this notation is adopted for the two approaches of the definition, and we will write

1. *For the first approach :*

$$(A_1, A_2)\, x = (A_1 x, A_2 x) \text{ for all } x \in D(A_1) \cap D(A_2).$$

2. *For the second approach :*

$$\left((A_1, A_2)\begin{pmatrix} h \\ k \end{pmatrix}\right) x = hA_1 x + kA_2 x \text{ for all } (h,k) \in \mathbb{R}^{+^2} \text{ and all } x \in D(A_1) \cap D(A_2).$$

Theorem 5.6. *Let $(T_\alpha(s,t))_{s,t\geq 0}$ be a two-parameter C_0-α-semigroup, and let A be its α-infinitesimal generator then for all $x \in D(A)$*

1. *For any $t \geq 0$ we have $T_\alpha(0,t)\, x \in D(A_1)$ and $A_1 T_\alpha(0,t)\, x = T_\alpha(0,t)\, A_1 x$.*
2. *For any $s \geq 0$ we have $T_\alpha(s,0)\, x \in D(A_2)$ and $A_2 T_\alpha(s,0)\, x = T_\alpha(s,0)\, A_2 x$.*
3. *For all $(s,t) \in \mathbb{R}^{+^2}$, $T_\alpha(s,t)\, x \in D(A)$ and we have*

$$\frac{\partial^\alpha}{\partial s^\alpha}(T(s,t)\, x) = A_1 T_\alpha(s,t)\, x = T_\alpha(s,t)\, A_1 x$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha}(T(s,t)\, x) = A_2 T_\alpha(s,t)\, x = T_\alpha(s,t)\, A_2 x.$$

4. *For all $(s,t) \in \mathbb{R}^{+^2}$, $T_\alpha(s,t)\, x \in D(A)$ and*

$$D^\alpha(T_\alpha(s,t)\, x)\begin{pmatrix} h \\ k \end{pmatrix} = \left((A_1, A_2)\begin{pmatrix} h \\ k \end{pmatrix}\right) T_\alpha(s,t)\, x = T_\alpha(s,t)\left((A_1, A_2)\begin{pmatrix} h \\ k \end{pmatrix}\right) x.$$

for all $(h,k) \in \mathbb{R}^2$.

Proof. 1. Let $x \in D(A) \subseteq D(A_1)$, then $\lim\limits_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, x)$ exists, so $D^\alpha(T_\alpha(s,0)\, x)$ exists in an open of the form $]0,a[$, $a > 0$. Let $s \in]0,a[$ and $t \geq 0$, we have

$$\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, T_\alpha(0,t)\, x) = T_\alpha(0,t)\, \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, x).$$

Therefore,

$$\begin{aligned} \lim_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, T_\alpha(0,t)\, x) &= T_\alpha(0,t)\left(\lim_{s\to 0^+} \frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, x)\right) \\ &= T_\alpha(0,t)\, A_1 x, \end{aligned}$$

then for any $t \geq 0$ we have $T_\alpha(0,t)\, x \in D(A_1)$ and $A_1 T_\alpha(0,t)\, x = T_\alpha(0,t)\, A_1 x$.

2. The same method as 1.

3. Let $(s,t) \in \mathbb{R}^{+^2}$ and $x \in D(A)$, we have from 1. that $T_\alpha(0,t)\, x \in D(A_1)$, then from Theorem 2.7 we have

$$\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,0)\, T_\alpha(0,t)\, x) = A_1 T_\alpha(s,0)\, T_\alpha(0,t)\, x = T_\alpha(s,0)\, A_1 T_\alpha(0,t)\, x$$

and from 1. we have

$$T_\alpha(s,0)\, A_1 T_\alpha(0,t)\, x = T_\alpha(s,0)\, T_\alpha(0,t)\, A_1 x.$$

Finally, we get for any $(s,t) \in \mathbb{R}^{+^2}$ and $x \in D(A)$

$$\frac{\partial^\alpha}{\partial s^\alpha}(T_\alpha(s,t)\, x) = A_1 T_\alpha(s,t)\, x = T_\alpha(s,t)\, A_1 x.$$

With the same method, we show that

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(T_{\alpha}(s,t)\,x\right) = A_2 T_{\alpha}(s,t)\,x = T_{\alpha}(s,t)\,A_2 x.$$

4. Let $(h,k) \in \mathbb{R}^2$, $(s,t) \in \mathbb{R}^{+^2}$ and $x \in D(A)$
From 3. we have

$$\begin{aligned}
D^{\alpha}\left(T_{\alpha}(s,t)\,x\right)\begin{pmatrix} h \\ k \end{pmatrix} &= \left(\frac{\partial^{\alpha}}{\partial s^{\alpha}}\left(T_{\alpha}(s,t)\,x\right), \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(T_{\alpha}(s,t)\,x\right)\right)\begin{pmatrix} h \\ k \end{pmatrix} \\
&= h\frac{\partial^{\alpha}}{\partial s^{\alpha}}\left(T_{\alpha}(s,t)\,x\right) + k\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(T_{\alpha}(s,t)\,x\right) \\
&= hA_1T_{\alpha}(s,t)\,x + kA_2T_{\alpha}(s,t)\,x \\
&= T_{\alpha}(s,t)\,hA_1x + T_{\alpha}(s,t)\,kA_2x \\
&= (hA_1 + kA_2)\,T_{\alpha}(s,t)\,x \\
&= T_{\alpha}(s,t)\,(hA_1 + kA_2)\,x \\
&= \left((A_1, A_2)\begin{pmatrix} h \\ k \end{pmatrix}\right)T_{\alpha}(s,t)\,x \\
&= T_{\alpha}(s,t)\left((A_1, A_2)\begin{pmatrix} h \\ k \end{pmatrix}\right)x.
\end{aligned}$$

