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Nonlinear maps preserving the mixed triple $*$-product between factors

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two factors. In this paper, it is proved that a not necessarily linear bijective map $\phi : \mathcal{A} \to \mathcal{B}$ satisfies $\phi([A, B]_* \bullet C) = [\phi(A), \phi(B)]_* \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$ if and only if ϕ is a linear $*$-isomorphism, a conjugate linear $*$-isomorphism, the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a map. We consider that ϕ preserves the mixed triple $*$-product if $\phi([A, B]_* \bullet C) = [\phi(A), \phi(B)]_* \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$, where $[A, B]_* = AB - BA^*$ is the skew Lie product and $A \bullet B = AB + BA^*$ is the Jordan $*$-product of A and B. Recently, some authors have considered the mixture of (skew) Lie product and Jordan $*$-product [3–17]. For example, Yang and Zhang [8] proved the nonlinear maps preserving the mixed skew Lie triple product $[[A, B]_*, C]$ on factors. Zhao et al. [17] proved the nonlinear maps preserving mixed product $[A \bullet B, C]$ on von Neumann algebras. Yang and Zhang [9] proved the nonlinear maps preserving the second mixed Lie triple product $[[A, B], C]_*$ on factors. In this article, motivated by the above results, we will obtain the structure of the nonlinear maps preserving the mixed triple $*$-product $[A, B]_* \bullet C$ on factors.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. $\mathcal{A}$ is a factor means that its center only contains the scalar operators. It is well known that the factor $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$, if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

Lemma 1.1. *[16] Let $\mathcal{A}$ be a factor and $A \in \mathcal{A}$. Then $AB + BA^* = 0$ for all $B \in \mathcal{A}$ implies that $A \in i\mathbb{R}I$ (i is the imaginary number unit).*

Lemma 1.2. *[7] Let $\mathcal{A}$ be a factor von Neumann algebra and $A \in \mathcal{A}$. If $[A, B]_* \in \mathbb{C}I$ for all $B \in \mathcal{A}$, then $A \in \mathbb{C}I$.*

Lemma 1.3. *([2, Problem 230]) Let $\mathcal{A}$ be a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $[A, B] = \lambda I$, where $[A, B] = AB - BA$, then $\lambda = 0$.*

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2. The main result and its proof

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with* $\dim \mathcal{A} \geq 2$. *Then a bijective map $\phi : \mathcal{A} \to \mathcal{B}$ satisfies $\phi([A, B]_* \bullet C) = [\phi(A), \phi(B)]_* \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$ if and only if ϕ is a linear $*$-isomorphism, a conjugate linear $*$-isomorphism, the negative of a linear $*$-isomorphism, or the negative conjugate linear $*$-isomorphism.*

Proof. Choose an arbitrary nontrivial projection $P_1 \in \mathcal{A}$, write $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j, i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. We can write every $A \in \mathcal{A}$ as $A = \sum_{i,j=1}^{2} A_{ij}$, where A_{ij} denotes an arbitrary element of $\mathcal{A}_{ij}$. We denote by $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ all projections of $\mathcal{A}$ and $\mathcal{B}$, respectively. Clearly, we only need to prove the necessity.

Claim 1. $\phi(0) = 0$.

Since ϕ is surjective, there exists $A \in \mathcal{A}$ such that $\phi(A) = 0$. Hence $\phi(0) = \phi([0, A]_* \bullet A) = [\phi(0), \phi(A)]_* \bullet \phi(A) = 0$.

Claim 2. $\phi(\Sigma_{i,j=1}^{2} A_{ij}) = \Sigma_{i,j=1}^{2} \phi(A_{ij})$ for all $A_{ij} \in \mathcal{A}_{ij}$.

Let $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{A}$ such that $\phi(X) = \Sigma_{i,j=1}^{2} \phi(A_{ij})$. We have $\phi([P_1, X]_* \bullet P_2) = \Sigma_{i,j=1}^{2} \phi([P_1, A_{ij}]_* \bullet P_2)$, i.e., $\phi(X_{12} + X_{12}^*) = \phi(A_{12} + A_{12}^*)$, which implies that $X_{12} = A_{12}$. In the same manner, $X_{21} = A_{21}$.

