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Finite-dimensional exact controllability of an abstract semilinear fractional composite relaxation equation

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Abstract. In Hilbert space, the finite-dimensional exact controllability of an abstract semilinear fractional composite relaxation equation is researched. We make assumptions about the parameters in the equation and suppose that the linear equation associated with the abstract semilinear fractional relaxation equation is approximately controllable. We apply the variational method, the resolvent theory and the fixed point trick to demonstrate the finite-dimensional exact controllability of the abstract semilinear equation. An application is given in the last paper to testify our results.

1. Introduction

The idea of controllability has a profound impact on control system. In 1963, Kalman first proposed the concept of controllability. Controllability is widely used in computing, science, biomedical, economy and so on. So far, many scholars have considered various type of linear and nonlinear dynamics controllability problems. We can refer to [1, 3, 11, 20, 25] and references therein. The study of controllability of differential systems includes exact controllability, approximate controllability, zero controllability and so on. It is the representation of exact controllability that choose an appropriate control function to control the final status of the system to any given status. Approximate controllability indicates that the final status of the system can be limited to the neighborhood of a given status, and zero controllability means that the system can reach zero at the terminal point under the limit of the control function. However, there are various errors in real life, which make it difficult for the system to achieve any given status accurately. Owing to the high requirements of exact controllability, the systems can not achieve it in the infinite dimensional space. The results of reference [10] show that the differential system is exact controllability in the finite dimensional space if the control operator is bounded linear and the operator semigroup is compact. For more information, we can refer to [9, 19, 20, 25].

In 1997, R. Triggiani examined a shortcoming of exact controllability in Banach spaces about the weak solutions, while approximate controllability is more suitable in application (see e.g. [25]). Therefore, the application of approximate controllability in nonlinear systems is particularly important. Many researches

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have discussed the different kinds of nonlinear systems which have the approximate controllability under different conditions. We can refer to the references [8, 12, 13, 24, 26]. In [21, 27], the authors gave an inequality about the range of the control operator B and obtained the system that is the approximately controllable. In [7], Fan, Dong and Li applied two ways which are the analytic resolvent theory and the fixed point trick to conclude that the abstract semilinear fractional system has the approximate controllability. In [5], the authors adopted a new Volterra integral equation and defined a control function with Gramian controllable operator. They proved that the parabolic non-autonomous evolution system is approximately controllable under nonlocal conditions.

It is worth noting that we can choose an approximate control function to make the final status of the system satisfies the approximate controllability condition as well as the quantity constraint condition, that is, the finite-dimensional exact controllability. In 1997, Lions and Zuazua demonstrated that finite-dimensional exact controllability in linear heat equations is the result of approximate controllability, but it is not true in nonlinear problems (see e.g. [14]). In general, finite-dimensional exact controllability needs much more stronger requirements than approximate controllability. Recently, Mahmudov researched the nonlinear systems which achieve finite-dimensional exactly controllable and approximately controllable in infinite-dimensional space. We can refer to the references [15]-[19]. For instance, based on employing the linear part to realize that the system has approximate controllability under natural conditions, Mahmudov in [16] proved that the semilinear system is finite-dimensional exact controllability.

So far as we know, no one has considered whether the abstract semilinear fractional composite relaxation equation has the finite-dimensional exact controllability. Thus, in this article, we main investigate the finite-dimensional exact controllability (finite-approximate controllability) of the abstract semilinear fractional composite relaxation equation which is given below

$$\begin{cases} \varphi'(t) = G\,{}^{c}D_0^{\gamma}\varphi(t) - \varphi(t) + g(t, \varphi(t)) + Hv(t), \quad t \in [0, a], \\ \varphi(0) = \varphi_0, \end{cases} \tag{1}$$

where the state variable $\varphi(t) \in Z$, $v \in L^2([0, a]; V)$ is a control function. Z and V are Hilbert spaces. φ' is the form of the first derivative φ with respect to t. $D(\cdot)$ represents the domain of definition of the operator, and $-G : D(G) \subset Z \to Z$ denotes that the operator is linear and closed. The resolvent $\{T_{1-\gamma}(t),\ t \geq 0\}$ is generated by $-G$. ${}^{c}D_0^{\gamma}$ indicates the derivative operator of Caputo with order γ, $0 < \gamma < 1$. $H \in \mathcal{L}(V; Z)$. $\mathcal{L}(V; Z) : V \to Z$ represents the set of linear and bounded operators. $g : [0, a] \times Z \to Z$ is the suitable function.

Inspired by literatures [15]-[19], the present paper solves the following finite-dimensional exact controllability problem. Given $\varepsilon > 0$, $a > 0$, and $\varphi_0, \varphi_a \in Z$, find a control function $v_\varepsilon \in L^2([0, a]; V)$ which is intimately correlated with the solution $\varphi(t; v_\varepsilon)$. The solution $\varphi(t; v_\varepsilon)$ satisfies the following conditions

$$\| \varphi(a; v_\varepsilon) - \varphi_a \| < \varepsilon, \tag{2}$$

$$\pi_E \varphi(a; v_\varepsilon) = \pi_E \varphi_a, \tag{3}$$

where $E \subset Z$ is the finite-dimensional and $\pi_E : Z \to E$ is the orthogonal projection. This shows that the appropriate control function v can be selected by the features of the system, so that the corresponding solution $\varphi(a; v)$ which satisfies the conditions (2) and (3).

The main ways introduced in this paper are variational method, resolvent theory and fixed point trick. Based on the assumption that the relevant linear equation has approximate controllability, we obtain that the equation (1) has finite-dimensional exact controllability. At last, we present the abstract results and give an example to prove.

The article is made up of three parts as follows. Part 2 reviews the fundamental basis of symbols and definitions. Part 3 makes some assumptions about the parameters in (1). Then, through the Schauder's fixed point trick, we prove the equation (1) has solutions. The sufficient conditions which guarantee the equation (1) to have finite-dimensional exact controllability are constructed. The last part shows that the results are feasible through a typical example.

2. Preliminaries

First, we introduce the notations which will be used in the whole text. Let $a > 0$ be a given constant, $\mathbb{R}_+$ and $\mathbb{N}$ be the sets of nonnegative real numbers and positive integers. $C([0,a];Z)$ is the space which defines all Z-valued continuous functions on $[0,a]$. $L^2([0,a];Z)$ is the space which defines Z-valued Bochner integrable functions on $[0,a]$. $L^\infty([0,a];Z)$ is the space which defines Z-valued essentially bounded functions on $[0,a]$. The norms of Z, $C([0,a];Z)$ and $L^2([0,a];Z)$ are defined as $\|\cdot\|_Z$, $\|\varphi\|_C = \sup_{t\in[0,a]} \|\varphi(t)\|$ and $\|g\|_{L^2} = (\int_0^a \|g(t)\|^2 dt)^{\frac{1}{2}}$, respectively. $\mathcal{L}(Z)$ represents $\mathcal{L}(Z;Z)$. Γ is recorded as the Gamma function. The set $B(0;r) := \{\varphi \in C([0,a];Z); \|\varphi\| \le r\}$, where 0 is the origin and r is the radius.

