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Discrete Rubio De Francia extrapolation theorems in the theory of $\mathcal{B}_p-$discrete weights

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Abstract. In this paper, we will prove some discrete Rubio De Francia extrapolation theorems for non-increasing sequences in the setting of the discrete $\mathcal{B}_p$-weights. We also present some extensions to the discrete B_∞-weights. The proof of the results based on the boundedness of the discrete Hardy operator and the self-improving property of the discrete $\mathcal{B}_p$ weights. To the best of the authors' knowledge the discrete extrapolation theorems in connection with $\mathcal{B}_p-$weights are essentially new.

1. Introduction

A weight w is a nonnegative locally integrable function defined on $\mathbb{R}^+ = [0, \infty)$. A nonnegative weight function w defined on a bounded interval $J \subset \mathbb{R}^+$ is called an A^p-Muckenhoupt weight for $1 < p < \infty$, if there exists a constant $C < \infty$ such that

$$\left(\frac{1}{|I|}\int_I w(t)dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}(t)dt\right)^{p-1} \leq C, \tag{1}$$

for every subinterval $I \subset J$. The necessary and sufficient condition for the boundedness of a series of classical operators in the weighted spaces $L^p_w(\mathbb{R}^+)$ is the A^p-Muckenhoupt condition on the function w. The proof of the boundedness of operators is based precisely on the applications of the self-improving property of the A^p-weights which states that: if $w \in A^p(C)$ then there exists a constant $\epsilon > 0$ and a positive constant C_1 such that $w \in A^{p-\epsilon}(C_1)$, and then

$$A^p(C) \subset A^{p-\epsilon}(C_1). \tag{2}$$

An important application of the A^p-Muckenhoupt weights is the extrapolation theorem due Rubio de Francia (see [19]), that is announced in [20] and the detailed proof is given in [21]. Since then, many results concerning this topic have been considered by several authors, see [9–16] and the references cited therein. In the following, we present the celebrated extrapolation theorem due to Rubio de Francia in the setting of the Muckenhoupt weights.

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Theorem 1.1. *Let T be a sublinear operator defined on a class of measurable functions in $\mathbb{R}^n$. Suppose that for some p_0, with $1 \leq p_0 < \infty$, and every weight $w \in A^{p_0}$, T satisfies the inequality*

$$\int_{\mathbb{R}^n} \left|Tf(x)\right|^{p_0} w(x)dx \leq C \int_{\mathbb{R}^n} \left|f(x)\right|^{p_0} w(x)dx, \tag{3}$$

for every f, where C depends only on $A^{p_0}-$constant of w. Then for every p with $1 < p < \infty$, and every $w \in A^p$, the operator satisfies the inequality

$$\int_{\mathbb{R}^n} \left|Tf(x)\right|^{p} w(x)dx \leq C \int_{\mathbb{R}^n} \left|f(x)\right|^{p} w(x)dx, \tag{4}$$

for every f, where C depends only on A^p-constant of w.

In the literature there are also parallel studies based on the applications of a certain B_p-class of weights. A weight w is said to be belong to the class $B_p(B)$ for $0 < p < \infty$ if it satisfies the condition

$$\int_t^\infty \frac{w(x)}{x^p} dx \leq \frac{B}{t^p} \int_0^t w(x)dx, \text{ for all } t > 0. \tag{5}$$

The smallest constant $B > 0$ satisfying (5) is called the B_p-constant of the weight w and is denoted by $B_p(w)$. Since $B_p \subset \mathcal{B}_q$ for every $0 < p \leq q < \infty$, we see that the class $\mathcal{B}_\infty$ (similarly to the Muckenhoupt weights) can be defined as the collection of weights belonging to some B_p that is $\mathcal{B}_\infty = \cup_{p>0} B_p$ and the norm $B_\infty(w) = \inf\{B_p(w) : w \in B_p\}$. The B_p class has been introduced by Ariňo and Muckenhoupt [1] in connection with the boundedness on $L^p_w[\mathbb{R}^+)$ of the Hardy operator

$$Hf(t) = \frac{1}{t} \int_0^t f(x)dx, \text{ for } t > 0.$$

In [8] Carro and Lorente proved a new version of Rubio de Francia extrapolation theorem in the setting of B_p-weights, instead of the A^p-weights for a pair of positive decreasing functions defined on $\mathbb{R}^+$. The theory has also been generalized to the case B_∞ weights and many interesting consequences have been derived from these results to characterize the boundedness of certain operators on $L^p_w(\mathbb{R}^+)$. For completeness, we present the basic results proved in [8] in the following two theorems.