For every $T_{12} \in \mathcal{A}_{12}$, we obtain $\phi([T_{12}, X]_* \bullet P_2) = \Sigma_{i,j=1}^{2} \phi([T_{12}, A_{ij}]_* \bullet P_2)$, i.e., $\phi(T_{12} X_{22} + X_{22}^* T_{12}^*) = \phi(T_{12} A_{22} + A_{22}^* T_{12}^*)$. By the injectivity of ϕ, we obtain $T_{12} X_{22} + X_{22}^* T_{12}^* = T_{12} A_{22} + A_{22}^* T_{12}^*$ for all $T_{12} \in \mathcal{A}_{12}$. By the primeness of $\mathcal{A}$, we get $X_{22} = A_{22}$. In the same manner, we obtain $X_{11} = A_{11}$.

Claim 3. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $\phi(A_{ij} + B_{ij}) = \phi(A_{ij}) + \phi(B_{ij})$ for all $A_{ij} \in \mathcal{A}_{ij}$ and $B_{ij} \in \mathcal{A}_{ij}$.

It follows from $A_{ij} + B_{ij} + A_{ij}^* + B_{ij} A_{ij}^* = [-\frac{i}{2} I, iP_i + iA_{ij}]_* \bullet (P_j + B_{ij})$ and Claim 2 that

$$
\begin{aligned}
& \phi(A_{ij} + B_{ij}) + \phi(A_{ij}^*) + \phi(B_{ij} A_{ij}^*) \\
= \; & \phi(A_{ij} + B_{ij} + A_{ij}^* + B_{ij} A_{ij}^*) \\
= \; & \phi([-\frac{i}{2} I, iP_i + iA_{ij}]_* \bullet (P_j + B_{ij})) \\
= \; & [\phi(-\frac{i}{2} I), \phi(iP_i + iA_{ij})]_* \bullet \phi(P_j + B_{ij}) \\
= \; & [\phi(-\frac{i}{2} I), \phi(iP_i) + \phi(iA_{ij})]_* \bullet (\phi(P_j) + \phi(B_{ij})) \\
= \; & \phi([-\frac{i}{2} I, iP_i]_* \bullet P_j) + \phi([-\frac{i}{2} I, iP_i]_* \bullet B_{ij}) \\
& + \phi([-\frac{i}{2} I, iA_{ij}]_* \bullet P_j) + \phi([-\frac{i}{2} I, iA_{ij}]_* \bullet B_{ij}) \\
= \; & \phi(B_{ij}) + \phi(A_{ij} + A_{ij}^*) + \phi(B_{ij} A_{ij}^*) \\
= \; & \phi(B_{ij}) + \phi(A_{ij}) + \phi(A_{ij}^*) + \phi(B_{ij} A_{ij}^*),
\end{aligned}
$$

which indicates that $\phi(A_{ij} + B_{ij}) = \phi(A_{ij}) + \phi(B_{ij})$.

Claim 4. Let $i \in \{1, 2\}$. Then $\phi(A_{ii} + B_{ii}) = \phi(A_{ii}) + \phi(B_{ii})$ for all $A_{ii} \in \mathcal{A}_{ii}$ and $B_{ii} \in \mathcal{A}_{ii}$.

Choose $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{A}$ such that $\phi(X) = \phi(A_{ii}) + \phi(B_{ii})$. We obtain

$$\phi(X_{ij} + X_{ij}^*) = \phi([P_i, X]_* \bullet P_j) = \phi([P_i, A_{ii}]_* \bullet P_j) + \phi([P_i, B_{ii}]_* \bullet P_j) = 0.$$

Thus we get $X_{ij} = 0$. In the same manner, $X_{ji} = 0$. For every $T_{ij} \in \mathcal{A}_{ij}, i \neq j$, we have

$$\phi(T_{ij} X_{jj} + X_{jj}^* T_{ij}^*) = \phi([T_{ij}, X]_* \bullet P_j) = \phi([T_{ij}, A_{ii}]_* \bullet P_j) + \phi([T_{ij}, B_{ii}]_* \bullet P_j) = 0,$$

which implies that $T_{ij} X_{jj} = X_{jj}^* T_{ij}^* = 0$. By the primeness of $\mathcal{A}$, we obtain $X_{jj} = 0$. Therefore,

$$\phi(X_{ii}) = \phi(A_{ii}) + \phi(B_{ii}). \tag{1}$$

For every $T_{ij} \in \mathcal{A}_{ij}, i \neq j$, it follows from Claims 2 and 3 that