At present, we recall some vital definitions and consequences in this article.

Definition 2.1 ([22]). *Let $g : [0,a] \to Z$, $t \in [0,a]$ and $\gamma \in (0,+\infty)$, the integral J_0^γ is expressed by*

$$J_0^\gamma g(t) = \int_0^t \frac{(t-\tau)^{\gamma-1} g(\tau)}{\Gamma(\gamma)} d\tau.$$

Definition 2.2 ([22]). *Let $g : [0,a] \to Z$, $t \in [0,a]$ and $\gamma \in (m-1,m]$, the expression of the derivative ${}^cD_0^\gamma$ is as follows*

$$ {}^cD_0^\gamma g(t) = \int_0^t \frac{(t-\tau)^{m-\gamma-1} g^{(m)}(\tau)}{\Gamma(m-\gamma)} d\tau,$$

where $m \in \mathbb{N}$ and ${}^cD_0^\gamma$ is the Caputo fractional differential operator.

The integrals can be understood in a Bochner sense.

Definition 2.3 ([23]). *Assume that $\{T_\gamma(t), t \ge 0\}$ represents the linear and bounded operators and has a generator G on a Hilbert space Z, $\{T_\gamma(t), t \ge 0\} \subseteq \mathcal{L}(Z)$ is a resolvent (or solution operator) if for any $t \ge 0$ and $\varphi \in D(G)$, the following conclusions hold:*
(A1) *$T_\gamma(0) = I$ and $T_\gamma(t)$ is strong continuous on $[0,+\infty)$;*
(A2) *$T_\gamma(t)D(G) \subseteq D(G)$ and $GT_\gamma(t)\varphi = T_\gamma(t)G\varphi$;*
(A3) *the resolvent equation holds*

$$T_\gamma(t)\varphi = \varphi + \int_0^t q_\gamma(t-\tau) G T_\gamma(\tau)\varphi d\tau,$$

where $q_\gamma(t) = \frac{t^{1-\gamma}}{\Gamma(\gamma)}$, $\gamma \in (0,1)$ and $t > 0$.

The literature [23] shows that for all $\varphi \in Z$, the above resolvent equation is also established.

Definition 2.4 ([23]). *Suppose $T_\gamma(t)$ is a resolvent, if the above definition is satisfied, the following conclusions are valid:*
(i) $\{T_\gamma(t)\}_{t\ge0}$ is called the analytic resolvent if the map $z \mapsto T_\gamma(z)$ is analytic in $\Sigma(0,\theta_0)$, where $0 < \theta_0 \le \pi/2$;
(ii) $\{T_\gamma(t)\}_{t\ge0}$ is called exponentially bounded if for each $\omega > \omega_0$ and $\theta < \theta_0$, find a constant $M_1 = M_1(\omega,\theta)$ such that

$$\|T_\gamma(z)\| \le M_1 e^{\omega Rez}, \quad z \in \Sigma(0,\theta),$$

more precisely, (ω_0,θ_0) is described as an analyticity type of $\{T_\gamma(t)\}_{t\ge0}$.

Definition 2.5. *For any $t > 0$, $\{T_\gamma(t)\}$ is a compact operator, then the resolvent $\{T_\gamma(t)\}_{t\ge0}$ is compact.*

Lemma 2.6 ([6]). *Suppose the equation (1) admits a compact analytic resolvent $\{T_{1-\gamma}(t)\}_{t\ge0}$ of analyticity type (ω_0,θ_0) iff $t \in (0,\infty)$ is arbitrary, then*
(i) $\lim_{\xi\to0} \|T_{1-\gamma}(t) - T_{1-\gamma}(\xi+t)\| = 0$;
(ii) $\lim_{\xi\to0^+} \|T_{1-\gamma}(t)T_{1-\gamma}(\xi) - T_{1-\gamma}(\xi+t)\| = 0$;
(iii) $\lim_{\xi\to0^+} \|T_{1-\gamma}(t-\xi)T_{1-\gamma}(\xi) - T_{1-\gamma}(t)\| = 0$.

Definition 2.7. *For any given $\varphi_0 \in Z$, $v \in L^2([0,a];V)$, the equation (1) has a mild solution $\varphi \in C([0,a];Z)$ if it is given below*

$$\varphi(t) = \varphi_0 - \int_0^t T_{1-\gamma}(t-\tau)\varphi(\tau)d\tau + \int_0^t T_{1-\gamma}(t-\tau)[g(\tau,\varphi(\tau)) + Hv(\tau,\varphi)]d\tau, \qquad t \in [0,a].$$

3. The main results

Owing the following assumptions, we prove that the existence of the solutions of the equation (1).
(B1) $\{T_{1-\gamma}(t)\}_{t\geq 0}$ represents a compact resolvent with $M := \sup\limits_{0\leq t\leq a} \|T_{1-\gamma}(t)\|$. $H : V \to Z$ represents a bounded linear operator with $M_H := \|H\|$.
(B2) The function $g : [0,a] \times Z \to Z$ is strongly measurable with respect to the first component t for $\varphi \in Z$; the second component φ is continuous for almost everywhere $t \in [0,a]$.
(B3) For any $(t,\varphi) \in [0,a] \times Z$, there are functions $m \in L^\infty([0,a];Z)$ and $\Lambda_g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|g(t,\varphi)\| \leq m(t)\Lambda_g(\|\varphi\|), \qquad \liminf_{r\to\infty} \frac{\Lambda_g(r)}{r} < \frac{1-Ma}{Ma\|m\|_{L^\infty}},$$

where m is the positive integrable and Λ_g is the non-decreasing and continuous function.

For the corresponding linear system, the hypothesis is as follows.
(F) The linear equation

$$\varphi(t) = \varphi_0 - \int_0^t T_{1-\gamma}(t-\tau)\varphi(\tau)d\tau + \int_0^t T_{1-\gamma}(t-\tau)Hv(\tau,\varphi)d\varphi, \ t \in [0,a], \tag{4}$$

is approximately controllable.

Remark 3.1. *By [4], we know that the linear equation (4) is approximately controllable if and only if $H^*T^*_{1-\gamma}(a-\tau)\mu = 0$, $0 \leq \tau \leq a$, where H^* and $T^*_{1-\gamma}$ are the adjoint of operators H and $T_{1-\gamma}$, which implies that $\mu = 0$.*

In fact, the problem of approximate controllability can be transformed into the problem of finding the limit of the optimal control. The method used here is inspired by the references [12, 16]. Therefore, we can utilize the variational way to solve the finite-dimensional exact controllability of the equation (1).

To start with, we briefly introduce the following functional. For any $h \in C([0,a];Z)$, $\varepsilon > 0$, then

$$J_\varepsilon(\cdot;h) : Z \to (-\infty,+\infty),$$

$$J_\varepsilon(\mu;h) = \frac{1}{2}\int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\mu\|^2 d\tau + \varepsilon\|(I-\pi_E)\mu\| - \langle \mu, p(h)\rangle, \qquad \mu \in Z, \tag{5}$$

where

$$p : C([0,a];Z) \to Z,$$

$$p(h) = \varphi_a - \varphi_0 + \int_0^a T_{1-\gamma}(a-\tau)[h(\tau) - g(\tau,h(\tau))]d\tau, \qquad \varphi_a \in Z.$$

Moreover, for convenience, we introduce some lemmas which are related to the properties of the mapping p and the functional J_ε as follows.