Theorem 1.2. *Let φ be an increasing function on $(0, \infty)$ and f and g are positive nonincreasing functions defined on $(0, \infty)$. Let $0 < p_0 < \infty$ and suppose for every $w \in B_{p_0}$ that*

$$\int_0^\infty f^{p_0}(s)w(s)ds \leq \varphi(B_{p_0}(w)) \int_0^\infty g^{p_0}(s)w(s)ds.$$

Then for every $0 < p < \infty$ and $w \in B_p$

$$\int_0^\infty f^{p}(s)w(s)ds \leq \varphi^*(B_{p}(w)) \int_0^\infty g^{p}(s)w(s)ds,$$

where φ^* is a function depends on φ and the constant $B_p(w)$.

Theorem 1.3. *Let φ be an increasing function on $(0, \infty)$, let f and g are positive nonincreasing functions defined on $(0, \infty)$ and let $0 < p_0 < \infty$. Suppose that for every $w \in B_\infty = \cup_{p>0} B_p$,*

$$\int_0^\infty f^{p_0}(s)w(s)ds \leq \varphi(B_\infty(w)) \int_0^\infty g^{p_0}(s)w(s)ds.$$

Then, for every $0 < p_0 < \infty$ and $w \in B_\infty$,

$$\int_0^\infty f^{p}(s)w(s)ds \leq \varphi^*(B_\infty(w)) \int_0^\infty g^{p}(s)w(s)ds,$$

where φ^* is a function depends on φ and the constant $B_\infty(w)$.

During the past few years there has been renewed interest in the area of discrete harmonic analysis and then it becomes an active field of research. For example, the study of regularity and boundedness of discrete operator on $\ell^p(\mathbb{Z}_+)$ analogues for $L^p(\mathbb{R}^+)-$regularity and boundedness has been considered by some authors, see for example [4–6, 18, 23–25, 28–30] and the references cited therein. This began with an observation of M. Riesz in his work on the Hilbert transform in 1928 that was carried over in the work of Calderón and Zygmund on singular integrals in 1952. In the following, we present the basic definitions and some facts concerning the discrete Muckenhoupt classes and the Ariňo and Muckenhoupt $\mathcal{B}_p-$ classes of weights which will be fundamental for our purpose. A discrete nonnegative weight u defined on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ belongs to the discrete Muckenhoupt class $\mathcal{A}^p$ for $p > 1$ if there exists a positive function $A < \infty$ such that

$$\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)\left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{-1}{p-1}}(k)\right)^{p-1} \leq A, \tag{6}$$

holds for every $n > 1$. For a given exponent $p > 1$, we define the $\mathcal{A}^p$-norm by the following quantity

$$\mathcal{A}^p(u) := \sup_{n>1}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)\left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{-1}{p-1}}(k)\right)^{p-1}, \tag{7}$$

where the supremum is taken over all $n > 1$. The boundedness of discrete Hardy-Littlewood maximal operator $\mathcal{M}f(n)$ defined by

$$\mathcal{M}f(n) := \sup_{n>1}\frac{1}{n}\sum_{k=1}^{n} f(k), \tag{8}$$

where f is nonnegative sequence has been characterized in [26] in terms of the Muckenhoupt weights. A nonnegative discrete sequence w defined on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is said to be belong to the discrete class $\mathcal{B}_p(B)$ for $p > 0$ and $B > 0$ if w satisfies the condition

$$\sum_{k=n}^{\infty}\frac{w(k)}{k^p} \leq \frac{B}{n^p}\sum_{k=1}^{n} w(n), \quad \text{for all } n \in \mathbb{Z}_+. \tag{9}$$

In [17] Heing and Kufner proved that the discrete Hardy operator

$$\mathcal{H}f(n) = \frac{1}{n}\sum_{k=1}^{n} f(k),$$

is bounded on $\ell^p_w(\mathbb{Z}_+)$ if and only if $w \in \mathcal{B}_p(B)$ and the weight w satisfies $\lim_{n\to\infty}(w(n+1)/w(n)) = c > 0$ and $\sum_{n=1}^{\infty} w(n) = \infty$. In [3] Bennett and Gross-Erdmann improved the result of Heing and Kufner by excluding the conditions on w and proved that $\mathcal{H}g(n)$ is bounded on $\ell^p_w(\mathbb{Z}_+)$ if and only if $w \in \mathcal{B}_p(B)$ for all decreasing sequence f. In [22] the authors proved the discrete analogy of Theorem 1.1 via the discrete Muckenhoupt weights. Since the action of $\mathcal{H}$ on characteristic functions $f(s) = \chi_{[1,k]}(s)$, together with (9) gives us that

$$\begin{aligned} \sum_{k=1}^{\infty}\left(\sum_{s=1}^{k}\frac{\chi_{[1,k]}(s)}{k}\right)^p w(n) &= \sum_{k=1}^{n} w(k) + n^p\sum_{k=n}^{\infty}\frac{w(k)}{k^p} \\ &\leq (1+B)\sum_{k=1}^{n} w(k), \end{aligned} \tag{10}$$

it is natural to express the dependence on the $\mathcal{B}_p$ condition (9) of the weight w in terms of the quantity

$$\mathcal{B}_p(w) :\simeq 1 + \sup_{n>0}\frac{n^p\sum_{k=n}^{\infty}\frac{w(k)}{k^p}}{\sum_{k=1}^{n} w(k)}. \tag{11}$$