$$\begin{aligned} & \phi(X_{ii}T_{ij} + T_{ij}^*X_{ii}^*) = \phi([X, T_{ij}]_* \bullet P_j) \\ = \ & \phi([A_{ii}, T_{ij}]_* \bullet P_j) + \phi([B_{ii}, T_{ij}]_* \bullet P_j) \\ = \ & \phi(A_{ii}T_{ij} + T_{ij}^*A_{ii}^*) + \phi(B_{ii}T_{ij} + T_{ij}^*B_{ii}^*) \\ = \ & \phi(A_{ii}T_{ij}) + \phi(T_{ij}^*A_{ii}^*) + \phi(B_{ii}T_{ij}) + \phi(T_{ij}^*B_{ii}^*) \\ = \ & \phi(A_{ii}T_{ij} + B_{ii}T_{ij}) + \phi(T_{ij}^*A_{ii}^* + T_{ij}^*B_{ii}^*) \\ = \ & \phi(A_{ii}T_{ij} + B_{ii}T_{ij} + T_{ij}^*A_{ii}^* + T_{ij}^*B_{ii}^*), \end{aligned}$$

which indicates that $X_{ii} = A_{ii} + B_{ii}$. This together with Eq. (1) shows that $\phi(A_{ii} + B_{ii}) = \phi(A_{ii}) + \phi(B_{ii})$.
Claim 5. ϕ is additive.

By Claims 2–4, ϕ is additive.
Claim 6. $\phi(\mathbb{R}I) = \mathbb{R}I, \phi(\mathbb{C}I) = \mathbb{C}I$ and ϕ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{R}$ be arbitrary. It is easily seen that

$$0 = \phi([\lambda I, B]_* \bullet C) = [\phi(\lambda I), \phi(B)]_* \bullet \phi(C)$$

holds true for any $B, C \in \mathcal{A}$. Since ϕ is surjective, by Lemma 1.1, which indicates that

$$[\phi(\lambda I), \phi(B)]_* \in \mathrm{i}\mathbb{R}I.$$

Then $[\phi(\lambda I), B]_* \in \mathbb{C}I$ for any $B \in \mathcal{B}$. We obtain from Lemma 1.2 that $\phi(\lambda I) \in \mathbb{C}I$, so exists $\lambda_0 \in \mathbb{C}$ such that $(\lambda_0 - \overline{\lambda_0})B \in \mathbb{C}I$ for any $B \in \mathcal{B}$, then $\phi(\lambda I) \in \mathbb{R}I$. Note that ϕ^{-1} has the same properties as ϕ. In the same manner, if $\phi(A) \in \mathbb{R}I$, then $A \in \mathbb{R}I$. Therefore, $\phi(\mathbb{R}I) = \mathbb{R}I$.

Due to $\phi(\mathbb{R}I) = \mathbb{R}I$, exists $\lambda \in \mathbb{R}$ such that $\phi(\lambda I) = I$. For any $A = A^* \in \mathcal{A}$ and $B \in \mathcal{A}$, we obtain

$$0 = \phi([A, \lambda I]_* \bullet B) = [\phi(A), I]_* \bullet \phi(B),$$

from the surjectivity of ϕ and Lemma 1.1, the above equation indicates $[\phi(A), I]_* \in \mathrm{i}\mathbb{R}I$. Then exists $\lambda \in \mathrm{i}\mathbb{R}$ such that $\phi(A)^* = \phi(A) + \lambda I$. However,

$$0 = \phi([A, A]_* \bullet B) = [\phi(A), \phi(A)]_* \bullet \phi(B)$$

for all $A = A^* \in \mathcal{A}$ and $B \in \mathcal{A}$. In the same manner, $[\phi(A), \phi(A)]_* \in \mathrm{i}\mathbb{R}I$. Then $\lambda\phi(A) \in \mathrm{i}\mathbb{R}I$. If $\lambda \neq 0$, then $\phi(A) \in \mathbb{R}I$. It follows from $\phi(\mathbb{R}I) = \mathbb{R}I$ that $A = A^* \in \mathbb{R}I$, which is contradiction. Thus $\lambda = 0$. Now we get that $\phi(A) = \phi(A)^*$. In the same manner, if $\phi(A) = \phi(A)^*$, then $A = A^* \in \mathcal{A}$. Therefore ϕ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{C}$ be arbitrary. For every $A = A^* \in \mathcal{A}$, we obtain

$$0 = \phi([A, \lambda I]_* \bullet B) = [\phi(A), \phi(\lambda I)]_* \bullet \phi(B)$$