Lemma 3.2. *The mapping $p : C([0,a];Z) \to Z$ is continuous.*

Proof. Suppose $h_n \to h$ as $n \to \infty$, from the definition of continuity, we just have to prove $\|p(h_n) - p(h)\| \to 0$. Since

$$\begin{aligned}\|p(h_n) - p(h)\| &\leq \int_0^a \|T_{1-\gamma}(a-\tau)[h_n(\tau) - h(\tau)]\|d\tau + \int_0^a \|T_{1-\gamma}(a-\tau)[g(\tau, h_n(\tau)) - g(\tau, h(\tau))]\|d\tau \\ &\leq M\int_0^a \|h_n(\tau) - h(\tau)\|d\tau + M\int_0^a \|g(\tau, h_n(\tau)) - g(\tau, h(\tau))\|d\tau \\ &\leq Ma\|h_n - h\|_C + M\int_0^a \|g(\tau, h_n(\tau)) - g(\tau, h(\tau))\|d\tau,\end{aligned}$$

where $h_n, h \in C([0,a]; Z)$. From $\lim_{n\to\infty} \|h_n - h\|_C = 0$, for any $0 \leq t \leq a$, we know $h_n(t) \to h(t)$. According to the hypothesis (B2), we get $\|g(\tau, h_n(\tau)) - g(\tau, h(\tau))\| \to 0, 0 \leq \tau \leq a$, which implies $\int_0^a \|g(\tau, h_n(\tau)) - g(\tau, h(\tau))\|d\tau \to 0$ as $n \to \infty$ in accordance with the essential theorem of Lebesgue dominated convergence. Obviously, we have $\|p(h_n) - p(h)\| \to 0$ as $n \to \infty$. The proof is completed. □

Lemma 3.3. *For any $\{h_n\} \subset B(0;r)$, $p(h_n)$ has a convergent subsequence in Z.*

Proof. We need to find a series of relatively compact sets to approximate $\{p(h) : h \in B(0;r)\}$. Thus, the relative compactness of p is proved.

For any $0 < \varepsilon < a$, we define

$$p_\varepsilon(h) = \varphi_a - \varphi_0 + T_{1-\gamma}(\varepsilon)\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau-\varepsilon)[h(\tau) - g(\tau, h(\tau))]d\tau$$

and

$$W(\varepsilon) = \int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau-\varepsilon)[h(\tau) - g(\tau, h(\tau))]d\tau.$$

Then

$$\begin{aligned}\|W(\varepsilon)\| &\leq M\int_0^{a-\varepsilon} \|h(\tau)\|d\tau + M\int_0^{a-\varepsilon} \|g(\tau, h(\tau))\|d\tau \\ &\leq Mar + Ma\|m\|_{L^\infty}\Lambda_g(r),\end{aligned}$$

from the boundedness of $W(\varepsilon)$ and the compactness of $T_{1-\gamma}(\varepsilon)$, the relatively compact of $\{p_\varepsilon(h) : h \in B(0;r)\}$ is proved in Z. In addition, we find

$$\begin{aligned}&\|p(h) - p_\varepsilon(h)\| \\ &= \|\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau)[h(\tau) - g(\tau, h(\tau))]d\tau + \int_{a-\varepsilon}^a T_{1-\gamma}(a-\tau)[h(\tau) - g(\tau, h(\tau))]d\tau \\ &\quad - T_{1-\gamma}(\varepsilon)\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau-\varepsilon)[h(\tau) - g(\tau, h(\tau))]d\tau\| \\ &\leq \|\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau)h(\tau)d\tau - T_{1-\gamma}(\varepsilon)\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau-\varepsilon)h(\tau)d\tau\| + \|\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau)g(\tau, h(\tau))d\tau \\ &\quad - T_{1-\gamma}(\varepsilon)\int_0^{a-\varepsilon} T_{1-\gamma}(a-\tau-\varepsilon)g(\tau, h(\tau))d\tau\| + M\varepsilon r + M\varepsilon\|m\|_{L^\infty}\Lambda_g(r) \\ &\leq \int_0^{a-\varepsilon} \|T_{1-\gamma}(a-\tau) - T_{1-\gamma}(\varepsilon)T_{1-\gamma}(a-\tau-\varepsilon)\|\|h(\tau)\|d\tau \\ &\quad + \int_0^{a-\varepsilon} \|T_{1-\gamma}(a-\tau) - T_{1-\gamma}(\varepsilon)T_{1-\gamma}(a-\tau-\varepsilon)\|\|g(\tau, h(\tau))\|d\tau + M\varepsilon r + M\varepsilon\|m\|_{L^\infty}\Lambda_g(r).\end{aligned}$$

From Lemma 2.6 (iii), we know

$$\lim_{\varepsilon\to 0^+}\|T_{1-\gamma}(a-\tau)-T_{1-\gamma}(\varepsilon)T_{1-\gamma}(a-\tau-\varepsilon)\|=0,\ \tau\in[0,a-\varepsilon].$$

Thus, by the arbitrariness of ε : $\|p(h)-p_\varepsilon(h)\|\to 0$. The proof is completed. □

Lemma 3.4. *For any $\mu\in Z$ and $h\in C([0,a];Z)$, $\mu\to J_\varepsilon(\mu;h)$ is a mapping which satisfies continuity and strict convexity.*

Proof. If μ_n, $\mu\in Z$ satisfying $\mu_n\to\mu$, from the definition of J_ε and the important Lebesgue dominated convergence theorem, one has

$$J_\varepsilon(\mu_n;h)\to J_\varepsilon(\mu;h),\quad n\to\infty.$$

Thus, J_ε is continuous.

By Remark 3.1, we conclude that $H^*T^*_{1-\gamma}(a-\tau)$ is injective. Thus, for any $\alpha\in(0,1)$, μ_1, $\mu_2\in Z$, we find

$$\begin{aligned}
&J_\varepsilon(\alpha\mu_1+(1-\alpha)\mu_2;h)\\
&=\frac{1}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)[\alpha\mu_1+(1-\alpha)\mu_2]\|^2d\tau+\varepsilon\|(I-\pi_E)[\alpha\mu_1+(1-\alpha)\mu_2]\|-\langle\alpha\mu_1+(1-\alpha)\mu_2,p(h)\rangle\\
&\le\frac{\alpha^2}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_1\|^2d\tau+\varepsilon\alpha\|(I-\pi_E)\mu_1\|+\frac{(1-\alpha)^2}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_2\|^2d\tau+\varepsilon(1-\alpha)\|(I-\pi_E)\mu_2\|\\
&\quad+\alpha(1-\alpha)\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_1\|\|H^*T^*_{1-\gamma}(a-\tau)\mu_2\|d\tau-\alpha\langle\mu_1,p(h)\rangle-(1-\alpha)\langle\mu_2,p(h)\rangle\\
&<\frac{\alpha^2}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_1\|^2d\tau+\varepsilon\alpha\|(I-\pi_E)\mu_1\|+\frac{(1-\alpha)^2}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_2\|^2d\tau+\varepsilon(1-\alpha)\|(I-\pi_E)\mu_2\|\\
&\quad+\frac{\alpha(1-\alpha)}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_1\|^2d\tau-\alpha\langle\mu_1,p(h)\rangle+\frac{\alpha(1-\alpha)}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_2\|^2d\tau-(1-\alpha)\langle\mu_2,p(h)\rangle\\
&=\alpha J_\varepsilon(\mu_1;h)+(1-\alpha)J_\varepsilon(\mu_2;h).
\end{aligned}$$