The natural questions arise now are the following:

Q_1). Is it possible to prove Theorems 1.2 and 1.3 for sequences in the discrete space $\ell_w^p(\mathbb{Z}_+)$?

Q_2). Is it possible to prove Theorems 1.2 and 1.3 for sequences in the discrete space $\ell_w^p\left(\mathbb{Z}_+^d\right)$?

Q_3). Is it possible to prove Theorems 1.2 and 1.3 for sequences in the discrete Lorentz space $\ell_w^{p,q}(\mathbb{Z}_+)$?

Q_4). Is it possible to prove Theorems 1.2 and 1.3 for sequences in the discrete Morrey space $\mathcal{M}_w^p(\mathbb{Z}_+)$?

Our aim in paper is to give an affirmative answer to the first question, which to the best of the authors' knowledge has not considered before. The paper is organized as follows: In Section 2, we consider the generalized operator

$$S_\lambda g(k) = \frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)g(k),$$

and prove that $S_\lambda g$ is bounded on $\ell_w^p(\mathbb{Z}_+)$, for $0 < p < \infty$, in connection with the discrete weights $w \in \mathcal{B}_p^\lambda(B)$, i.e., when w satisfies the condition

$$\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^p(k)} \leq \frac{B}{\Lambda^p(n)}\sum_{k=1}^{n}w(n), \quad \text{for all } n \in \mathbb{Z}_+. \tag{12}$$

where $\Lambda(n) = \sum_{k=1}^{n}\lambda(k)$. Next, we will prove some fundamental properties of the discrete class $\mathcal{B}_p$ of weights and prove that the self-improving property holds, i.e., we will prove that if $w \in \mathcal{B}_p$ then $w \in \mathcal{B}_{p-\epsilon}$ for $\epsilon > 0$ and establish exact values of ϵ and prove that $\mathcal{B}_\infty$ is the collection of the discrete weights belong to some $\mathcal{B}_p$ classes for $p > 0$. In Section 3, we will prove the discrete extrapolation theorems for nonincreasing sequence, which give the affirmative answer to the first question. Finally, we apply the results to prove some extrapolation theorems for discrete operators of nonincreasing sequences.

2. Fundamental properties of $\mathcal{B}_p$−weights and basic inequalities

In this section, we will prove some properties of the discrete class $\mathcal{B}_p$ of sequences that will be needed later in the proofs and also are important for their own. The sequences in the statements of theorems that follow are assumed to be nonnegative defined on $\mathbb{Z}_+$. In addition, in our proofs, we will use the convention $0 \cdot \infty = 0$ and $0/0 = 0$ and $\sum_{k=a}^{b}y(k) = 0$, whenever $a > b$.

We shall denote by C the universal constant depending only on p, p_0 but independent on w and C might be not be the same in all the instances. We write $A \lesssim B$ if there exists a universal constant C such that $A \leq CB$ and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. The following theorem is adapted from [27].

Theorem 2.1. *Let $1 \leq p < \infty$, w and f be non-negative sequences such that f is a nonincreasing sequence. Then the operator*

$$\mathcal{S}_\lambda f(n) := \frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)f(k), \tag{13}$$

is bounded on $\ell_w^p(\mathbb{Z}_+)$ if and only if the weight $w \in \mathcal{B}_p^\lambda(B)$ for $B > 0$ and there exists a constant $C > 0$ such that the inequality

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)f(k)\right)^p \leq C\sum_{n=1}^{\infty}w(n)u^p(n). \tag{14}$$

Moreover, if C and B are chosen best-possible then, we have $C \leq p^p(B+1)^p$.

In the following, we consider the case when $0 < p < 1$. To the best of the author's knowledge the proof of this case is new and complement the results due Bennett and Gross-Erdmann [3]. The technique can be applied to the continuous case for the case when $p < 1$ and then the results improve the results due Carro and Soria [7, Theorem 4.1] and Carro and Lorente [8, Lemma 2.4], in the sense that our technique do not require the monotonicity of $\lambda(k)$ and does not depend on the distribution function. To prove this case, we need the following lemmas.