for any $B \in \mathcal{A}$. By the surjectivity of ϕ and Lemma 1.1 again, the above equation indicates $[\phi(A), \phi(\lambda I)]_* \in \mathrm{i}\mathbb{R}I$. Due to $A = A^*$, we have $\phi(A) = \phi(A)^*$. Hence $[\phi(A), \phi(\lambda I)] \in \mathrm{i}\mathbb{R}I$. We obtain from Lemma 1.3 that $[\phi(A), \phi(\lambda I)] = 0$, and then $B\phi(\lambda I) = \phi(\lambda I)B$ for any $B = B^* \in \mathcal{B}$. Thus for any $B \in \mathcal{B}$, since $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, we get

$$B\phi(\lambda I) = \phi(\lambda I)B$$

for any $B \in \mathcal{B}$. Hence $\phi(\lambda I) \in \mathbb{C}I$. In the same manner, if $\phi(A) \in \mathbb{C}I$, then $A \in \mathbb{C}I$. Therefore, $\phi(\mathbb{C}I) = \mathbb{C}I$.
Claim 7. $\phi(\mathcal{P}(\mathcal{A})) = \mathcal{P}(\mathcal{B})$.

Fix a nontrivial projection $P \in \mathcal{P}(\mathcal{B})$. Based on Claim 6, exists $A = A^* \in \mathcal{A}$ such that $\phi(A) = P + \mathbb{R}I$. For any $B = B^* \in \mathcal{A}$ and $C \in \mathcal{A}$, we obtain

$$\begin{aligned} & \phi([A, B]_* \bullet C) = [\phi(A), \phi(B)]_* \bullet \phi(C) \\ = \ & [P, \phi(B)]_* \bullet \phi(C) = [([P, \phi(B)]_* \bullet P), P]_* \bullet \phi(C) \\ = \ & [([\phi(A), \phi(B)]_* \bullet \phi(A)), \phi(A)]_* \bullet \phi(C) = \phi([([A, B]_* \bullet A), A]_* \bullet C). \end{aligned}$$

By the injectivity of ϕ, we concur that $[([A,B]_* \bullet A), A]_* \bullet C = [A,B]_* \bullet C$ for all $C \in \mathcal{A}$, from Lemma 1.1, we obtain

$$[([A,B]_* \bullet A), A]_* - [A,B]_* \in i\mathbb{R}I \tag{2}$$

for all $B = B^* \in \mathcal{A}$. For every $X \in \mathcal{A}$, we have $X = X_1 + iX_2$, where $X_1 = \frac{X+X^*}{2}$ and $X_2 = \frac{X-X^*}{2i}$ are self-adjoint. From Eq. (2), we obtain $[A,[A,[A,X]]] - [A,X] \in \mathbb{C}I$, i.e.,

$$A^3X - 3A^2XA + 3AXA^2 - XA^3 - AX + XA \in \mathbb{C}I \tag{3}$$

for all $X \in \mathcal{A}$.

Let $\mathcal{U}$ be the group of unitary operators of $\mathcal{A}$ and let φ be the set of the functions $U \to f(U)$ defined on $\mathcal{U}$ with non-negative real values, zero except on a finite subset of $\mathcal{U}$ and such that $\sum_{U\in\mathcal{U}} f(U) = 1$. For $A \in \mathcal{A}$ and $f \in \varphi$, we define $f \cdot A = \sum_{U\in\mathcal{U}} f(U)UAU^*$.

For all $U \in \mathcal{U}$, by Eq. (3),

$$(A^3 - A)U - 3A^2UA + 3AUA^2 - U(A^3 - A) = \alpha I \tag{4}$$

for certain $\alpha \in \mathbb{C}I$. Multiplying by U^* from the right of Eq. (4) gives

$$A^3 - A - 3A^2UAU^* + 3AUA^2U^* - U(A^3 - A)U^* = \alpha U^*,$$

then $A^3 - A - 3A^2 f\cdot A + 3Af\cdot A^2 - f\cdot A^3 + f\cdot A = \alpha U^*$ for any $f \in \varphi$. Due to $\mathcal{A}$ is a factor, from [1, Lemma 5 (Part III, Chapter 5)], exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

$$A^3 - A - 3\lambda_1A^2 + 3\lambda_2A - (\lambda_3 - \lambda_1)I = \alpha U^*.$$