Then J_ε is strictly convex. The proof is completed. □

Lemma 3.5. *For any $h\in B(0;r)$, we have the following formula*

$$\varliminf_{\mu\to\infty}\inf_{h\in B(0;r)}\frac{J_\varepsilon(\mu;h)}{\|\mu\|}\ge\varepsilon. \tag{6}$$

Proof. Hypothesis (6) is not valid. Then, there exists subsequences $\{\mu_n\}\subset Z$, $\{h_n\}\subset B(0;r)$, with $\|\mu_n\|\to\infty$, such that

$$\varliminf_{\mu_n\to\infty}\inf_{h_n\in B(0;r)}\frac{J_\varepsilon(\mu_n;h_n)}{\|\mu_n\|}<\varepsilon. \tag{7}$$

By (5), we have

$$J_\varepsilon(\mu_n;h_n)=\frac{1}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\mu_n\|^2d\tau+\varepsilon\|(I-\pi_E)\mu_n\|-\langle\mu_n,p(h_n)\rangle.$$

If normalized for μ_n : $\widetilde{\mu_n}=\frac{\mu_n}{\|\mu_n\|}$, we acquire

$$\frac{J_\varepsilon(\mu_n;h_n)}{\|\mu_n\|}=\frac{\|\mu_n\|}{2}\int_0^a\|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n}\|^2d\tau+\varepsilon\|(I-\pi_E)\widetilde{\mu_n}\|-\langle\widetilde{\mu_n},p(h_n)\rangle.$$

Since $\|\widetilde{\mu_n}\| = 1$, we can extract a subsequence (still expressed by $\widetilde{\mu_n}$) such that $\widetilde{\mu_n} \rightharpoonup \widetilde{\mu}$ in Z. As a matter of fact, because $\{T_{1-\gamma}(t)\}_{t\geq 0}$ represents a compact resolvent and H is continuous, one has

$$\sup_{0\leq\tau\leq a} \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n} - H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu}\| \to 0.$$

Note that,

$$\underline{\lim}_{n\to\infty} \int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n}\|^2 d\tau = 0,$$

according to the lemma of Fatou, then

$$\begin{aligned}\int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu}\|^2 d\tau &= \int_0^a \underline{\lim}_{n\to\infty} \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n}\|^2 d\tau \\ &\leq \underline{\lim}_{n\to\infty} \int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n}\|^2 d\tau = 0.\end{aligned}$$

By assumption (F) and Remark 3.1, we obtain $\widetilde{\mu} = 0$ and infer that

$$\widetilde{\mu_n} \rightharpoonup 0.$$

Moreover, since the compactness of π_E, $\pi_E\widetilde{\mu_n} \to 0$ in Z and therefore

$$\begin{aligned}\lim_{n\to\infty} \|(I-\pi_E)\widetilde{\mu_n}\| &= \lim_{n\to\infty} \sqrt{\langle (I-\pi_E)\widetilde{\mu_n}, (I-\pi_E)\widetilde{\mu_n}\rangle} \\ &= \lim_{n\to\infty} \sqrt{\|\widetilde{\mu_n}\|^2 + \|\pi_E\widetilde{\mu_n}\|^2} \\ &= 1.\end{aligned}$$

In fact, from Lemma 3.3 and $\widetilde{\mu_n} \rightharpoonup 0$, we obtain

$$\langle \widetilde{\mu_n}, p(h_n)\rangle \to 0.$$

Hence

$$\begin{aligned}\underline{\lim}_{n\to\infty} \frac{J_\varepsilon(\mu_n; h_n)}{\|\mu_n\|} &= \underline{\lim}_{n\to\infty} (\frac{\|\mu_n\|}{2} \int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\widetilde{\mu_n}\|^2 d\tau + \varepsilon\|(I-\pi_E)\widetilde{\mu_n}\| - \langle \widetilde{\mu_n}, p(h_n)\rangle) \\ &\geq \underline{\lim}_{n\to\infty} (\varepsilon\|(I-\pi_E)\widetilde{\mu_n}\| - \langle \widetilde{\mu_n}, p(h_n)\rangle) \\ &= \varepsilon.\end{aligned}$$

The contradiction we have arrived at completes the proof. □

Next, we need to verify that the equation (1) has the solution, which ensures the equation can achieve finite-dimensional exactly controllable.

According to Lemma 3.4, we know that the functional J_ε has continuity and strict convexity. We can define a map

$$\begin{aligned} R_\varepsilon : C([0,a]; Z) &\to Z \\ h &\mapsto \widehat{\mu}_\varepsilon,\end{aligned}$$

where $\widehat{\mu}_\varepsilon$ is a unique minimum of $J_\varepsilon(\cdot; h)$, for all $h \in C([0,a]; Z)$. We further construct the following operator $\Theta_\varepsilon : C([0,a]; Z) \to C([0,a]; Z)$,

$$\begin{aligned}&(\Theta_\varepsilon\varphi)(t) = \varphi_0 - \int_0^t T_{1-\gamma}(t-\tau)[\varphi(\tau) - g(\tau,\varphi(\tau)) - Hv_\varepsilon(\tau,\varphi)]d\tau, \quad 0 \leq t \leq a, \\ &v_\varepsilon(\tau,\varphi) := H^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi). \qquad (8)\end{aligned}$$

For the convenience of the following proof, we introduce some properties of $R_\varepsilon(\varphi)$ as follows.

