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Regulated functions space $R(\mathbb{R}_+, \mathbb{R}^\infty)$ and its application to some infinite systems of fractional differential equations via family of measures of noncompactness

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Abstract. We study the solvability of following infinite systems of fractional boundary value problem

$$\begin{cases} {}^cD^\rho u_i(t) = f_i(t, u_i(t))), \ \rho \in (n-1, n), \ 0 < t < +\infty, \\ u_i(0) = 0, \ u_i^q(0) = 0, \ {}^cD^{\rho-1}u_i(\infty) = \displaystyle\sum_{j=1}^{m-2} \beta_j u_i(\xi_j). \end{cases}$$

The purpose of this work is to present a new family of measures of noncompactness in the regulated function spaces $R(\mathbb{R}_+, \mathbb{R}^\infty)$ on unbounded interval and a fixed point theorem of Darbo type. Finally, we give an example to show the effectiveness of the obtained result.

1. Introduction

Fractional differential equations (FDEs) rise in the fields of engineering, chemistry, physics, economics and etc., [22, 24, 25]. Also, some basic theory for the boundary value problems (BVP) of (FDEs) has been discussed in [7, 8, 17, 18].

The measure of noncompactness (MNC) which was first introduced by Kuratowski [16] is a powerful tool for studying IODEs. In recent times, the regular MNC for certain Banach and Fréchet spaces defined on an unbounded or a bounded interval and by applying fixed point theorems have many applications, see [2–5, 10, 11, 20, 23].

The implication of a regulated function was presented in twentieth-century [6]. Moreover, some researchers introduced this notion from different perspectives and represented some of its applications

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[12–14]. Particularly the approach offered in [12] seems to be transparent and appropriate. In, (2018) Banas [9] formulated a standard for relative compactness in regulated functions on closed interval $[a,b]$, so-called regulated functions, and proved that the mentioned criterion is tantamount to standard obtained by D. Frankova. Next, in (2019) Leszek Olszowy [21] build and investigate two arithmetically convenient MNC in the spaces of regulated functions $R(J)$ and $R(J,E)$.

The aim of this paper is to formulate standard relative compactness in the space of functions regulated on unbounded interval and investigate the multi-point (BVP) for the infinite systems of (FDEs)

$$\begin{cases} {}^cD^\rho u_i(t) = f_i(t,u_i(t)),\ \rho \in (n-1,n),\ 0 < t < +\infty, \\ u_i(0) = 0,\ u_i^q(0) = 0,\ {}^cD^{\rho-1}u_i(\infty) = \sum_{j=1}^{m-2} \beta_j u_i(\xi_j), \end{cases} \tag{1}$$

where ${}^cD^\rho$ and ${}^cD^{\rho-1}$ are the Caputo fractional derivatives, $n-1 < \rho \leq n$ $(2 < n)$, $q = 2,3,\dots,n-1$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \infty$, and $\beta_j > 0$, $j = 1,2,\dots,m-2$, $m \geq 3$ satisfy $0 < \sum_{j=1}^{m-2} \beta_j \xi_j^{\rho-1} < \Gamma(\rho)$. via a new family of MNC in the regulated function space $R(\mathbb{R}_+,\mathbb{R}^\infty)$, using a fixed point theorem of Darbo type. Now, we organize the paper as follows: Section 2 consists of some related preliminary material. Section 3 to characterize the compact subsets of $R(\mathbb{R}_+,\mathbb{R}^\infty)$ and we present a new family of MNC in this space, and we prove a version of Darbo's fixed point theorem in $R(\mathbb{R}_+,\mathbb{R}^\infty)$. Finally, we give existence result for problem (1) with an example.

2. preliminaries

Let $(\Upsilon, \|\cdot\|)$ be a real Banach space containing zero element. We mean by $D(x,r)$ the closed ball centered at x with radius r. For $\emptyset \neq \mathcal{V} \subset \Upsilon$, the symbols $\overline{\mathcal{V}}$ and $\mathrm{Conv}\mathcal{V}$ denote the closure and closed convex hull of $\mathcal{V}$, respectively. We denote by $\mathfrak{M}_\Upsilon$ the family of all non-empty, bounded subsets of Υ and by $\mathfrak{N}_\Upsilon$ its subfamily consisting of non-empty relatively compact subsets of Υ.

Theorem 2.1. *([1]) Let $\emptyset \neq G \subseteq U$ be convex of Hausdorff locally convex linear topological space U and $H : G \to U$ be a continuous mapping so that*

$$H(G) \subseteq B \subseteq G,$$

with B compact. Then H has at least one fixed point.

Definition 2.2. *([22]) The fractional integral of order ρ is defined by*

$$I^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_0^t \frac{f(s)}{(t-s)^{1-\rho}} ds,\ \rho > 0,$$

that $\Gamma(.)$ is the gamma function.

Definition 2.3. *([22]) For at least n-times continuously differentiable function $f : [0,\infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\rho > 0$ is defined by*

$${}^cD^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\rho-n+1}} ds,$$

where $n-1 = [\rho]$.

Lemma 2.4. *[19] Let $f(t) \in L^1(\mathbb{R}_+)$ be a continuous function. Then the boundary value problem of FDEs*

$$\begin{cases} {}^cD^\rho u(t) = f(t),\ \rho \in (n-1,n),\ 0 < t < +\infty, \\ u(0) = 0,\ u^q(0) = 0,\ {}^cD^{\rho-1}u(+\infty) = \sum_{j=1}^{m-2} \beta_j u(\xi_j), \end{cases}$$

has a unique solution

$$u(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s) ds + \frac{t}{\sum_{j=1}^{m-2} \beta_j \xi_j} \int_0^\infty f(s) ds - \frac{t \sum_{j=1}^{m-2} \beta_j}{\Gamma(\rho) \sum_{j=1}^{m-2} \beta_j \xi_j} \int_0^{\xi_j} (\xi_j - s)^{\rho-1} f(s) ds.$$

Definition 2.5. *[9] The function $y : [c, d] \to \mathbb{R}$ is regulated function if for every $\tau \in [c, d)$ the right-sided limit $y(\tau^+) := \lim_{s\to\tau^+} y(s)$ exists and for every $\tau \in (c, d]$ the left-sided limit $y(\tau^-) := \lim_{s\to\tau^-} y(s)$ exists.*

Theorem 2.6. *[9] Suppose that $V \subseteq R([c, d])$ is bounded. The set V is relatively compact in $R([c, d])$ iff V is equiregulated on $[c, d]$ i.e $(a) - (b)$ hold:*
(a) $\forall\varepsilon > 0, \exists\delta > 0$, so that $\forall v \in V, \tau \in (c, d]$ and $\varsigma, \nu \in (\tau - \delta, t) \cap [c, d]$, we have $|v(\varsigma) - v(\nu)| \le \varepsilon$.
(b) $\forall\varepsilon > 0, \exists\delta > 0$, so that $\forall v \in V, \tau \in [c, d)$ and $\varsigma, \nu \in (\tau, \tau + \delta) \cap [c, d]$, we have $|v(\varsigma) - v(\nu)| \le \varepsilon$.