Lemma 2.2. *[3]Assume that ψ, ϕ, g be nonnegative sequences and g is nonincreasing. If*

$$\sum_{k=1}^{n}\psi(k) \le \sum_{k=1}^{n}\phi(k), \text{ for all } n \in \mathbb{Z}_+, \tag{15}$$

then

$$\sum_{n=1}^{\infty}\psi(n)g(n) \le \sum_{n=1}^{\infty}\phi(n)g(n).$$

Lemma 2.3. *[3] If $p \ge 1$, then for all $n \in \mathbb{Z}_+$*

$$\sum_{k=1}^{N} a(k)\left(\sum_{s=1}^{k} a(s)\right)^{p-1} \le \left(\sum_{k=1}^{N} a(k)\right)^{p} \le p\sum_{k=1}^{N} a(k)\left(\sum_{s=1}^{k} a(s)\right)^{p-1}. \tag{16}$$

The inequalities reverse direction if $0 < p < 1$ and $a(1) > 0$. The constants (1 and p) are best possible.

Theorem 2.4. *[27].Assume that φ, ψ are nonnegative sequences. Then*

$$\sum_{n=1}^{\infty}\varphi(n)\left(\sum_{k=n}^{\infty}\psi(k)\right) = \sum_{n=1}^{\infty}\psi(n)\left(\sum_{k=1}^{n}\varphi(k)\right).$$

Theorem 2.5. *Let $0 < p < 1$, $w(n)$ and $f(n)$ be a non-negative sequences such that f is a nonincreasing sequence. Then the operator $\mathcal{S}_\lambda f(n)$ is bounded in $l_w^p(\mathbb{Z}_+)$ if and only if the weight $w \in \mathcal{B}_p^\lambda(B)$ for $B > 0$ and there exists a constant $C > 0$ such that*

$$\sum_{n=1}^{\infty} w(n)\left(\frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p} \le C\sum_{n=1}^{\infty} w(n)f^{p}(n). \tag{17}$$

Moreover, if C and B are chosen best-possible then, we have $C \le (B+1)/p$.

Proof. First, we assume that $w \in \mathcal{B}_p^\lambda(B)$ for $B > 0$, i.e.,

$$\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)} \le \frac{B}{\Lambda^{p}(n)}\sum_{k=1}^{n} w(k), \text{ for all } n \in \mathbb{Z}_+. \tag{18}$$

Now, the left hand side of the inequality (17) takes the from

$$\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}. \tag{19}$$

From Lemma 2.3, we see that

$$\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p} \le \sum_{k=1}^{n}\lambda(k)f(k)\left(\sum_{s=1}^{k}\lambda(s)f(s)\right)^{p-1}.$$

Substituting the last inequality into (19), we get that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}\\ \leq\ &\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\left(\sum_{s=1}^{k}\lambda(s)f(s)\right)^{p-1}\right).\end{aligned}$$

By using Fubini's Theorem 2.4, we have that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}\\ \leq\ &\sum_{n=1}^{\infty}\lambda(n)f(n)\left(\sum_{s=1}^{n}\lambda(s)f(s)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right).\end{aligned} \tag{20}$$

Since f is nonincreasing, we have that $\mathcal{S}_{\lambda}f(n)$ is also nonincreasing, and

$$f(n)\leq\mathcal{S}_{\lambda}f(n)=\frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)f(k),$$

that is

$$\Lambda(n)f(n)\leq\sum_{k=1}^{n}\lambda(k)f(k).$$

Since $p-1<0$, we have that

$$\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p-1}\leq\left(\Lambda(n)f(n)\right)^{p-1}. \tag{21}$$

By combining (21) and (20), we have that

$$\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}\leq\sum_{n=1}^{\infty}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)f^{p}(n). \tag{22}$$

Now, we give an estimate for the term

$$\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right),$$

as follows

$$\begin{aligned}&\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)\\ =\ &\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)}+\sum_{k=N}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)\\ =\ &\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)}\right)\\ &+\left(\sum_{k=N}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}.\end{aligned} \tag{23}$$

Applying summation by parts formula on the term

$$\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)}\right),$$

with

$$u(n)=\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)},\ \text{and}\ \ \Delta v(k)=\lambda(n)\left(\Lambda(n)\right)^{p-1},$$

we get that

$$\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)}\right)=\ u(k)v(k)|_{1}^{N+1}-\sum_{n=1}^{N}\Delta u(n)v(n+1),$$

where $v(n)=\sum_{k=1}^{n-1}\lambda(n)\left(\Lambda(n)\right)^{p-1}$. Since $u(N+1)=v(1)=0$, then we obtain

$$\begin{aligned}\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{\Lambda^{p}(k)}\right) &= -\sum_{n=1}^{N}\Delta u(n)v(n+1)\\ &= \sum_{n=1}^{N}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(n)\left(\Lambda(n)\right)^{p-1}\right). \qquad (24)\end{aligned}$$