Lemma 3.6. *For all $h \in B(0;r)$, the set $\{R_\varepsilon(h)\}$ of minimum points is bounded. In other words, there is a constant $M_\varepsilon^r > 0$, such that $\|R_\varepsilon(h)\| \le M_\varepsilon^r$.*

Proof. Let us suppose that there are a constant $K_\varepsilon^r > 0$ and $h_0 \in B(0;r)$, such that $\|\mu\| > K_\varepsilon^r$, and argue from this to a contradiction. Indeed, in this case, by Lemma 3.5 we have,

$$\inf_{h\in B(0;r)} \frac{J_\varepsilon(\mu;h)}{\|\mu\|} \ge \frac{\varepsilon}{2}.$$

From the assumptions, one has

$$\frac{J_\varepsilon(R_\varepsilon(h_0);h_0)}{\|R_\varepsilon(h_0)\|} \ge \inf_{h_0\in B(0;r)} \frac{J_\varepsilon(\mu;h_0)}{\|\mu\|} \ge \frac{\varepsilon}{2}.$$

Moreover, according to the definition of R_ε, we have

$$J_\varepsilon(R_\varepsilon(h);h) \le J_\varepsilon(0;h) = 0, \quad h \in B(0;r),$$

which implies a contradiction. The proof is completed. □

Lemma 3.7. *If $h_n \to h \in B(0;r)$ as $n \to \infty$, we have $R_\varepsilon(h_n) \xrightarrow{\omega} R_\varepsilon(h)$.*

Proof. From the previous work, we obtain that the boundedness of $\{\widehat{\mu}_{\varepsilon,n}\} = \{R_\varepsilon(h_n)\}$. we just need to prove $\widehat{\mu}_{\varepsilon,n} \xrightarrow{\omega} \widehat{\mu}_\varepsilon$, as $n \to \infty$, where $\widehat{\mu}_\varepsilon$ is the minimum of $J(\cdot;h)$ and $\widehat{\mu}_\varepsilon = R_\varepsilon(h)$. Suppose one of the subsequences (still denoted by $\widehat{\mu}_{\varepsilon,n}$) weakly converges to $\widetilde{\mu}_\varepsilon$. Thus, by the optimality of $\{\widehat{\mu}_{\varepsilon,n}\} = \{R_\varepsilon(h_n)\}$ and $\widehat{\mu}_\varepsilon = R_\varepsilon(h)$, one has

$$J_\varepsilon(\widehat{\mu}_\varepsilon;h) \le J_\varepsilon(\widetilde{\mu}_\varepsilon;h) \le \underline{\lim}_{n\to\infty} J_\varepsilon(\widehat{\mu}_{\varepsilon,n};h_n) \le \overline{\lim}_{n\to\infty} J_\varepsilon(\widehat{\mu}_{\varepsilon,n};h_n) \le \lim_{n\longrightarrow\infty} J_\varepsilon(\widehat{\mu}_\varepsilon;h_n) = J_\varepsilon(\widehat{\mu}_\varepsilon;h).$$

Then, as easily seen,

$$J_\varepsilon(\widehat{\mu}_\varepsilon;h) = J_\varepsilon(\widetilde{\mu}_\varepsilon;h).$$

Inasmuch as the uniqueness of the minimum, this implies that $\widehat{\mu}_\varepsilon = \widetilde{\mu}_\varepsilon$. The proof is completed. □

Lemma 3.8. *If $h_n \to h \in B(0;r)$ as $n \to \infty$, one has $\lim_{n\to\infty} \|R_\varepsilon(h_n)\| = \|R_\varepsilon(h)\|$.*

Proof. From (5), we get

$$J_\varepsilon(\mu_n;h_n) = \frac{1}{2}\int_0^a \|H^*T_{1-\gamma}^*(a-\tau)\mu_n\|^2 d\tau + \varepsilon\|(I-\pi_E)\mu_n\| - \langle \mu_n, p(h_n)\rangle.$$

On one hand, from Lemma 3.7, we find

$$\lim_{n\to\infty} J_\varepsilon(\widehat{\mu}_{\varepsilon,n};h_n) = J_\varepsilon(\widehat{\mu}_\varepsilon;h).$$

On the other hand, these important conditions which are compactness of $T_{1-\gamma}(t)$, continuity of p, and $\widehat{\mu}_{\varepsilon,n} \xrightarrow{\omega} \widehat{\mu}_\varepsilon$, imply that

$$\lim_{n\to\infty} \int_0^a \|H^*T_{1-\gamma}^*(a-\tau)\widehat{\mu}_{\varepsilon,n}\|^2 d\tau = \int_0^a \|H^*T_{1-\gamma}^*(a-\tau)\widehat{\mu}_\varepsilon\|^2 d\tau,$$

$$\lim_{n\to\infty} \langle \widehat{\mu}_{\varepsilon,n}, p(h)\rangle = \langle \widehat{\mu}_\varepsilon, p(h)\rangle.$$

Consider the limit of $\|(I-\pi_E)\widehat{\mu}_\varepsilon\|$ exists, we acquire

$$\|(I-\pi_E)\widehat{\mu}_\varepsilon\| \le \lim_{n\to\infty} \|(I-\pi_E)\widehat{\mu}_{\varepsilon,n}\|.$$

By the weak convergence and the weak lower semicontinuity of the norm in Z,

$$\lim_{n\to\infty} \|(I-\pi_E)\widehat{\mu}_{\varepsilon,n}\| = \|(I-\pi_E)\widehat{\mu}_\varepsilon\|.$$

In addition, because the compactness of π_E, it follows that $\widehat{\mu}_{\varepsilon,n} \xrightarrow{\omega} \widehat{\mu}_\varepsilon$ and

$$\begin{aligned}\lim_{n\to\infty} \|\widehat{\mu}_{\varepsilon,n}\|^2 &= \lim_{n\to\infty} \|(I-\pi_E)\widehat{\mu}_{\varepsilon,n}\|^2 + \lim_{n\to\infty} \|\pi_E\widehat{\mu}_{\varepsilon,n}\|^2 \\ &= \|(I-\pi_E)\widehat{\mu}_\varepsilon\|^2 + \|\pi_E\widehat{\mu}_\varepsilon\|^2 = \|\widehat{\mu}_\varepsilon\|^2.\end{aligned}$$

The proof is completed. □

Finally, we prove the solution φ and the corresponding control $v_\varepsilon(\tau,\varphi) = H^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi)$ which satisfy the finite-dimensional exact controllability conditions

$$\| \varphi(a; v_\varepsilon) - \varphi_a \| \leq \varepsilon, \qquad \pi_E\varphi(a; v_\varepsilon) = \pi_E\varphi_a.$$

Theorem 3.9. *Under the conditions (B1), (B2), (B3) and (F), the equation (1) has solutions.*

Proof. The method depends upon standard Schauder's fixed point trick. We construct the nonlinear operator $\Theta_\varepsilon : C([0,a];Z) \to C([0,a];Z)$ satisfying the following assertions:
(1) Θ_ε is continuous;
(2) There is a closed convex set $B(0;r_\varepsilon) \subset C([0,a];Z)$ such that $\Theta_\varepsilon(B(0;r_\varepsilon)) \subseteq B(0;r_\varepsilon)$;
(3) Θ_ε is a compact mapping.
Here $B(0;r_\varepsilon) = \{\varphi \in C([0,b];Z) : \|\varphi\| \leq r_\varepsilon\}$. Then, nonlinear operator Θ_ε has one or more than one fixed points.

At present, we prove Theorem 3.9 by the three assertions given below.
Assertion 1 To prove the continuity of nonlinear operator $\Theta_\varepsilon : C([0,a];Z) \to C([0,a];Z)$.