Firstly, we remind the Fréchet space $\mathbb{R}^\infty$ the linear space of all real sequences equipped with the distance

$$d_{\mathbb{R}^\infty}(v, w) = \sup\Big\{\frac{1}{2^j}\frac{|v_j - w_j|}{(1 + |v_j - w_j|)} : j \in \mathbb{N}\Big\},$$

for $v = (v_j), w = (w_j) \in \mathbb{R}^\infty$.
Now, we denote by $R([0, T], \mathbb{R}^\infty)$ the space consisting of all regulated function defined on $[0, T]$ with values in the space $\mathbb{R}^\infty$.
For $v = (v_j(\tau)) \in R([0, T], \mathbb{R}^\infty)$, we put $\pi_j(v) = v_j$. Obviously $\pi_j(v) \in R([0, T], \mathbb{R})$.
If $V \subset R([0, T], \mathbb{R}^\infty)$ then for a fixed $j \in \mathbb{N}$ we denote by $\pi_j(v)$ the following set situated in $R([0, T], \mathbb{R})$

$$\pi_j(V) = \{\pi_j(v) : \ v \in V\}.$$

The space $R([0, T], \mathbb{R}^\infty)$ will be equipped with the distance

$$d_{R_T}(v, w) = \sup\{d_{\mathbb{R}^\infty}(v(\tau), w(\tau)) : \ \tau \in [0, T]\},$$

for $v, w \in R([0, T], \mathbb{R}^\infty)$.

3. Main results

Let $R(\mathbb{R}_+, \mathbb{R}^\infty)$ be the space of all regulated function defined on $\mathbb{R}_+$ with values in $\mathbb{R}^\infty$. This space equipped with the family of seminorms

$$|v|_T = \sup\{|\pi_i(v)(\tau)| : \ i \le T, \ \tau \in [0, T]\},$$

and distance

$$d(v, w) = \sup\Big\{\frac{1}{2^T}\min\{1, |v - w|_T\} : T \in \mathbb{N}\Big\},$$

becomes a Fréchet space.

Remark 3.1.
(a) The sequence (v_n) is convergent to v in $R(\mathbb{R}_+, \mathbb{R}^\infty)$ if and only if $\pi_i(v_n)$ is uniformly convergent to $\pi_i(v)$ on $[0, T]$ for each $i, T \in \mathbb{N}$.
(b) The $\emptyset \ne V \subset R(\mathbb{R}_+, \mathbb{R}^\infty)$ is bounded if the functions of the set $\pi_i(V)$ are uniformly bounded on $[0, T]$ for each $i, T \in \mathbb{N}$ i.e.

$$\sup\{|\pi_i(v)| : \ \tau \in [0, T], \ v \in V\} < \infty \text{ for } i, T \in \mathbb{N}.$$

By similarly way in [9, 13] we can prove

Theorem 3.2. *Let $V \subseteq R(\mathbb{R}_+, \mathbb{R}^\infty)$ be bounded. The set V is relatively compact in $R(\mathbb{R}_+, \mathbb{R}^\infty)$ iff $\pi_i(V)$ are relatively compact in $R([0, T])$ for each $i, T \in \mathbb{N}$ i.e.*
(a) $\forall\varepsilon > 0, \exists\delta > 0$, such that $\forall v \in V, \tau \in (0, T]$ and $\varsigma, \nu \in (\tau - \delta, \tau) \cap [0, T]$, we have $|\pi_i(v)(\varsigma) - \pi_i(v)(\nu)| \le \varepsilon$, for $i, T \in \mathbb{N}$.
(b) $\forall\varepsilon > 0, \exists\delta > 0$, such that $\forall v \in V, \tau \in [0, T)$ and $\varsigma, \nu \in (\tau, \tau + \delta) \cap [0, T]$, we have $|\pi_i(v)(\varsigma) - \pi_i(v)(\nu)| \le \varepsilon$, for $i, T \in \mathbb{N}$.

Now, we define $\emptyset \ne \mathfrak{M}_{R(\mathbb{R}_+, \mathbb{R}^\infty)} \subseteq R(\mathbb{R}_+, \mathbb{R}^\infty)$ the family of bounded and $\emptyset \ne \mathfrak{N}_{R(\mathbb{R}_+, \mathbb{R}^\infty)} \subseteq R(\mathbb{R}_+, \mathbb{R}^\infty)$ the family of relatively compact.

Definition 3.3. *The family of mappings $\{\overline{\mu}\}_{T\in\mathbb{N}}$, $\overline{\mu} : \mathfrak{M}_{R(\mathbb{R}_+, \mathbb{R}^\infty)} \to \mathbb{R}_+$, is a family regular measures of noncompactness (MNC) in $R(\mathbb{R}_+, \mathbb{R}^\infty)$ if $1^\circ - 10^\circ$ hold:*

1° $\emptyset \neq \ker\{\overline{\mu}\} = \{V \in \mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)} : \overline{\mu}(V) = 0$ *for each* $T \in \mathbb{N}\} \subseteq \mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$.

2° $V \subset U$ *implies that* $\overline{\mu}(V) \leq \overline{\mu}(U)$ *for* $T \in \mathbb{N}$.

3° $\overline{\mu}(\overline{V}) = \overline{\mu}(V)$ *for* $T \in \mathbb{N}$.