By applying Lemma 2.3 since $p<1$, we have

$$\sum_{k=1}^{n}\lambda(k)\left(\Lambda(k)\right)^{p-1}\leq\frac{1}{p}\left(\Lambda(n)\right)^{p}. \qquad (25)$$

By combining (23), (24) and (25), we have that

$$\begin{aligned}&\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)\\ \leq\ &\frac{1}{p}\sum_{n=1}^{N}\frac{w(n)}{\Lambda^{p}(n)}+\frac{1}{p}\left(\sum_{k=N}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)\left(\Lambda(N)\right)^{p}.\end{aligned}$$

Applying the condition (18) for the second term, we get that

$$\begin{aligned}\sum_{n=1}^{N}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right) &\leq \frac{1}{p}\sum_{n=1}^{N}\frac{w(n)}{\Lambda^{p}(n)}+\frac{B}{p}\sum_{n=1}^{N}w(k)\\ &= \frac{B+1}{p}\sum_{n=1}^{N}w(n). \qquad (26)\end{aligned}$$

Since f is nonincreasing, we see that $f^{p}(n)$ is also nonincreasing. So, by applying Lemma 2.2 with

$$\psi=\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right),\ \text{and}\ \phi=\frac{B+1}{p}w(n),$$

and $g=f^{p}$, we obtain from (26) that

$$\sum_{n=1}^{\infty}\lambda(n)\left(\Lambda(n)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p}(k)}\right)f^{p}(n)\leq\frac{B+1}{p}\sum_{n=1}^{\infty}w(n)f^{p}(n). \qquad (27)$$

Substituting (27) into (22), we have that

$$\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}\leq\frac{B+1}{p}\sum_{n=1}^{\infty}w(n)f^{p}(n),$$

which gives the desired inequality (17). Now, we consider the reverse and suppose that

$$\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{n}\lambda(k)f(k)\right)^{p}\leq D\sum_{k=1}^{\infty}w(k)f^{p}(k), \tag{28}$$

holds for some constant $D>0$. Then (28) holds when

$$f(k)=\chi_{[1,s]}(k)=\begin{cases}1, & k\in[1,s],\\ 0, & k\notin[1,s].\end{cases}$$

For this f in (28), we obtain

$$\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{s}\lambda(k)\right)^{p}\leq D\sum_{k=1}^{s}w(k). \tag{29}$$

By noting that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{w(n)}{\Lambda^{c}(n)}\left(\sum_{k=1}^{s}\lambda(s)\right)^{p}&\geq\sum_{n=s}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\left(\sum_{k=1}^{s}\lambda(k)\right)^{p}\\ =\left(\sum_{k=1}^{s}\lambda(k)\right)^{p}\sum_{n=s}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}&=\Lambda^{p}(s)\sum_{n=s}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}.\end{aligned} \tag{30}$$

we have from (29) that

$$\sum_{n=s}^{\infty}\frac{w(n)}{\Lambda^{p}(n)}\leq\frac{D}{\Lambda^{p}(s)}\sum_{k=1}^{s}w(k),$$

which implies that $w\in\mathcal{B}_{p}^{\lambda}(B)$ with a constant $\mathcal{B}_{p}(w)\leq D$. This proves the necessary condition. The proof is complete. □

By replacing f by g^{p_0} and p by p/p_0 in Theorems 2.1 and 2.5 we have the following theorems which play the crucial rules in the proofs of the next main results.

Theorem 2.6. *Let $p_0, p>0$ such that $p/p_0>1$ and g be a nonnegative and nonincreasing sequence and define*

$$\mathcal{S}_{\lambda}g^{p_0}(n)=\frac{1}{\Lambda(n)}\sum_{k=1}^{n}\lambda(k)g^{p_0}(k).$$

Then

$$\sum_{n=1}^{\infty}w(n)\left(S_{\lambda}g^{p_0}(n)\right)^{p/p_0}\leq C\sum_{n=1}^{\infty}w(n)g^{p}(n),\ p/p_0\geq 1, \tag{31}$$

if and only if

$$\sum_{k=n}^{\infty}\frac{w(k)}{\Lambda^{p/p_0}(k)}\leq\frac{B}{\Lambda^{p/p_0}(n)}\sum_{k=1}^{n}w(k),\ \textit{for all } n\in\mathbb{Z}_{+},\ \textit{and } B>0. \tag{32}$$