Choose a subsequence $\{\varphi^m\}_{m\in N} \subset C([0,a];Z)$ such that $\|\varphi^m - \varphi\|_C \to 0$ as $m \to \infty$. By the assumption (B2), it follows that, for almost everywhere $\tau \in [0,a]$, the formula

$$\|g(\tau,\varphi^m(\tau)) - g(\tau,\varphi(\tau))\| \to 0, \quad as\ m \to \infty.$$

Moreover

$$v_\varepsilon(\tau,\varphi) := H^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi),$$

$$\begin{aligned}\lim_{m\to\infty} \|v_\varepsilon(\tau,\varphi^m) - v_\varepsilon(\tau,\varphi)\| &= \lim_{m\to\infty} \|H^*T^*_{1-\gamma}(a-\tau)[R_\varepsilon(\varphi^m) - R_\varepsilon(\varphi)]\| \\ &\leq M_H M \lim_{m\to\infty} \|R_\varepsilon(\varphi^m) - R_\varepsilon(\varphi))\|.\end{aligned}$$

By Lemma 3.7 and Lemma 3.8, we acquire

$$\|v_\varepsilon(\tau,\varphi^m) - v_\varepsilon(\tau,\varphi)\| \to 0,\ as\ m \to \infty,$$

where τ holds almost everywhere on $[0,a]$. Then,

$$\begin{aligned}&\|(\Theta_\varepsilon\varphi^m)(t) - (\Theta_\varepsilon\varphi)(t)\| \\ &\leq \int_0^t \|T_{1-\gamma}(t-\tau)[\varphi^m(\tau) - \varphi(\tau)]\|d\tau + \int_0^t \|T_{1-\gamma}(t-\tau)[g(\tau,\varphi^m(\tau)) - g(\tau,\varphi(\tau))]\|d\tau \\ &\quad + \int_0^t \|T_{1-\gamma}(t-\tau)[Hv_\varepsilon(\tau,\varphi^m) - Hv_\varepsilon(\tau,\varphi)]\|d\tau \\ &\leq M\int_0^a \|\varphi^m(\tau) - \varphi(\tau)\|d\tau + M\int_0^a \|g(\tau,\varphi^m(\tau)) - g(\tau,\varphi(\tau))\|d\tau + M_H M\int_0^a \|v_\varepsilon(\tau,\varphi^m) - v_\varepsilon(\tau,\varphi)\|d\tau \\ &\leq Ma\|\varphi^m - \varphi\|_C + M\int_0^a \|g(\tau,\varphi^m(\tau)) - g(\tau,\varphi(\tau))\|d\tau + M_H M\int_0^a \|v_\varepsilon(\tau,\varphi^m) - v_\varepsilon(\tau,\varphi)\|d\tau, \quad t \in [0,a].\end{aligned}$$

By the essential theorem of Lebesgue dominated convergence, we can conclude that $\|\Theta_\varepsilon\varphi^m - \Theta_\varepsilon\varphi\| \to 0$, as $m \to \infty$. This means that the operator Θ_ε is continuous.

Assertion 2 For any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that $\Theta_\varepsilon(B(0;r_\varepsilon)) \subseteq B(0;r_\varepsilon)$, that is, Θ_ε maps $B(0;r_\varepsilon)$ into itself.

Let $\varphi \in B(0;r_\varepsilon)$, $t \in [0,a]$, we find

$$\begin{aligned}\|(\Theta_\varepsilon\varphi)(t)\| &= \|\varphi_0 - \int_0^t T_{1-\gamma}(t-\tau)[\varphi(\tau) - g(\tau,\varphi(\tau)) - Hv_\varepsilon(\tau,\varphi)]d\tau\| \\ &\le \|\varphi_0\| + Ma\kappa,\end{aligned}$$

where $\kappa := r_\varepsilon + \|m\|_{L^\infty}\Lambda_g(r_\varepsilon) + MM_H^2M_\varepsilon^r$. We take supremum of t on both sides of the above formula. Finally, by assumption (B3), we know that $\|\Theta_\varepsilon\varphi\| \le r_\varepsilon$, where $r_\varepsilon > 0$ is sufficiently large.

Assertion 3 We can use the following two steps to prove that Θ_ε is a compact mapping.

Step 1 For any $t \in [0,a]$, $\{\Theta_\varepsilon\varphi : \varphi \in B(0;r_\varepsilon)\}$ is equicontinuous on $[0,a]$.
Let $\varphi \in B(0;r_\varepsilon)$, and $0 \le t \le \zeta \le a$, then

$$\begin{aligned}\|(\Theta_\varepsilon\varphi)(\zeta) - (\Theta_\varepsilon\varphi)(t)\| \le& \int_0^t \|[T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)]\varphi(\tau)\|d\tau + \int_t^\zeta \|T_{1-\gamma}(\zeta-\tau)\varphi(\tau)\|d\tau \\ &+ \int_t^\zeta \|T_{1-\gamma}(\zeta-\tau)[g(\tau,\varphi(\tau)) + Hv_\varepsilon(\tau,\varphi)]\|d\tau \\ &+ \int_0^t \|[T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)][g(\tau,\varphi(\tau)) + Hv_\varepsilon(\tau,\varphi)]\|d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(\Theta_\varepsilon\varphi)(\zeta) - (\Theta_\varepsilon\varphi)(0)\| &\le \int_0^\zeta \|T_{1-\gamma}(\zeta-\tau)\varphi(\tau)\|d\tau + \int_0^\zeta \|T_{1-\gamma}(\zeta-\tau)[g(\tau,\varphi(\tau)) + Hv_\varepsilon(\tau,\varphi)]\|d\tau \\ &\le M\kappa\zeta \\ &\to 0,\end{aligned}$$

as $\zeta \to 0$.

For $t \in (0,a]$, we take $\delta \in (0,t)$, then

$$\begin{aligned}&\|(\Theta_\varepsilon\varphi)(\zeta) - (\Theta_\varepsilon\varphi)(t)\| \\ &\le \int_0^t \|[T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)]\varphi(\tau)\|d\tau + \int_t^\zeta \|T_{1-\gamma}(\zeta-\tau)\varphi(\tau)\|d\tau \\ &\quad + \int_t^\zeta \|T_{1-\gamma}(\zeta-\tau)[g(\tau,\varphi(\tau)) + Hv_\varepsilon(\tau,\varphi)]\|d\tau \\ &\quad + \int_0^t \|[T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)][g(\tau,\varphi(\tau)) + Hv_\varepsilon(\tau,\varphi)]\|d\tau \\ &\le \kappa\int_0^t \|T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau + M\kappa(\zeta - t) \\ &\le \kappa\left[\int_0^{t-\delta} \|T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau + \int_{t-\delta}^t \|T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau\right] + M\kappa(\zeta - t) \\ &\le \kappa\int_0^{t-\delta} \|T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau + 2M\delta\kappa + M\kappa(\zeta - t).\end{aligned}$$

From Lemma 2.6(i), we get $\lim_{\zeta\to t}\|T_{1-\gamma}(\zeta-\tau) - T_{1-\gamma}(t-\tau)\| = 0$, where $\tau \in [0, t-\delta]$. Then, as seen earlier, δ is arbitrary. Hence, according to the essential theorem of Lebesgue dominated convergence, we conclude that $\|(\Theta_\varepsilon\varphi)(\zeta) - (\Theta_\varepsilon\varphi)(t)\| \to 0$ as $\zeta \to t$.