Thus $U(A^3-A)U^* - 3\lambda_1UA^2U^* + 3\lambda_2UAU^* - (\lambda_3-\lambda_1)I = \alpha U^*$ and then $f\cdot A^3 - f\cdot A - 3\lambda_1 f\cdot A^2 + 3\lambda_2 f\cdot A - (\lambda_3-\lambda_1)I = \alpha U^*$ for any $f \in \varphi$. From [1, Lemma 5 (Part III, Chapter 5)], we obtain $\alpha U^* = 0$ for any $U \in \mathcal{U}$. Hence $\alpha = 0$. Thus we obtain

$$(A^3 - A)U - 3A^2UA + 3AUA^2 - U(A^3 - A) = 0 \tag{5}$$

and

$$A^3 - A = 3\lambda_1A^2 - 3\lambda_2A + (\lambda_3 - \lambda_1)I \tag{6}$$

for any $U \in \mathcal{U}$. From Eqs. (5)–(6), we conclude that

$$(\lambda_1A^2 - \lambda_2A)U - A^2UA + AUA^2 - U(\lambda_1A^2 - \lambda_2A) = 0. \tag{7}$$

Multiplying by AU^* from the right of Eq. (7) gives

$$(\lambda_1A^2 - \lambda_2A)UAU^* - A^2UA^2U^* + AUA^3U^* - U(\lambda_1A^2 - \lambda_2A)AU^* = 0$$

for any $U \in \mathcal{U}$. Thus

$$(\lambda_1A^2 - \lambda_2A)f\cdot A - A^2 f\cdot A^2 + Af\cdot A^3 - \lambda_1 f\cdot A^3 + \lambda_2 f\cdot A^2 = 0$$

for any $f \in \varphi$. By applying [1, Lemma 5 (Part III, Chapter 5)] again, we obtain

$$\lambda_1(\lambda_1A^2 - \lambda_2A) - \lambda_2A^2 + \lambda_3A + (\lambda_2^2 - \lambda_1\lambda_3)I = 0,$$

i.e.,

$$(\lambda_1^2 - \lambda_2)A^2 + (\lambda_3 - \lambda_1\lambda_2)A + (\lambda_2^2 - \lambda_1\lambda_3)I = 0. \tag{8}$$

If $\lambda_2 = \lambda_1^2$, we obtain from $\phi(\mathbb{C}I) = \mathbb{C}I$ and $\phi(A) = P + \mathbb{R}I \notin \mathbb{C}I$ that $A \notin \mathbb{C}I$, then $\lambda_3 = \lambda_1\lambda_2 = \lambda_1^3$. From Eq. (6), we have $(A - \lambda_1I)^3 = A - \lambda_1I$. Take $B = A - \lambda_1I$, we obtain

$$B^3 = B \text{ and } [B,[B,[B,X]]] = [B,X]$$

for any $X \in \mathcal{A}$, which indicates that

$$B^2XB - BXB^2 = 0 \tag{9}$$

for any $X \in \mathcal{A}$. Take $E_1 = \frac{1}{2}(B^2 + B)$ and $E_2 = \frac{1}{2}(B^2 - B)$. We obtain from $B^3 = B$ that E_1 and E_2 are idempotents of $\mathcal{A}$, then

$$B = E_1 - E_2, \ \ B^2 = E_1 + E_2, \ \ E_1E_2 = E_2E_1 = 0.$$

This along with Eq. (9) shows that $E_1XE_2 = 0$ for any $X \in \mathcal{A}$. Thus $E_1 = 0$ or $E_2 = 0$. Therefore $A = \lambda_1 I + E_1$ or $A = \lambda_1 I - E_2$.

If $\lambda_2 \neq \lambda_1^2$, from Eq. (8), we obtain $A^2 = \lambda A + \mu I$ for certain $\lambda, \mu \in \mathbb{C}$. This along with Eq. (5) indicates that

$$(\lambda^2 + 4\mu - 1)(AU - UA) = 0 \tag{10}$$

for any $U \in \mathcal{U}$. From $A \notin \mathbb{C}I$, we obtain $AU - UA \neq 0$ for some $U \in \mathcal{U}$. By Eq. (10), we obtain $\lambda^2 + 4\mu - 1 = 0$. Take $E = A + \frac{1}{2}(1-\lambda)I$, we have

$$\begin{aligned} E^2 &= A^2 + (1-\lambda)A + \frac{1}{4}(1-\lambda)^2 I = \lambda A + \mu I + (1-\lambda)A + \frac{1}{4}(1-\lambda)^2 I \\ &= A + \frac{1}{4}(\lambda^2 + 4\mu - 2\lambda + 1)I = A + \frac{1}{2}(1-\lambda)I = E. \end{aligned}$$