4° $\overline{\mu}(ConvV) = \overline{\mu}(V)$ *for* $T \in \mathbb{N}$.

5° $\overline{\mu}(\vartheta V + (1-\vartheta)U) \leq \vartheta\overline{\mu}(V) + (1-\vartheta)\overline{\mu}(U)$ *for* $\vartheta \in [0,1]$, *and* $T \in \mathbb{N}$.

6° *If* $\{V_j\} \in \mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$, $V_j = \overline{V_j}$, $V_{j+1} \subset V_j$ *for* $j \in \mathbb{N}$ *and if* $\lim_{j\to\infty}\overline{\mu}(V_j) = 0$ *for each* $T \in \mathbb{N}$, *then* $V_\infty = \bigcap_{j=1}^{\infty} V_j \neq \emptyset$.

7° $\overline{\mu}(V \cup U) = \max\{\overline{\mu}(V), \overline{\mu}(U)\}$ *for* $T \in \mathbb{N}$.

8° $\overline{\mu}(V + U) \leq \overline{\mu}(V) + \overline{\mu}(U)$ *for* $T \in \mathbb{N}$.

9° $\overline{\mu}(\vartheta V) = |\vartheta|\overline{\mu}(V)$ *for* $T \in \mathbb{N}$ *and* $\vartheta \in \mathbb{R}$.

10° $\ker\{\overline{\mu}\} = \mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$ *for* $T \in \mathbb{N}$.

Assume that $p_i : \mathbb{R}_+ \to (0,\infty)$ $(i \in \mathbb{N})$ is a sequence of functions. for $Z \in \mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$ and $T \in \mathbb{N}$ putting

$$\omega_T^-(\pi_i(z),\tau,\varepsilon) = \sup\{|\pi_i(z)(u) - \pi_i(z)(v)| : u,v \in (\tau-\varepsilon,\tau) \cap [0,T]\},\ \tau \in (0,T],$$
$$\omega_T^+(\pi_i(z),\tau,\varepsilon) = \sup\{|\pi_i(z)(u) - \pi_i(z)(v)| : u,v \in (\tau,\tau+\varepsilon) \cap [0,T]\},\ \tau \in [0,T),$$

The quantities $\omega_T^-(\pi_i(z),\tau,\varepsilon)$ and $\omega_T^+(\pi_i(z),\tau,\varepsilon)$ can be interpreted as left hand and right hand sided moduli of convergence of the function z at the point τ, for $T \in \mathbb{N}$. Further,

$$\omega_T^-(\pi_i(Z),\tau,\varepsilon) = \sup\{\omega_T^-(\pi_i(z),\tau,\varepsilon) : z \in Z\},\ \tau \in (0,T],$$
$$\omega_T^+(\pi_i(Z),\tau,\varepsilon) = \sup\{\omega_T^+(\pi_i(z),\tau,\varepsilon) : z \in Z\},\ \tau \in [0,T),$$

and

$$\omega_T^-(\pi_i(Z),\varepsilon) = \sup_{\tau\in(0,T]} \omega_T^-(\pi_i(Z),\tau,\varepsilon),$$
$$\omega_T^+(\pi_i(Z),\varepsilon) = \sup_{\tau\in[0,T)} \omega_T^+(\pi_i(Z),\tau,\varepsilon),$$
$$\omega_T^-(\pi_i(Z)) = \lim_{\varepsilon\to0^+} \omega_T^-(\pi_i(Z),\varepsilon),$$
$$\omega_T^+(\pi_i(Z)) = \lim_{\varepsilon\to0^+} \omega_T^+(\pi_i(Z),\varepsilon),$$

Now, we define

$$\overline{\omega_T^-}(Z) = \sup\{p_i(T)\omega_T^-(\pi_i(Z)),\ i \in \mathbb{N}\},\ \tau \in (0,T],$$
$$\overline{\omega_T^+}(Z) = \sup\{p_i(T)\omega_T^+(\pi_i(Z)),\ i \in \mathbb{N}\},\ \tau \in [0,T),$$

and

$$\mu^-(Z) = \sup\{\overline{\omega_T^-}(Z),\ T \in \mathbb{N}\},$$
$$\mu^+(Z) = \sup\{\overline{\omega_T^+}(Z),\ T \in \mathbb{N}\},$$

Finally, we define

$$\overline{\mu}(Z) = \mu^-(Z) + \mu^+(Z). \tag{2}$$

Theorem 3.4. *The family of mappings* $\{\overline{\mu}\}_{T\in\mathbb{N}}$, $\overline{\mu} : \mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)} \to [0,+\infty)$ *given by (2) fulfills the assumptions* 1°-10° *of Definition 3.3.*

Proof. Assume that $Z \in \ker\{\overline{\mu}\}$, then $\overline{\mu}(Z) = \mu^-(Z) + \mu^+(Z) = 0$ since $\forall\, T, p_i(T) \neq 0$ therefore, $\lim_{\varepsilon\to0^+}\omega_T^-(\pi_i(Z),\varepsilon) = 0$ and $\lim_{\varepsilon\to0^+}\omega_T^+(\pi_i(Z),\varepsilon) = 0$. Fix an arbitrary $\eta > 0$. Then $\omega_T^-(\pi_i(Z),\varepsilon) < \frac{\eta}{2}$ and $\omega_T^+(\pi_i(Z),\varepsilon) < \frac{\eta}{2}$ for enough small $\varepsilon > 0$. So by definition of $\overline{\mu}(Z)$, we get

$$\omega_T^+(\pi_i(Z),\varepsilon)) + \omega_T^-(\pi_i(Z),\varepsilon) < \eta.$$

Hence, we have

$$\omega_T^-(\pi_i(z),\tau,\varepsilon)=\sup\{|\pi_i(z)(u)-\pi_i(z)(v)|:\ u,v\in(\tau-\varepsilon,\tau)\cap[0,T]\}<\frac{\eta}{2},\ \tau\in(0,T],$$

and

$$\omega_T^+(\pi_i(z),\tau,\varepsilon)=\sup\{|\pi_i(z)(u)-\pi_i(z)(v)|:\ u,v\in(\tau,\tau+\varepsilon)\cap[0,T]\}<\frac{\eta}{2},\ \tau\in[0,T),$$