Moreover, if C and B are chosen best-possible then, we have

$$C \leq \left(\frac{p}{p_0}\right)^{p/p_0} (B+1)^{p/p_0}.$$

Theorem 2.7. *Let $p_0, p > 0$ such that $p/p_0 < 1$ and g be a nonnegative and nonincreasing sequences and define*

$$\mathcal{S}_\lambda g^{p_0}(n) = \frac{1}{\Lambda(n)} \sum_{k=1}^{n} \lambda(k) g^{p_0}(k).$$

Then

$$\sum_{n=1}^{\infty} w(n) \left(S_\lambda g^{p_0}(n)\right)^{p/p_0} \leq C \sum_{n=1}^{\infty} w(n) g^p(n), \tag{33}$$

where if and only if

$$\sum_{k=n}^{\infty} \frac{w(k)}{\Lambda^{p/p_0}(k)} \leq \frac{B}{\Lambda^{p/p_0}(n)} \sum_{k=1}^{n} w(k), \quad \text{for all } n \in \mathbb{Z}_+, \text{ and } B > 0. \tag{34}$$

Moreover, if C and B are chosen best-possible then, we have $C \leq p_0(B+1)/p$.

In the following, we will prove that the self-improving property of the class $\mathcal{B}_p$ for the nonincreasing weights holds.

Theorem 2.8. *Suppose that $0 < p, B < \infty$ and w is a nonnegative weight. If $w \in \mathcal{B}_p(B)$, then $w \in \mathcal{B}_{p-\epsilon}$ for $0 < \epsilon < p/(B+1)$ with a constant*

$$\mathcal{B}_{p-\epsilon}(w) < \frac{pB}{p - \epsilon(B+1)}. \tag{35}$$

Proof. Since $w \in \mathcal{B}_p(B)$ for $0 < p, B < \infty$, then

$$\sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p} \leq \frac{B}{k^p} \sum_{\tau=1}^{k} w(\tau), \text{ for all } k \in \mathbb{Z}_+. \tag{36}$$

Multiplying (36) by $k^{\epsilon-1}$ and summing from m to ∞, we have that

$$\sum_{k=m}^{\infty} k^{\epsilon-1} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p} \leq B \sum_{k=m}^{\infty} \frac{1}{k^{p-\epsilon+1}} \sum_{\tau=1}^{k} w(\tau). \tag{37}$$

By setting

$$\varphi(k) = k^{\epsilon-1}, \ \psi(\tau) = \frac{w(\tau)}{\tau^p}, \text{ and } \Psi(k) = \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p},$$

we see that the left hand side of (37) can rewritten in the form

$$\sum_{k=m}^{\infty} k^{\epsilon-1} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p} = \sum_{k=m}^{\infty} \varphi(k)\Psi(k). \tag{38}$$

Applying the summation by parts formula

$$\sum_{k=m}^{\infty} u(k)\Delta v(k) = u(k)v(k)|_{k=m}^{\infty} - \sum_{k=m}^{\infty} \Delta u(k)v(k+1), \tag{39}$$

with $u(k) = \Psi(k)$ and $\Delta v(k) = \varphi(k)$, we have that

$$\sum_{k=m}^{\infty} \varphi(k)\Psi(k) = \sum_{k=m}^{\infty} \varphi(k)\Psi(k) = \Psi(k)v(k)|_{m}^{\infty} - \sum_{k=m}^{\infty} \Delta\Psi(k)v(k+1),$$

where $v(k) = \sum_{\tau=m}^{k-1} \varphi(\tau)$. Using $v(m) = \Psi(\infty) = 0$ (recall all summations are assumed to be convergent), we obtain that

$$\sum_{k=m}^{\infty} \varphi(k)\Psi(k) = \sum_{k=m}^{\infty} \psi(k)\left(\sum_{\tau=m}^{k} \varphi(\tau)\right). \tag{40}$$

By combining (38) and (40), we get that

$$\sum_{k=m}^{\infty} k^{\epsilon-1} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p} = \sum_{k=m}^{\infty} \frac{w(k)}{k^p}\left(\sum_{\tau=m}^{k} \tau^{\epsilon-1}\right) \geq \sum_{k=m}^{\infty} \frac{w(k)}{k^p}\left(\sum_{\tau=m}^{k-1} \tau^{\epsilon-1}\right).$$