Therefore $A = \frac{1}{2}(\lambda - 1)I + E$. Since $A = A^*$, then $A = \alpha I + E, \alpha \in \mathbb{R}I, E \in \mathcal{P}(\mathcal{A})$. If $E = 0$ or $E = I$, from $\phi(A) = P + \mathbb{R}I$, we obtain $\phi(\mathbb{R}I) = P + \mathbb{R}I$. It follows $\phi(\mathbb{R}I) = \mathbb{R}I$ that $P = 0$ or $P = I$, since P is a nontrivial projection, which is a contradiction. Thus, A is the sum of a real number and a nontrivial projection of $\mathcal{A}$. Applying the same argument to ϕ^{-1}, we can obtain the reverse inclusion and $\phi(\mathcal{P}(\mathcal{A}) + \mathbb{R}I) = \mathcal{P}(\mathcal{B}) + \mathbb{R}I$. By Claims 5 and 6, we obtain $\phi(\mathcal{P}(\mathcal{A})) = \mathcal{P}(\mathcal{B})$.

remark 1. Since $[P_1, B]_* \bullet C = [B, P_2]_* \bullet C$ for all $B = B^* \in \mathcal{A}$ and $C \in \mathcal{A}$, from Claim 7, we obtain

$$[Q_1, \phi(B)]_* \bullet \phi(C) = [\phi(B), Q_2]_* \bullet \phi(C),$$

where $Q_i \in \mathcal{P}(\mathcal{B}), i = 1, 2$. The surjectivity of ϕ indicates that $[Q_1, \phi(B)]_* - [\phi(B), Q_2]_* \in \mathrm{i}\mathbb{R}I$. It follows from Claim 6 that $[Q_1 + Q_2, B] \in \mathrm{i}\mathbb{R}I$ holds true for all $B = B^* \in \mathcal{B}$. By Lemma 1.3, $[Q_1 + Q_2, B] = 0$. Thus for every $B \in \mathcal{B}$, because $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, we get $[Q_1 + Q_2, B] = 0$ for all $B \in \mathcal{B}$. From this, exists $\lambda \in \mathbb{R}$ such that

$$Q_1 + Q_2 = \lambda I.$$

Multiplying by Q_1 and Q_2 from the left and right respectively in the above equation, we obtain $Q_1 + Q_1Q_2 = \lambda Q_1$ and $Q_1Q_2 + Q_2 = \lambda Q_2$. Therefore, we can concur that $(1-\lambda)(Q_1 - Q_2) = 0$ by subtracting the above two equations. By the injectivity of ϕ, exists $P_1 \neq P_2$ such that $Q_1 \neq Q_2$. Thus $\lambda = 1$ and then $Q_2 = I - Q_1$.

Claim 8. $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}, \phi(\mathcal{A}_{jj}) \subseteq \mathcal{B}_{jj}, 1 \leq i \neq j \leq 2$.

Let $i, j \in \{1, 2\}$ with $i \neq j$ and $A_{ij} \in \mathcal{A}_{ij}$. By the fact $\mathrm{i}A_{ij} = [\frac{\mathrm{i}}{2}I, P_i]_* \bullet A_{ij}$, we obtain

$$\phi(\mathrm{i}A_{ij}) = (\phi(\frac{\mathrm{i}}{2}I) - \phi(\frac{\mathrm{i}}{2}I)^*)Q_i\phi(A_{ij}) + (\phi(\frac{\mathrm{i}}{2}I)^* - \phi(\frac{\mathrm{i}}{2}I))\phi(A_{ij})Q_i.$$

From this and Remark 1, we get $Q_i\phi(\mathrm{i}A_{ij})Q_i = Q_j\phi(\mathrm{i}A_{ij})Q_j = 0$. Thus

$$\phi(\mathrm{i}A_{ij}) = Q_i\phi(\mathrm{i}A_{ij})Q_j + Q_j\phi(\mathrm{i}A_{ij})Q_i. \tag{11}$$