$\forall\ z\in Z$ and $\forall\ T\in\mathbb{N}$. By Theorem 3.2, we deduce that the closure of Z is compact and $\ker\{\overline{\mu}\}\subseteq\mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$. So 1° holds.
The prove of 2° is clearly.
We prove 3°. Let $Z\in\mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$ and $z\in\overline{Z}$. So, $\exists$ a sequence $\{z_n\}\subseteq Z$ so that $\{z_n\}$ converges to z in $R(\mathbb{R}_+,\mathbb{R}^\infty)$. Thus for every $\xi>0$ $\exists$ $n_0\in\mathbb{N}$ so that $\forall$ $n\geq n_0$, $|\pi_i(z_n)-\pi_i(z)|_T\leq\xi$, for $T\in\mathbb{N}$. So, for each $\tau\in[0,T]$ we get

$$\lim_{n\to\infty}\pi_i z_n(\tau)=\pi_i z(\tau).$$

In addition, let us fix arbitrarily $\varepsilon>0$. So, for a fixed $\tau\in(0,T]$ and for $u,v\in(\tau-\varepsilon,\tau)\cap[0,T]$, we get

$$\lim_{n\to\infty}|\pi_i z_n(u)-\pi_i z_n(v)|=|\pi_i z(u)-\pi_i z(v)|,$$

as the sequence (z_n) is uniformly convergent to the function z on $[0,T]$ for $T\in\mathbb{N}$. So for each $\varepsilon>0$, we get

$$\mu^-(\overline{Z})\leq\mu^-(Z)+\varepsilon.$$

By taking $\varepsilon\to0$ and combined with the assumption 2° we have

$$\mu^-(\overline{Z})=\mu^-(Z).\tag{3}$$

Also, for a fixed $\tau\in[0,T)$ and for $u,v\in(\tau,\tau+\varepsilon)\cap[0,T]$, we obtain

$$\lim_{n\to\infty}|\pi_i z_n(u)-\pi_i z_n(v)|=|\pi_i z(u)-\pi_i z(v)|,$$

so $\mu^+(\overline{Z})\leq\mu^+(Z)$ and axiom 2° we obtain

$$\mu^+(\overline{Z})=\mu^+(Z),\tag{4}$$

by (3) and (4) we deduce $\overline{\mu}(\overline{Z})=\overline{\mu}(Z)$.
Now, for arbitrary functions $z,w\in R(\mathbb{R}_+,\mathbb{R}^\infty)$ we obtain

$$\omega_T^-(\pi_i(z+w),\tau,\varepsilon)\leq\omega_T^-(\pi_i(z),\tau,\varepsilon)+\omega_T^-(\pi_i(w),\tau,\varepsilon),\tag{5}$$

$$\omega_T^+(\pi_i(z+w),\tau,\varepsilon)\leq\omega_T^+(\pi_i(z),\tau,\varepsilon)+\omega_T^+(\pi_i(w),\tau,\varepsilon),\tag{6}$$

By (5) and (6) we have $\mu^+(Z+W)\leq\mu^+(Z)+\mu^+(W)$ and $\mu^-(Z+W)\leq\mu^-(Z)+\mu^-(W)$, and for arbitrary function $z\in R(\mathbb{R}_+,\mathbb{R}^\infty)$ and $\vartheta\in\mathbb{R}$, we have

$$\omega_T^-(\pi_i(\vartheta z),\tau,\varepsilon)=|\vartheta|\omega_T^-(\pi_i(z),\tau,\varepsilon),\tag{7}$$

$$\omega_T^+(\pi_i(\vartheta z),\tau,\varepsilon)=|\vartheta|\omega_T^+(\pi_i(z),\tau,\varepsilon).\tag{8}$$

And by (7) and (8) we have $\mu^+(\vartheta Z)=|\vartheta|\mu^+(Z)$ and $\mu^-(\vartheta Z)=|\vartheta|\mu^-(Z)$. So, we can easily see that for an arbitrary set $Z\in\mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$ and $\vartheta\in\mathbb{R}$

$$\overline{\mu}(Z+W)\leq\overline{\mu}(Z)+\overline{\mu}(W),\ \overline{\mu}(\vartheta Z)=|\vartheta|\overline{\mu}(Z).$$

Then, the axioms 7°, 8° and 9° hold.
By the same reasoning as above we have

$$\overline{\mu}(convZ)\leq\overline{\mu}(Z),$$

for an arbitrary set $Z\in\mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$. Combining the above inequality and axiom 2°, we obtain

$$\overline{\mu}(convZ)=\overline{\mu}(Z),$$

therefore assumption 4° holds, by similar way the assumption 5° holds.
We prove 6°, let $Z_j\in\mathfrak{M}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$, $Z_j=\overline{Z_j}$, $Z_{j+1}\subset Z_j$ for $j=1,2,\dots$ and $\lim_{j\to\infty}\overline{\mu}(Z_j)=0$ for each T. $\forall$ $j\in\mathbb{N}$, take an $z_j\in Z_j$. Claim: $F=\overline{\{z_j\}}$ is compact in $R(\mathbb{R}_+,\mathbb{R}^\infty)$. Suppose that $\varepsilon>0$ be fixed and take any $T\in\mathbb{N}$. Since $\lim_{j\to\infty}\overline{\mu}(Z_j)=0$, then $\exists\zeta\in\mathbb{N}$ sufficiently large so that for each $T\in\mathbb{N}$

$$\overline{\mu}(Z_\zeta)<\varepsilon.$$

Since $\forall\, T$, $p_i(T) \neq 0$, so, exists $\delta_1 > 0$ enough small so that

$$\omega_T^-(\pi_i(Z_m), \delta_1) < \varepsilon \ \forall\, m \geq \zeta.$$

and

$$\omega_T^+(\pi_i(Z_m), \delta_1) < \varepsilon \ \forall\, m \geq \zeta.$$

So, $\forall\, m \geq \zeta$ we have

$$\sup\{|\pi_i(z_m)(u) - \pi_i(z_m)(v)| : \ u, v \in (\tau - \delta_1, \tau) \cap [0, T]\} < \varepsilon \ \tau \in (0, T],$$

and

$$\sup\{|\pi_i(z_m)(u) - \pi_i(z_m)(v)| : \ u, v \in (\tau, \tau + \delta_1) \cap [0, T]\} < \varepsilon \ \tau \in [0, T).$$