Step 2 Now it needs to certificate that $\{(\Theta_\varepsilon\varphi)(t) : \varphi \in B(0; r_\varepsilon)\}$ has relative compactness in Z. Define

$$(\Theta_\varepsilon\varphi)(t) := (\Theta^1_\varepsilon\varphi)(t) + (\Theta^2_\varepsilon\varphi)(t), \quad 0 \le t \le a,$$

where

$$(\Theta^1_\varepsilon\varphi)(t) = \varphi_0 - \int_0^t T_{1-\gamma}(t-\tau)[\varphi(\tau) - g(\tau,\varphi(\tau))]d\tau,$$

$$(\Theta^2_\varepsilon\varphi)(t) = \int_0^t T_{1-\gamma}(t-\tau)Hv_\varepsilon(\tau,\varphi)d\tau.$$

(i) If $t = 0$, $(\Theta_\varepsilon\varphi)(0) = (\Theta^1_\varepsilon\varphi)(0) + (\Theta^2_\varepsilon\varphi)(0) = \varphi_0$, then it is obviously relatively compact.

(ii) For any $t \in (0, a]$, we observe that $(\Theta^1_\varepsilon\varphi)(t)$ is relatively compact by Lemma 3.3. This implies that we only need to testify that the operator $(\Theta^2_\varepsilon\varphi)(t)$ has relative compactness. Take $\eta \in (0, t)$ and $\delta \in (\eta, a)$, define operators $\Theta^2_{\varepsilon,\eta}$ and $Q(\eta)$, one has

$$(\Theta^2_{\varepsilon,\eta}\varphi)(t) = T_{1-\gamma}(\eta)\int_0^{t-\eta} T_{1-\gamma}(t-\eta-\tau)Hv_\varepsilon(\tau,\varphi)d\tau,$$

$$Q(\eta) = \int_0^{t-\eta} T_{1-\gamma}(t-\eta-\tau)HH^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi)d\tau.$$

Consequently,

$$\begin{aligned}\|Q(\eta)\| &\le \int_0^{t-\eta} \|T_{1-\gamma}(t-\eta-\tau)HH^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi)\|d\tau \\ &\le M^2M_H^2M_\varepsilon^r a.\end{aligned}$$

Since the compactness of $T_{1-\gamma}(\eta)$ and the boundedness of $Q(\eta)$, $(\Theta^2_{\varepsilon,\eta}\varphi)(t)$ has relative compactness in Z. Inasmuch as $\varphi \in B(0; r_\varepsilon)$, we find

$$\begin{aligned}&\|(\Theta^2_{\varepsilon,\eta}\varphi)(t) - (\Theta^2_\varepsilon\varphi)(t)\| \\ &\le \|T_{1-\gamma}(\eta)\int_0^{t-\eta} T_{1-\gamma}(t-\eta-\tau)Hv_\varepsilon(\tau,\varphi)d\tau - \int_0^{t-\eta} T_{1-\gamma}(t-\tau)Hv_\varepsilon(\tau,\varphi)d\tau\| + \|\int_{t-\eta}^t T_{1-\gamma}(t-\tau)Hv_\varepsilon(\tau,\varphi)d\tau\| \\ &\le \|T_{1-\gamma}(\eta)\int_0^{t-\eta} T_{1-\gamma}(t-\eta-\tau)Hv_\varepsilon(\tau,\varphi)d\tau - \int_0^{t-\eta} T_{1-\gamma}(t-\tau)Hv_\varepsilon(\tau,\varphi)d\tau\| + M_H^2M^2M_\varepsilon^r\eta \\ &\le M_H^2MM_\varepsilon^r\int_0^{t-\delta} \|T_{1-\gamma}(\eta)T_{1-\gamma}(t-\eta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau \\ &\quad + M_H^2MM_\varepsilon^r\int_{t-\delta}^{t-\eta} \|T_{1-\gamma}(\eta)T_{1-\gamma}(t-\eta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau + M_H^2M^2M_\varepsilon^r\eta \\ &\le M_H^2MM_\varepsilon^r\int_0^{t-\delta} \|T_{1-\gamma}(\eta)T_{1-\gamma}(t-\eta-\tau) - T_{1-\gamma}(t-\tau)\|d\tau + (\delta-\eta)M_H^2MM_\varepsilon^r(M^2+M) + M_H^2M^2M_\varepsilon^r\eta,\end{aligned}$$

and therefore by Lemma 2.6(iii), we find

$$\lim_{\eta\to 0}\|T_{1-\gamma}(\eta)T_{1-\gamma}(t-\eta-\tau) - T_{1-\gamma}(t-\tau)\| = 0, \quad \tau \in [0, t-\delta].$$

For any $\delta \in (\eta, a)$, it turns out that if the above inequality satisfies the essential theorem of Lebesgue dominated convergence, we have

$$\lim_{\eta\to 0}\|(\Theta^2_{\varepsilon,\eta}\varphi)(t) - (\Theta^2_\varepsilon\varphi)(t)\| = 0.$$

On the basis of Arzelà-Ascoli theorem, Θ_ε is the operator with compactness. We can apply the technique of Schauder's fixed point to conclude that φ_ε is a fixed point of Θ_ε. □

Theorem 3.10. *Under the conditions (B1), (B2), (B3) and (F), the equation (1) has the finite-dimensional exact controllability on* $[0, a]$.

Proof. We consider the critical point $\widehat{\mu}_\varepsilon$ of J_ε. It follows by Lemma 3.4 that

$$J_\varepsilon(\widehat{\mu}_\varepsilon, \varphi_\varepsilon) = \min_{\mu \in Z} J_\varepsilon(\mu, \varphi_\varepsilon), \quad \widehat{\mu}_\varepsilon \in Z.$$

This yields

$$J_\varepsilon(\widehat{\mu}_\varepsilon; \varphi_\varepsilon) \le J_\varepsilon(\widehat{\mu}_\varepsilon + \lambda\nu; \varphi_\varepsilon), \quad for\ any\ \nu \in Z,\ \lambda \in R.$$

Then

$$J_\varepsilon(\widehat{\mu}_\varepsilon + \lambda\nu; \varphi_\varepsilon) - J_\varepsilon(\widehat{\mu}_\varepsilon; \varphi_\varepsilon) \ge 0,$$

$$\frac{\lambda^2}{2}\int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\nu\|^2 d\tau + \lambda\int_0^a \langle H^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, H^*T^*_{1-\gamma}(a-\tau)\nu\rangle d\tau + \varepsilon|\lambda|\|(I-\pi_E)\nu\| - \lambda\langle \nu, p(\varphi_\varepsilon)\rangle \ge 0,$$

and hence

$$\lambda\langle \nu, p(\varphi_\varepsilon)\rangle - \frac{\lambda^2}{2}\int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\nu\|^2 d\tau - \lambda\int_0^a \langle H^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, H^*T^*_{1-\gamma}(a-\tau)\nu\rangle d\tau \le \varepsilon|\lambda|\|(I-\pi_E)\nu\|.$$