For every $B \in \mathcal{A}$, we obtain from the fact $[\mathrm{i}A_{ij}, P_i]_* \bullet B = 0$ that $[\phi(\mathrm{i}A_{ij}), Q_i]_* \bullet \phi(B) = 0$. Thus $[\phi(\mathrm{i}A_{ij}), Q_i]_* \in \mathrm{i}\mathbb{R}I$, which together with Eq. (11) indicates that $Q_j\phi(\mathrm{i}A_{ij})Q_i - Q_i\phi(\mathrm{i}A_{ij})^*Q_j \in \mathrm{i}\mathbb{R}I$. Multiplying by Q_j and Q_i from the left and right respectively in the above equation, we have $Q_j\phi(\mathrm{i}A_{ij})Q_i = 0$. It follows from Eq. (11) that $\phi(\mathrm{i}A_{ij}) = Q_i\phi(\mathrm{i}A_{ij})Q_j$. Since A_{ij} is arbitrary, we obtain $\phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$. Applying the same argument to ϕ^{-1}, we obtain $\mathcal{B}_{ij} \subseteq \phi(\mathcal{A}_{ij})$. Thus $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}, i \neq j$.

Let $A_{jj} \in \mathcal{A}_{jj}$ and $i \neq j$. It follows from Claim 7 and Remark 1 that

$$0 = \phi([P_i, A_{jj}]_* \bullet P_j) = [Q_i, \phi(A_{jj})]_* \bullet Q_j = Q_i\phi(A_{jj})Q_j + Q_j\phi(A_{jj})^*Q_i$$

and

$$0 = \phi([P_j, A_{jj}]_* \bullet P_i) = [Q_j, \phi(A_{jj})]_* \bullet Q_i = Q_j\phi(A_{jj})Q_i + Q_i\phi(A_{jj})^*Q_j,$$

which indicates that $Q_i\phi(A_{jj})Q_j = Q_j\phi(A_{jj})Q_i = 0$. Now we obtain

$$\phi(A_{jj}) = Q_i\phi(A_{jj})Q_i + Q_j\phi(A_{jj})Q_j. \tag{12}$$

For every $A_{ji} \in \mathcal{A}_{ji}$ and $C \in \mathcal{A}$, we have $T_{ji} = \phi(A_{ji}) \in \mathcal{A}_{ji}$. Therefore

$$0 = \phi([A_{ji}, A_{jj}]_* \bullet C) = [T_{ji}, \phi(A_{jj})]_* \bullet \phi(C).$$

Using the surjectivity of ϕ, the above equation indicates $[T_{ji}, \phi(A_{jj})]_* \in \mathrm{i}\mathbb{R}I$. It follows from Eq. (12) that

$$T_{ji}\phi(A_{jj})Q_i - Q_i\phi(A_{jj})T_{ji}^* \in \mathrm{i}\mathbb{R}I. \tag{13}$$

By Remark 1, multiplying by Q_j and Q_i from the left and right respectively in Eq. (13), we can get that $T_{ji}\phi(A_{jj})Q_i = 0$ for all $T_{ji} \in \mathcal{B}_{ji}$. By the primeness of $\mathcal{B}$, we obtain that $Q_i\phi(A_{jj})Q_i = 0$, thus $\phi(\mathcal{A}_{jj}) \subseteq \mathcal{B}_{jj}$.
Claim 9. $\phi(AB) = \phi(A)\phi(B)$ for all $A, B \in \mathcal{A}$.

It follows from Remark 1 and Claim 8 that

$$\phi([P_i, A_{ij}]_* \bullet B_{ji}) = [\phi(P_i), \phi(A_{ij})]_* \bullet \phi(B_{ji}) = [Q_i, \phi(A_{ij})]_* \bullet \phi(B_{ji}).$$

Thus

$$\phi(A_{ij}B_{ji}) = \phi(A_{ij})\phi(B_{ji}). \tag{14}$$

For $T_{ji} \in \mathcal{B}_{ji}$, we have $X_{ji} = \phi^{-1}(T_{ji}) \in \mathcal{A}_{ji}$ by Claim 8. Therefore

$$\phi(A_{ii}B_{ij})T_{ji} = \phi(A_{ii}B_{ij}X_{ji}) = \phi([A_{ii}, B_{ij}]_* \bullet X_{ji}) = \phi(A_{ii})\phi(B_{ij})T_{ji}.$$