□

6. Two-parameter α-Abstract Cauchy Problem

Let $A_i : D(A_i) \subseteq X \to X$, $i = 1,2$, be a linear operator. We consider the following two-parameter α-Cauchy Problem

$$\text{2-}\alpha\text{-ACP}\begin{cases} \frac{\partial^{\alpha}}{\partial t_i^{\alpha}}u(t_1,t_2) = A_iu(t_1,t_2), \ t_i > 0, i = 1,2, \\ \\ u(0,0) = x, \qquad x \in D(A_1) \cap D(A_2). \end{cases}$$

We mean by a solution a function $u : [0,+\infty[\times [0,+\infty[\to X$ which satisfies the following :

1. $u(.,.)$ is continuous on $[0,+\infty[\times [0,+\infty[$.
2. u has continuous partial α-derivative.
3. $\forall s,t \geq 0$, $u(s,t) \in D(A_i)$ for $i = 1,2$.
4. u satisfies the 2-α-ACP.

Theorem 6.1. *Suppose that (A_1, A_2) is the α-infinitesimal generator of a two-parameter C_0-α-semigroup $(T_{\alpha}(s,t))_{s,t\geq 0}$. Then the 2-$\alpha$-ACP has the unique solution $u(s,t;x) = T_{\alpha}(s,t)\,x$ for all $x \in D(A_1) \cap D(A_2)$.*

Proof. It is clear from Theorem 5.6 that $u(s,t;x) = T_{\alpha}(s,t)\,x$ is a solution of 2-α-ACP. It remains to show that the 2-α-ACP has a unique solution, for that it is enough to show that the 2-α-ACP has a solution $u(s,t) = 0$, for the initial value $x = 0$.

Based on the case of one parameter [2], we know that the systems

$$\text{1-}\alpha\text{-ACP}\begin{cases} \frac{\partial^{\alpha}}{\partial t^{\alpha}}v(t) = A_1v(t), \ t > 0, \\ v(0) = 0, \end{cases}$$

and

$$\text{1-}\alpha\text{-ACP}\begin{cases} \frac{\partial^{\alpha}}{\partial t^{\alpha}}w(t) = A_2w(t), \ t > 0, \\ w(0) = 0, \end{cases}$$

have the unique solution $v = 0$ and $w = 0$.

Now, suppose that $u(s,t;0)$ is a solution of the 2-α-ACP for the initial value $x = 0$, then its clear that

$$v_1(s) = T_\alpha(s,0)\,u(0,t;0) \text{ and } v_2(s) = u(s,t;0)$$

are two solutions of the 1-α-ACP

$$\text{1-}\alpha\text{-ACP}\begin{cases} \frac{\partial^\alpha}{\partial s^\alpha} v(s) = A_1 v(s), & s > 0, \\ v(0) = u(0,t;0). \end{cases}$$

Indeed, we have

$$\begin{aligned} \frac{\partial^\alpha}{\partial s^\alpha} v_1(s) &= \frac{\partial^\alpha}{\partial s^\alpha} T_\alpha(s,0)\,u(0,t;0) \\ &= A_1 T_\alpha(s,0)\,u(0,t;0) \\ &= A_1 v_1(s) \end{aligned}$$

and

$$\begin{aligned} v_1(0) &= T_\alpha(0,0)\,u(0,t;0) \\ &= u(0,t;0). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial^\alpha}{\partial s^\alpha} v_2(s) &= \frac{\partial^\alpha}{\partial s^\alpha} u(s,t;0) \\ &= A_1 u(s,t;0) \end{aligned}$$

and

$$v_2(0) = u(0,t;0).$$

By uniqueness of the solution, we get for any $s,t \geq 0$

$$u(s,t;0) = T_\alpha(s,0)\,u(0,t;0).$$

With the same method, we show that

$$w_1(t) = T_\alpha(0,t)\,u(s,0;0) \text{ and } v_2(t) = u(s,t;0)$$

are two solutions of the 1-α-ACP

$$\text{1-}\alpha\text{-ACP}\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} w(t) = A_2 w(t), & t > 0, \\ v(0) = u(s,0;0). \end{cases}$$

Through the uniqueness of the solution, we obtain for all $s,t \geq 0$

$$u(s,t;0) = T_\alpha(0,t)\,u(s,0;0).$$

Finally, we have

$$\begin{aligned} u(s,t;0) &= T_\alpha(s,0)\,u(0,t;0) \\ &= T_\alpha(s,0)\,(T_\alpha(0,t)\,u(0,0;0)) \\ &= T_\alpha(s,0)\,T_\alpha(0,t)\,(0) \\ &= 0. \end{aligned}$$

□

As an application of our discussion, we conclude with a simple example.

Example 6.2. *Let A and B be two bounded commuting operators and consider the following two-parameter α-Cauchy Problem*

$$\text{2-}\alpha\text{-}ACP^{*}\begin{cases} \frac{\partial^{\alpha}}{\partial s^{\alpha}}u(s,t)=Au(s,t)\,,\ \ s,t>0, \\ \frac{\partial^{\alpha}}{\partial t^{\alpha}}u(s,t)=Bu(s,t)\,,\ \ s,t>0, \\ u(0,0)=x,\, x\in D(A)\cap D(B)\,. \end{cases}$$

Then for all $s,t\geq 0$ *and* $x\in D(A)\cap D(B)$ *the* 2-α-ACP^{*} *has the unique solution* $u(s,t;x)=e^{\frac{s^{\alpha}}{\alpha}A+\frac{t^{\alpha}}{\alpha}B}x$.

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