The above inequality is divided by $\lambda > 0$, one obtains

$$\langle \nu, p(\varphi_\varepsilon)\rangle - \frac{\lambda}{2}\int_0^a \|H^*T^*_{1-\gamma}(a-\tau)\nu\|^2 d\tau - \int_0^a \langle H^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, H^*T^*_{1-\gamma}(a-\tau)\nu\rangle d\tau \le \frac{\varepsilon|\lambda|\|(I-\pi_E)\nu\|}{\lambda}.$$

Letting $\lambda \to 0^+$ and $\lambda \to 0^-$, respectively, we find that

$$\left|\langle \nu, p(\varphi_\varepsilon)\rangle - \int_0^a \langle H^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, H^*T^*_{1-\gamma}(a-\tau)\nu\rangle d\tau\right| \le \varepsilon\|(I-\pi_E)\nu\|. \tag{9}$$

In fact, according to the $v_\varepsilon = H^*T^*_{1-\gamma}(a-\tau)R_\varepsilon(\varphi)$, we know

$$\int_0^a \langle T_{1-\gamma}(a-\tau)HH^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, \nu\rangle d\tau = \langle \int_0^a T_{1-\gamma}(a-\tau)Hv_\varepsilon d\tau, \nu\rangle,$$

$$p(\varphi_\varepsilon) = \varphi_a - \varphi_0 + \int_0^a T_{1-\gamma}(a-\tau)[\varphi_\varepsilon(\tau) - g(\tau, \varphi_\varepsilon(\tau))]d\tau, \quad \varphi_a \in Z. \tag{10}$$

In view of (9), (10) and Definition 2.7, then

$$\begin{aligned}
&\left|\int_0^a \langle H^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, H^*T^*_{1-\gamma}(a-\tau)\nu\rangle d\tau - \langle \nu, p(\varphi_\varepsilon)\rangle\right| \\
&= \left|\int_0^a \langle T_{1-\gamma}(a-\tau)HH^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon, \nu\rangle d\tau - \langle \nu, p(\varphi_\varepsilon)\rangle\right| \\
&= \left|\langle \int_0^a T_{1-\gamma}(a-\tau)[HH^*T^*_{1-\gamma}(a-\tau)\widehat{\mu}_\varepsilon - \varphi_\varepsilon(\tau) + g(\tau, \varphi_\varepsilon(\tau))]d\tau + \varphi_0 - \varphi_a, \nu\rangle\right| \\
&\le \varepsilon\|(I-\pi_E)\nu\| \\
&\le \varepsilon\|\nu\|,
\end{aligned}$$

holds for any $\nu \in Z$, which implies that

$$\|\varphi(a; v_\varepsilon) - \varphi_a\| \le \varepsilon.$$

By taking $\nu \in E$, ν is arbitrary, we have

$$\langle \varphi_\varepsilon(a) - \varphi_a, \pi_E\varphi_\varepsilon(a) - \pi_E\varphi_a\rangle = 0,$$

which implies that

$$\pi_E\varphi_\varepsilon(a) = \pi_E\varphi_a.$$

We can get the finite-dimensional exact controllability of the equation (1). □

4. Application

Let us briefly give a simple example, which is the Basset problem in $(0, \infty)$:

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,\varsigma) = \frac{\partial^2}{\partial \varsigma^2} D_t^{\frac{1}{2}}\psi(t,\varsigma) + R(t,\psi(t,\varsigma)) + v(t,\varsigma), \quad t \in (0,1],\ \varsigma \in [0,\pi], \\ \psi(t,0) = \psi(t,\pi) = 0, \quad t \in [0,1], \\ \psi(0,\varsigma) = \psi_0(\varsigma), \quad \varsigma \in [0,\pi], \end{cases} \tag{11}$$

where $D_t^{\frac{1}{2}}$ is the derivative operator of Caputo with the order 1/2. The equation (11) can be used to describe the basic problem in fluid mechanics.

Let $Z = V = L^2([0,\pi];\mathbb{R})$. The generator $(G, D(G))$ is given by

$$G\varphi = \varphi'',$$

the notation of domain is recorded as

$$D(G) := \{\varphi \in Z : \varphi, \varphi' \text{ are absolutely continuous}, \varphi(0) = \varphi(\pi) = 0, \varphi'' \in Z\}.$$

We can conclude that G is closed and densely defined, and it can be expressed as

$$G\varphi = -\sum_{m=1}^{\infty} \frac{m^2}{1+m^2}\langle x, e_m\rangle e_m, \quad m = 1, 2, \cdots,$$

where $e_m(\varsigma) = \sqrt{\frac{2}{\pi}}\sin m\varsigma$ is an orthonormal basis in Z. Moreover, a self-adjoint, analytic and compact semigroup $\{T(t)\}_{t\geq 0}$ is generated by G on Z. For any $\varphi \in Z$, we give

$$T(t)\varphi = \sum_{m=1}^{\infty} exp(\frac{-m^2 t}{1+m^2})\langle x, e_m\rangle e_m, \quad \varphi \in Z.$$

According to the principle of subordination in [2, 6], we obtain that a compact 1/2-order analytic resolvent $T_{1/2}(t)$. For some ω_0 and θ_0, we have

$$T_{1/2}(t) = \int_0^{\infty} \Psi_{1/2}(\tau)T(\tau t^{1/2})d\tau, \quad t \in [0, \infty),$$

where

$$\Psi_\gamma(\tau) := -\frac{s}{\Gamma(1-2\gamma)} + \frac{s^2}{2!\Gamma(1-3\gamma)} - \frac{s^3}{3!\Gamma(1-4\gamma)} + \cdots + (-1)^l \frac{s^l}{l!\Gamma(1-(l+1)\gamma)} + \cdots,$$

$l = 1, 2, \cdots, \gamma \in (0,1)$, and its analyticity type is (ω_0, θ_0).

In addition,

$$\begin{aligned} HT_{1/2}(a-\tau)\varphi &= T_{1/2}(a-\tau)\varphi \\ &= \sum_{m=1}^{\infty}\int_0^{\infty} \Psi_{1/2}(\eta) exp\frac{-m^2(a-\tau)^{\frac{1}{2}}\eta}{1+m^2} d\eta \langle \varphi, e_m\rangle e_m. \end{aligned}$$

From Remark 3.1, we infer that $\Psi_{1/2}(\eta) \geq 0$ and $exp\frac{-m^2(a-\tau)^{1/2}\eta}{1+m^2} > 0$. It follows clearly that if $HT_{1/2}(a-\tau)\varphi = 0$, $\tau \in [0, a]$, this implies that $\varphi = 0$. Therefore, the linear part of the equation (11) is approximately controllable, which indicates that the hypothesis condition (F) is satisfied.

On the other hand, according to Theorem 3.9, it follows that $g(t, \varphi(t)) = R(t, \varphi(t,\varsigma))$. In addition, the mapping $g : [0,1] \times Z \to Z$ satisfies the assumptions (B1)-(B3). Consequently, the finite-dimensional exact controllability of the equation (11) is proved.

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