Since the set $\{z_1, z_2, \ldots, z_{\zeta-1}\}$ is compact, then for each $j \in \{1, 2, \ldots, \zeta - 1\}$ $\exists \delta_2 > 0$ so that

$$\{|\pi_i(z_j)(u) - \pi_i(z_j)(v)| : \ u, v \in (\tau - \delta_2, \tau) \cap [0, T]\} < \varepsilon \ \tau \in (0, T],$$

and

$$\{|\pi_i(z_j)(u) - \pi_i(z_j)(v)| : \ u, v \in (\tau, \tau + \delta_2) \cap [0, T]\} < \varepsilon \ \tau \in [0, T),$$

Hence, by taking $\delta := \min\{\delta_1, \delta_2\}$ the assumptions of Theorem 3.2 hold so $\{z_j\}$ is relatively compact. Therefore, a subsequence $\{z_{n_j}\}$ and $z_0 \in R(\mathbb{R}_+, \mathbb{R}^\infty)$ exist such that $\{z_{n_j}\}$ converges to z_0. Since $z_j \in Z_j$, $Z_j = \overline{Z_j}$ and $Z_{j+1} \subset Z_j$ $\forall\, j \in \mathbb{N}$, we have

$$z_0 \in \bigcap_{j=1}^{\infty} Z_j = Z_\infty,$$

that completes the proof of 6°.

Finally, we check $\ker\{\overline{\mu}\} = \mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$. Take $T \in \mathbb{N}$ and $Z \in \mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$, then Z is relatively compact in $R(\mathbb{R}_+, \mathbb{R}^\infty)$. According Theorem 3.2, $\forall\, \varepsilon > 0\ \exists\, 0 < \delta' < \varepsilon$ so that

$$\{|\pi_i(z)(u) - \pi_i(z)(v)| : \ u, v \in (\tau - \delta', \tau) \cap [0, T]\} < \varepsilon \ \tau \in (0, T],$$

$\forall\, z \in Z$. By applying Theorem 3.2, for any $\varepsilon > 0\ \exists\, 0 < \delta'' < \varepsilon$ so that

$$\{|\pi_i(z)(u) - \pi_i(z)(v)| : \ u, v \in (\tau, \tau + \delta'') \cap [0, T]\} < \varepsilon \ \tau \in [0, T),$$

$\forall\, z \in Z$. Putting $\delta = \min\{\delta', \delta''\}$. Then, $\forall\, z \in Z$, we get

$$\omega_T^-(\pi_i(z), \tau, \delta) = \sup\{|\pi_i(z)(u) - \pi_i(z)(v)| : \ u, v \in (\tau - \delta, \tau) \cap [0, T]\} \leq \varepsilon \ \tau \in (0, T],$$
$$\omega_T^+(\pi_i(z), \tau, \delta) = \sup\{|\pi_i(z)(u) - \pi_i(z)(v)| : \ u, v \in (\tau, \tau + \delta) \cap [0, T]\} \leq \varepsilon \ \tau \in [0, T),$$

It in turn implies that

$$\overline{\mu}(Z) = \mu^-(Z) + \mu^+(Z) \leq 2\varepsilon.$$

Taking $\varepsilon \to 0$, then $\delta \to 0$ and $\overline{\mu}(Z) = 0$, $\forall\, T \in \mathbb{N}$. By condition 1°, we have $\ker\{\overline{\mu}\} = \mathfrak{N}_{R(\mathbb{R}_+,\mathbb{R}^\infty)}$. □

Theorem 3.5. *Let $\emptyset \neq C = \overline{C} \subseteq R(\mathbb{R}_+, \mathbb{R}^\infty)$ is bounded, convex and the mapping $F : C \to C$ is continuous. If for each $T \in \mathbb{N}\ \exists\, 0 \leq L_T < 1$ so that*

$$\overline{\mu}(FZ) \leq L_T \overline{\mu}(Z), \tag{9}$$

for each $Z \subset C$. Then F has at least one fixed point in the set C.

Proof. First, we define a sequence $\{C_m\}$ by taking $C_0 = C$ and $C_m = \mathrm{Conv}(FC_{m-1})$, $m \geq 1$. We have $C_1 = \mathrm{Conv}(FC_0) \subseteq C_0$, therefore by continuing this process we get

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots .$$

If $\overline{\mu}(C_N) = 0$ for some $N > 0$ and $\forall\, T$, then C_N is relatively compact and Theorem 2.1 grantees that F has a fixed point. Otherwise, let $T \geq 0$, so that $\overline{\mu}(C_m) \neq 0$ for any $m \geq 0$. From relation (9) we have

$$\overline{\mu}(C_{m+1}) = \overline{\mu}(\mathrm{Conv}(FC_m)) = \overline{\mu}(FC_m) \leq L_T \overline{\mu}(C_m). \tag{10}$$

Since $L_T \in [0, 1)$, then $\{\overline{\mu}(C_m)\}$ is a positive decreasing sequence of real numbers. So, there is an $r \geq 0$ so that $\overline{\mu}(C_m) \to r$ as $m \to \infty$. We show that $r = 0$. Suppose, to the contrary that $r > 0$. Then by (10) we get

$$\limsup_{m\to\infty} \overline{\mu}(C_{m+1}) \leq \limsup_{m\to\infty} L_T \overline{\mu}(C_m).$$

It enforces that $1 \leq L_T$, which is a contradiction. Consequently $r = 0$, and so $\overline{\mu}(C_m) \to 0$, as $m \to \infty$. Employing condition 6° of Definition 3.3, we deduce that $\emptyset \neq \bigcap_{m=1}^{\infty} C_m = C_\infty \subset C$ is convex and closed. Furthermore, C_∞ is invariant under F, and $C_\infty \in \ker\{\overline{\mu}\}$. By using Theorem 2.1 F has a fixed point. □