By the primeness of $\mathcal{B}$, we obtain

$$\phi(A_{ii}B_{ij}) = \phi(A_{ii})\phi(B_{ij}). \tag{15}$$

It follows from Eqs. (14)–(15) that

$$\phi(A_{ij}B_{jj})T_{ji} = \phi(A_{ij}B_{jj}X_{ji}) = \phi(A_{ij})\phi(B_{jj}X_{ji}) = \phi(A_{ij})\phi(B_{jj})T_{ji}.$$

In the same manner, we obtain

$$\phi(A_{ij}B_{jj}) = \phi(A_{ij})\phi(B_{jj}). \tag{16}$$

From Eq. (15), we have

$$\phi(A_{jj}B_{jj})T_{ji} = \phi(A_{jj}B_{jj}X_{ji}) = \phi(A_{jj})\phi(B_{jj}X_{ji}) = \phi(A_{jj})\phi(B_{jj})T_{ji}.$$

Thus

$$\phi(A_{jj}B_{jj}) = \phi(A_{jj})\phi(B_{jj}). \tag{17}$$

From Eqs. (14)–(17) and Claim 5, we obtain $\phi(AB) = \phi(A)\phi(B)$ for all $A, B \in \mathcal{A}$.
Claim 10. ϕ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

It follows from Claims 5 and 9 that ϕ is a ring isomorphism. By Claim 6, exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\phi(I) = \lambda I$. By the equality $\phi(I^3) = \phi(I)^3$, we concur that $\phi(I) = I$ or $\phi(I) = -I$. In the rest of this section, we deal with these two cases respectively.
Case 1. $\phi(I) = I$.

For every rational number q, we obtain $\phi(qI) = qI$. Take A be a positive element in $\mathcal{A}$. Then $A = B^2, B^* = B \in \mathcal{A}$. It follows that $\phi(A) = \phi(B)^2$ and $\phi(B) = \phi(B)^*$. We concur $\phi(A)$ is positive, i.e., ϕ preserves positive elements.

Take $\lambda \in \mathbb{R}$. Choose sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. From $a_nI \leq \lambda I \leq b_nI$, we concur $a_nI \leq \phi(\lambda I) \leq b_nI$. Taking the limit, we get $\phi(\lambda I) = \lambda I$ for any $\lambda \in \mathbb{R}$. Then for every $A \in \mathcal{A}$, we obtain $\phi(\lambda A) = \phi((\lambda I)A) = \phi(\lambda I)\phi(A) = \lambda\phi(A)$.

For every $A \in \mathcal{A}$, it follows from $-\phi(A) = \phi(\mathrm{i}^2A) = \phi(\mathrm{i}I)^2\phi(A)$ that $\phi(\mathrm{i}I)^2 = -1$, which indicates that $\phi(\mathrm{i}I) = \mathrm{i}I$ or $\phi(\mathrm{i}I) = -\mathrm{i}I$. From Claim 9, we obtain that $\phi(\mathrm{i}A) = \mathrm{i}\phi(A)$ or $\phi(\mathrm{i}A) = -\mathrm{i}\phi(A)$ for all $A \in \mathcal{A}$.

For all $A \in \mathcal{A}, A = A_1 + \mathrm{i}A_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2\mathrm{i}}$ are self-adjoint elements. If $\phi(\mathrm{i}A) = \mathrm{i}\phi(A)$, then

$$\phi(A^*) = \phi(A_1 - \mathrm{i}A_2) = \phi(A_1) - \phi(\mathrm{i}A_2) = \phi(A_1) - \mathrm{i}\phi(A_2) = \phi(A_1)^* - \mathrm{i}\phi(A_2)^* = \phi(A_1)^* + (\mathrm{i}\phi(A_2))^* = \phi(A)^*.$$

In the same manner, if $\phi(\mathrm{i}A) = -\mathrm{i}\phi(A)$, we also obtain $\phi(A^*) = \phi(A)^*$. Therefore ϕ is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

Case 2. $\phi(I) = -I$.

Consider that the map $\psi : \mathcal{A} \to \mathcal{B}$ defined by $\psi(A) = -\phi(A)$ for all $A \in \mathcal{A}$. We concur that ψ satisfies $\psi([A, B]_* \bullet C) = [\psi(A), \psi(B)]_* \bullet \psi(C)$ for all $A, B, C \in \mathcal{A}$ and $\psi(I) = I$. From Case 1, ϕ is either the negative of a linear $*$-isomorphism or the negative of a conjugate linear $*$-isomorphism. □

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