By employing the inequality

$$\gamma x^{\gamma-1}(x-y) \leq x^{\gamma} - y^{\gamma} \leq \gamma y^{\gamma-1}(x-y), \text{ for } x \geq y > 0,\ 0 \leq \gamma \leq 1, \tag{41}$$

with $\gamma = \epsilon < 1$, we have that $\epsilon(\tau+1)^{\epsilon-1} \leq \Delta\tau^{\epsilon} \leq \epsilon\tau^{\epsilon-1}$, and then

$$\sum_{\tau=m}^{k-1} \tau^{\epsilon-1} \geq \frac{1}{\epsilon}\sum_{\tau=m}^{k-1} \Delta(\tau)^{\epsilon} = \frac{k^{\epsilon}}{\epsilon} - \frac{m^{\epsilon}}{\epsilon}.$$

This implies that

$$\sum_{k=m}^{\infty} k^{\epsilon-1} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^p} \geq \frac{1}{\epsilon}\sum_{k=m}^{\infty} \frac{w(k)}{k^p}\left[k^{\epsilon} - m^{\epsilon}\right]. \tag{42}$$

Next, we consider the right hand side of (37). By applying summation by parts, by setting

$$\Delta v(k) = \frac{1}{k^{p-\epsilon+1}}, \text{ and } u(k) = \sum_{\tau=1}^{k} w(\tau),$$

we get that

$$\begin{aligned}
&\sum_{k=m}^{\infty} \frac{1}{k^{p-\epsilon+1}}\left(\sum_{\tau=1}^{k} w(\tau)\right) \\
= &\left(\sum_{\tau=m}^{\infty} \frac{1}{\tau^{p-\epsilon+1}}\right)\left(\sum_{\tau=1}^{m} w(\tau)\right) + \sum_{k=m}^{\infty}\left(\sum_{\tau=k+1}^{\infty} \frac{1}{\tau^{p-\epsilon+1}}\right)w(k+1) \\
\leq &\sum_{\tau=m}^{\infty} \frac{1}{\tau^{p-\epsilon+1}}\left(\sum_{\tau=1}^{m} w(\tau)\right) + \sum_{k=m}^{\infty}\left(\sum_{\tau=k+1}^{\infty} \frac{1}{\tau^{p-\epsilon+1}}\right)w(k).
\end{aligned} \tag{43}$$

By applying the inequality

$$\gamma y^{\gamma-1}(x-y) \leq x^{\gamma} - y^{\gamma} \leq \gamma x^{\gamma-1}(x-y), \tag{44}$$

for $x \geq y > 0$, $\gamma > 1$ or $\gamma < 0$, with $\gamma = -(p-\epsilon) < 0$, we see that

$$-(p-\epsilon)(\tau-1)^{-(p-\epsilon+1)} \leq \Delta(\tau-1)^{-(p-\epsilon)} \leq -(p-\epsilon)\tau^{-(p-\epsilon+1)},$$

and then

$$\sum_{\tau=k+1}^{\infty} \frac{1}{\tau^{p-\epsilon+1}} \leq -\sum_{\tau=k+1}^{\infty} \frac{\Delta(\tau-1)^{-(p-\epsilon)}}{(p-\epsilon)} = \frac{1}{(p-\epsilon)k^{p-\epsilon}}. \tag{45}$$

Substituting (45) into (43), we have that

$$B\sum_{k=m}^{\infty} k^{-(p-\epsilon+1)} \sum_{\tau=1}^{k} w(\tau) \leq \frac{B}{(p-\epsilon)} \sum_{k=m}^{\infty} \frac{w(k)}{k^{p-\epsilon}} + \frac{B\sum_{\tau=1}^{k} w(\tau)}{(p-\epsilon)m^{p-\epsilon}}. \tag{46}$$

By combining (42) and (46), we have that

$$\frac{1}{\epsilon}\sum_{k=m}^{\infty} \frac{w(k)}{k^{p}} \left[k^{\epsilon} - m^{\epsilon}\right] \leq \frac{B}{(p-\epsilon)} \sum_{k=m}^{\infty} \frac{w(k)}{k^{p-\epsilon}} + \frac{B}{(p-\epsilon)} \sum_{\tau=1}^{k} \frac{w(\tau)}{m^{p-\epsilon}}.$$