4. Application

In the following part, we prove the solvability of equation (1) in the Fréchet spaces $R(\mathbb{R}_+, \mathbb{R}^\infty)$. Finally, we give an example to show the usefulness of our result.
Assume that:
(i) The functions $f_i : \mathbb{R}_+ \times \mathbb{R}^\infty \to \mathbb{R}$ ($i \in \mathbb{N}$) are continuous and regulated and $\exists$ increasing functions $\varphi_i, \theta_i : \mathbb{R}_+ \to [0, +\infty)$ so that $\varphi_i(\tau) \to 0$, and $\theta_i(\tau) \to 0$ as $\tau \to 0$, $\varphi_i \in L^1([0, \infty))$ and the inequalities

$$|f_i(s, u_i) - f_i(s, v_i)| \le \varphi_i(|u_i - v_i|),$$

$$\int_0^\infty |f_i(s, u_i) - f_i(s, v_i)|ds \le M\theta_i(|u_i - v_i|),$$

$\forall\, s \in \mathbb{R}_+$, $u_i, v_i \in \mathbb{R}$ and $M > 0$ hold. Also

$$\overline{N} = \sup\{|f_i(s, 0)| : \ s \in [0, \infty), \ i \in \mathbb{N}\} < \infty, \text{ and } \overline{G} = \int_0^\infty |f_i(s, 0)|ds < \infty.$$

(ii) For each $T \in \mathbb{N}$, $\exists\, r_i(T) > 0$ that is a solution of the inequality

$$(\varphi_i(r_i(T)) + \overline{N})\Big(\frac{T^\rho}{\rho\Gamma(\rho)} + \frac{T(\sum_{j=1}^{m-2}\beta_j)\xi_j^\rho}{\rho\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\Big) + (M\theta_i(r_i(T)) + \overline{G})\frac{T}{\sum_{j=1}^{m-2}\beta_j\xi_j} \le r_i(T).$$

Theorem 4.1. *Under conditions (i) and (ii) the equation (1) has at least one solution in the $R(\mathbb{R}_+, \mathbb{R}^\infty)$.*

Proof. Define the operator $F : R(\mathbb{R}_+, \mathbb{R}^\infty) \to R(\mathbb{R}_+, \mathbb{R}^\infty)$ by:

$$\begin{aligned}(Fu)(t) &= (\pi_i(Fu)(t)) \\ &= \Big(\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1} f_i(s, u_i)ds + \frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^\infty f_i(s, u_i)ds - \frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j - s)^{\rho-1} f_i(s, u_i)ds\Big).\end{aligned}$$

where $u(t) = (u_i(t))_{i=1}^\infty \in R(\mathbb{R}_+, \mathbb{R}^\infty)$. First, we prove that $Fu \in R(\mathbb{R}_+, \mathbb{R}^\infty)$, for $u \in R(\mathbb{R}_+, \mathbb{R}^\infty)$. Select arbitrary $T \in \mathbb{N}$, $t \in [0, T]$ and $i \in \mathbb{N}$. By using assumption (i), we have

$$\begin{aligned}|\pi_i(Fu)(t)| &= \Big|\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1} f_i(s, u_i)ds + \frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^\infty f_i(s, u_i)ds - \frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j - s)^{\rho-1} f_i(s, u_i)ds\Big| \\ &\le \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}(|f_i(s, u_i) - f_i(s, 0)| + |f_i(s, 0)|)ds + \frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^\infty (|f_i(s, u_i) - f_i(s, 0)| + |f_i(s, 0)|)ds \\ &\quad + \frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j - s)^{\rho-1}(|f_i(s, u_i) - f_i(s, 0)| + |f_i(s, 0)|)ds \\ &\le \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}(\varphi_i(|u_i(s)|) + \overline{N})ds + \frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}(M\theta_i(|u_i(s)|) + \overline{G})\end{aligned}$$

$$+\frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j-s)^{\rho-1}(\varphi_i(|u_i(s)|)+\overline{N})ds$$

$$\leq (\varphi_i(|u_i(s)|)+\overline{N})\Big(\frac{t^\rho}{\rho\Gamma(\rho)}+\frac{t(\sum_{j=1}^{m-2}\beta_j)\xi_j^\rho}{\rho\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\Big)+(M\theta_i(|u_i(s)|)+\overline{G})\frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}.$$

So by supremum on t we obtain

$$|(Fu)|_T \leq (\varphi_i(|u_i|_T)+\overline{N})\Big(\frac{T^\rho}{\rho\Gamma(\rho)}+\frac{T(\sum_{j=1}^{m-2}\beta_j)\xi_j^\rho}{\rho\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\Big)+(M\theta_i(|u_i|_T)+\overline{G})\frac{T}{\sum_{j=1}^{m-2}\beta_j\xi_j}. \quad (11)$$

Also, for $u\in R(\mathbb{R}_+,\mathbb{R}^\infty)$, $t\in[0,T)$, $\varepsilon>0$ for $T\in\mathbb{N}$ and $t_1,t_2\in(t,t+\varepsilon)\cap[0,T]$, $t_1\leq t_2$. We get

$$\begin{aligned}
&|\pi_i(Fu)(t_2)-\pi_i(Fu)(t_1)|\\
&\leq \frac{1}{\Gamma(\rho)}\Big(\int_0^{t_1}((t_2-s)^{\rho-1}-(t_1-s)^{\rho-1})(|f_i(s,u_i)-f_i(s,0)|+|f_i(s,0)|)ds\\
&\quad+\int_{t_1}^{t_2}(t_2-s)^{\rho-1}(|f_i(s,u_i)-f_i(s,0)|+|f_i(s,0)|)ds\\
&\quad+\frac{|t_2-t_1|}{\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\infty}(|f_i(s,u_i)-f_i(s,0)|+|f_i(s,0)|)ds\\
&\quad+\frac{|t_2-t_1|\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j-s)^{\rho-1}(|f_i(s,u_i)-f_i(s,0)|+|f_i(s,0)|)ds\\
&\leq \frac{(\varphi_i(|u_i(s)|)+\overline{N})}{\Gamma(\rho)}\Big(\int_0^{t_1}(t_2-s)^{\rho-1}-(t_1-s)^{\rho-1}ds+\int_{t_1}^{t_2}(t_2-s)^{\rho-1}ds\Big)\\
&\quad+\frac{(M\theta_i(|u_i(s)|)+\overline{G})}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|+\frac{(\varphi_i(|u_i(s)|)+\overline{N})\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|\int_0^{\xi_j}(\xi_j-s)^{\rho-1}ds.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&|\pi_i(Fu)(t_2)-\pi_i(Fu)(t_1)|\\
&\leq \frac{(\varphi_i(|u_i(s)|)+\overline{N})}{\rho\Gamma(\rho)}\Big(2(t_2-t_1)^\rho+t_1^\rho-t_2^\rho+\frac{\xi_j^\rho\sum_{j=1}^{m-2}\beta_j}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|\Big)+\frac{(M\theta_i(|u_i(s)|)+\overline{G})}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|.
\end{aligned}$$

Since, $t_1,t_2\in(t,t+\varepsilon)\cap[0,T]$ so $|t_2-t_1|\to 0$, $(t_2-t_1)^\rho\to 0$ as $\varepsilon\to 0$, and we have used the fact that $t_1^\rho-t_2^\rho\leq 0$ (because $t_1\leq t_2$). Then we deduce

$$|\pi_i(Fu)(t_2)-\pi_i(Fu)(t_1)|\to 0. \quad (12)$$

Similarly, let us fix $t \in (0, T]$, $\varepsilon > 0$ for $T \geq 0$ and for $t_1, t_2 \in (t-\varepsilon, t) \cap [0, T]$ $(t_1 \leq t_2)$ we have

$$|\pi_i(Fu)(t_2) - \pi_i(Fu)(t_1)|$$

$$\leq \frac{(\varphi_i(|u_i(s)|) + \overline{N})}{\rho\Gamma(\rho)}\Big(2(t_2-t_1)^\rho + t_1^\rho - t_2^\rho + \frac{\xi_j^\rho \sum_{j=1}^{m-2}\beta_j}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2 - t_1|\Big) + \frac{(M\theta_i(|u_i(s)|) + \overline{G})}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2 - t_1|.$$

Since, $t_1, t_2 \in (t-\varepsilon, t) \cap [0, T]$ so $|t_2 - t_1| \to 0$, $(t_2 - t_1)^\rho \to 0$ as $\varepsilon \to 0$, and we have used the fact that $t_1^\rho - t_2^\rho \leq 0$ (because $t_1 \leq t_2$). Then we get

$$|\pi_i(Fu)(t_2) - \pi_i(Fu)(t_1)| \to 0. \tag{13}$$

So by (11), (12) and (13) we obtain $Fu \in R(\mathbb{R}_+, \mathbb{R}^\infty)$. Relation (11) implies that the operator F transforms of $R(\mathbb{R}_+, \mathbb{R}^\infty)$ into itself. Now, if we define the subset $B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$ of $R(\mathbb{R}_+, \mathbb{R}^\infty)$ by:

$$B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t)) = \{u = (u_i) \in R(\mathbb{R}_+, \mathbb{R}^\infty) : \ |u|_T \leq r_i(t) \text{ for, } t > 0\},$$

then the $\emptyset \neq B = \overline{B} \subseteq R(\mathbb{R}_+, \mathbb{R}^\infty)$ is bounded and convex and assumption (*ii*) ensures that F transforms $B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$ into itself.

Now, we prove that F is continuous on B. Fix $u = (u_i) \in B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$ and take a sequence $(u_{n,i}) \in B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$ such that $u_n = (u_{n,i}) \to u = (u_i)$. For $t \in [0, T]$, $T \in \mathbb{N}$ we get

$$|\pi_i(Fu_n)(t) - \pi_i(Fu)(t)|$$

$$\leq \Big|\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}(f_i(s, u_{n,i}) - f_i(s, u_i))ds\Big| + \frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}\Big|\int_0^\infty (f_i(s, u_{n,i}) - f_i(s, u_i))ds\Big|$$

$$+\frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\Big|\int_0^{\xi_j}(\xi_j - s)^{\rho-1}(f_i(s, u_{n,i}) - f_i(s, u_i))ds\Big|$$

$$\leq \frac{\varphi_i(|u_{n,i}(s) - u_i(s)|)}{\Gamma(\rho)}\Big(\int_0^t (t-s)^{\rho-1}ds + \frac{t\sum_{j=1}^{m-2}\beta_j}{\Gamma(\rho)\sum_{j=1}^{m-2}\beta_j\xi_j}\int_0^{\xi_j}(\xi_j - s)^{\rho-1}ds\Big)$$

$$+M\theta_i(|u_{n,i}(s) - u_i(s)|)\frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}$$

$$\leq \frac{\varphi_i(|u_{n,i}(s) - u_i(s)|)}{\rho\Gamma(\rho)}\Big(t^\rho + \frac{t(\sum_{j=1}^{m-2}\beta_j)\xi_j^\rho}{\sum_{j=1}^{m-2}\beta_j\xi_j}\Big) + M\theta_i(|u_{n,i}(s) - u_i(s)|)\frac{t}{\sum_{j=1}^{m-2}\beta_j\xi_j}.$$

Then we get

$$|(Fu_n) - (Fu)|_T \leq \frac{\varphi_i(|u_n - u)|_T}{\rho\Gamma(\rho)}\Big(T^\rho + \frac{T(\sum_{j=1}^{m-2}\beta_j)\xi_j^\rho}{\sum_{j=1}^{m-2}\beta_j\xi_j}\Big) + M\theta_i(|u_n - u|_T)\frac{T}{\sum_{j=1}^{m-2}\beta_j\xi_j}.$$

Since $u_n \to u$ and by condition (*i*) $\varphi_i(t) \to 0$, $\theta_i(t) \to 0$, as $t \to 0$. Then $(Fu_n) \to (Fu)$ *i.e.* F is continuous.

Eventually, we show that F satisfying the relation (9). Let $\emptyset \neq U \subseteq B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$ be bounded. Next, fix arbitrarily $t \in [0,T)$ and $\varepsilon > 0$. Select a function $u \in U$ and $t_1, t_2 \in (t, t+\varepsilon) \cap [0,T]$. Then, by (12) we have

$$\omega_T^+(\pi_i(Fu), t, \varepsilon) \leq \frac{(\varphi_i(|u_i|_T + \overline{N})}{\rho\Gamma(\rho)}\Big(2(t_2-t_1)^\rho + \frac{\xi_j^\rho \sum_{j=1}^{m-2}\beta_j}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|\Big) + \frac{(M\theta_i(|u_i|)_T + \overline{G})}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|.$$

Taking $\varepsilon \to 0$ we obtain

$$\omega_T^+(\pi_i(Fu), t) \leq 0. \tag{14}$$

Similarly, for $t \in (0,T]$ and $t_1, t_2 \in (t-\varepsilon, t) \cap [0,T]$ by virtue of (13) we have

$$\omega_T^-(\pi_i(Fu), t, \varepsilon) \leq \frac{(\varphi_i(|u_i|_T + \overline{N})}{\rho\Gamma(\rho)}\Big(2(t_2-t_1)^\rho + \frac{\xi_j^\rho \sum_{j=1}^{m-2}\beta_j}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|\Big) + \frac{(M\theta_i(|u_i|)_T + \overline{G})}{\sum_{j=1}^{m-2}\beta_j\xi_j}|t_2-t_1|.$$

Taking $\varepsilon \to 0$ we obtain

$$\omega_T^-(\pi_i(Fu), t) \leq 0. \tag{15}$$

By supremum on t of (14) and (15) we get

$$\omega_T^+(\pi_i(Fu)) \leq 0, \text{ and } \omega_T^-(\pi_i(Fu)) \leq 0.$$

Also, for $t \in (0,T]$ we get

$$\overline{\omega}_T^-(FU) = \sup\{p_i(T)\omega_T^-(\pi_i(FU)),\ i \in \mathbb{N}\} \leq 0,$$

and for $t \in [0,T)$ we get

$$\overline{\omega}_T^+(FU) = \sup\{p_i(T)\omega_T^+(\pi_i(FU)),\ i \in \mathbb{N}\} \leq 0.$$

Hence

$$\mu^-(FU) = \sup\{\overline{\omega}_T^-(FU),\ T > 0\} \leq 0,$$

and

$$\mu^+(FU) = \sup\{\overline{\omega}_T^+(FU),\ T > 0\} \leq 0.$$

Finally,

$$\overline{\mu}(FU) = \mu^-(FU) + \mu^+(FU) = 0,$$

or equivalently,

$$\overline{\mu}(FU) \leq L_T\overline{\mu}(U),$$

where $L_T = 0$. From Theorem 3.5, F has a fixed point $u(t) = u_i(t)$ in $R(\mathbb{R}_+, \mathbb{R}^\infty)$ belonging to the set $B_{R(\mathbb{R}_+,\mathbb{R}^\infty)}(0, r_i(t))$, which implies that the equation (1) has at least one solution in $R(\mathbb{R}_+, \mathbb{R}^\infty)$. □

Example 4.2. *Consider the following equation*

$$\begin{cases} {}^cD^{\frac{3}{2}}u_i(t) = \frac{\sin(u_i(t)+1)\cos(t+3)}{1+s^2}\sum_{k=i}^{i+1}\frac{1}{(k+1)k}, \\ u(0) = 0,\ u''(0) = 0,\ \lim_{t\to+\infty}{}^cD^{\frac{1}{2}}u_i(+\infty) = \sum_{j=1}^{3}\frac{1}{2j}u_i((j+1)^2), \end{cases} \tag{16}$$

see that Eq. (16) is a particular case of the Eq. (1) when $\rho = \frac{3}{2}$, $m = 5$, $\beta_j = \frac{1}{2j}$, $\xi_j = (j+1)^2$, and $f_i(t,u_i) = \frac{\sin(u_i+1)\cos(t+3)}{1+t^2}\sum_{k=i}^{i+1}\frac{1}{(k+1)k}$ ($t \in [0,+\infty)$, and $u_i \in \mathbb{R}$). Take $\varphi_i(t) = \theta_i(t) = \frac{1}{2}t$ and $M = \frac{\pi}{2}$, then the condition (i) of Theorem 4.1 holds. Since, for $s \in \mathbb{R}_+$ and $u_i, v_i \in \mathbb{R}$, we get

$$|f_i(s,u_i) - f_i(s,v_i)| \quad = \quad |\sum_{k=i}^{i+1}\frac{1}{(k+1)k}\frac{\cos(s+3)}{1+s^2}(\sin(u_i+1) - \sin(v_i+1))|$$

$$\leq \frac{1}{2}|\sin(u_i+1)-\sin(v_i+1)|$$
$$\leq \frac{1}{2}|u_i-v_i| = \varphi_i(|u_i-v_i|),$$

and also we have

$$\begin{aligned}\int_0^\infty |f_i(s,u_i)-f_i(s,v_i)|ds &= \int_0^\infty |\sum_{k=i}^{i+1}\frac{1}{(k+1)k}\frac{\cos(s+3)}{1+s^2}(\sin(u_i+1)-\sin(v_i+1))| \\ &\leq \frac{1}{2}|u_i-v_i|\int_0^\infty \frac{1}{1+s^2}ds = \frac{1}{2}|u_i-v_i|\lim_{t\to+\infty}\int_0^t \frac{1}{1+s^2}ds \\ &= \frac{1}{2}|u_i-v_i|\lim_{t\to+\infty}\arctan s\Big|_0^t \\ &= \frac{\pi}{2}\theta_i(|u_i-v_i|).\end{aligned}$$

Note that $f_i(t,u_i(t)) \in L^1([0,+\infty))$ *and regulated functions. Next, we have*

$$\overline{N} = \sup\Big\{|\sum_{k=i}^{i+1}\frac{1}{(k+1)k}\frac{\cos(s+3)\sin(1)}{1+s^2}|,\ s\in\mathbb{R}_+\Big\} = \frac{0.017}{2},$$

and

$$\overline{G} = \int_0^\infty |\sum_{k=i}^{i+1}\frac{1}{(k+1)k}\frac{\cos(s+3)(\sin(1)}{1+s^2}|ds \leq \frac{1}{2}\sin(1)\int_0^\infty \frac{1}{1+s^2}ds = \frac{0.017\pi}{4} < \infty.$$

Also, the condition (ii) *holds. Then, Theorem 4.1 grantees that Eq. (16) has at least one solution in* $R(\mathbb{R}_+,\mathbb{R}^\infty)$.

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