This gives us after once more using (36), that

$$\begin{aligned} \left(\frac{1}{\epsilon} - \frac{B}{p-\epsilon}\right)\sum_{k=m}^{\infty} \frac{w(k)}{k^{p-\epsilon}} &\leq \frac{(m)^{\epsilon}}{\epsilon} \sum_{k=m}^{\infty} \frac{w(k)}{k^{p}} + \frac{B\sum_{\tau=1}^{m} w(\tau)}{(p-\epsilon)m^{p-\epsilon}} \\ \leq \quad & \frac{Bm^{\epsilon}}{\epsilon m^{p}} \sum_{\tau=1}^{m} w(\tau) + \frac{B\sum_{\tau=1}^{m} w(\tau)}{(p-\epsilon)m^{p-\epsilon}} \leq B\left(\frac{1}{\epsilon} + \frac{1}{(p-\epsilon)}\right) \frac{\sum_{\tau=1}^{m} w(\tau)}{m^{p-\epsilon}}, \end{aligned}$$

that is

$$\left(\frac{(p-\epsilon)-\epsilon B}{(p-\epsilon)\,\epsilon}\right)\sum_{k=m}^{\infty} \frac{w(k)}{k^{p-\epsilon}} \leq \frac{Bp}{\epsilon(p-\epsilon)} \frac{1}{m^{p-\epsilon}} \sum_{\tau=1}^{m} w(\tau),$$

and thus

$$\sum_{k=m}^{\infty} \frac{w(k)}{k^{p-\epsilon}} \leq \frac{pB}{(p-\epsilon)-\epsilon B} \frac{1}{m^{p-\epsilon}} \sum_{\tau=1}^{m} w(\tau),$$

which implies that $w \in B_{p-\epsilon}$ for $\epsilon < p/(B+1)$ and with a constant

$$B_{p-\epsilon}(w) < \frac{pB}{p-\epsilon(B+1)},$$

which is the desired result. The proof is complete. □

Finally, for the sharpness of our constants, we shall need the following estimate for a power low sequence weight $w_{\alpha}(n) = n^{\alpha}$, for $-1 < \alpha < p-1$.

Lemma 2.9. *For the power low sequence weight $w_{\alpha}(n)$, the following estimate*

$$\mathcal{B}_p(w_{\alpha}) := 1 + \sup_{r>0} \frac{r^p \sum_{n=r}^{\infty} \frac{w_{\alpha}(n)}{n^p}}{\sum_{n=1}^{r} w_{\alpha}(n)} \simeq \frac{p}{p-\alpha-1}. \tag{47}$$

holds.

Proof. We directly substitute $w_\alpha(n) = n^\alpha$ in the definition of $\mathcal{B}_p(w_\alpha)$ (11) as follows

$$\mathcal{B}_p(w_\alpha) ::= 1 + \sup_{r>0} \frac{r^p \sum_{n=r}^{\infty} n^{\alpha-p}}{\sum_{n=1}^{r} n^\alpha}. \tag{48}$$

Using the inequality (16) to calculate the two summations in (48), we get that

$$\sum_{n=r}^{\infty} n^{\alpha-p} \simeq \frac{1}{\alpha-p+1}\, n^{\alpha-p+1}\Big|_r^\infty = \frac{-1}{\alpha-p+1} r^{\alpha-p+1}, \tag{49}$$

and

$$\sum_{n=r}^{\infty} n^{\alpha} \simeq \frac{1}{\alpha+1}\, n^{\alpha+1}\Big|_1^r = \frac{1}{\alpha+1} r^{\alpha+1}. \tag{50}$$

Substituting (49) and (50) in (48), we obtain that

$$\mathcal{B}_p(w_\alpha) :\simeq 1 + \sup_{r>0} \frac{r^p \left(\frac{1}{p-\alpha-1} r^{\alpha-p+1}\right)}{\frac{1}{\alpha+1} r^{\alpha+1}} = 1 + \frac{\alpha+1}{p-\alpha-1} = \frac{p}{p-\alpha-1}.$$

The proof is complete. □

Theorem 2.10. *Let $0 < p_1 < p$. A nonnegative weight $w \in \mathcal{B}_p(B)$ if and only if there exists a constant $C > 0$ such that*

$$\sum_{\tau=1}^{k} w(\tau) \geq C \left(\frac{k}{s}\right)^{p_1} \sum_{\tau=1}^{s} w(\tau), \text{ for } s \geq k \geq 1. \tag{51}$$

If $C_{p_1}(w)$ is the maximal C for which (51) holds, then

$$C_{p_1}(w) \geq \frac{1}{2\mathcal{B}_p(w)+1}, \text{ and } \mathcal{B}_p(w) \leq \frac{p}{C_{p_1}(w)(p-p_1)}, \tag{52}$$

for

$$p_1 > \frac{2\mathcal{B}_p(w)+1}{2\mathcal{B}_p(w)+2} p.$$

Proof. Assume that $w \in \mathcal{B}_p(B)$ and put $\mathcal{B}_p(w) = B$. By Theorem 2.8, we have by choosing $\epsilon = p/2(B+1)$, that

$$p_1 = p - \epsilon = \frac{(2B+1)}{2(B+1)} p < p, \; B_{p-\epsilon} = \frac{pB}{p - \frac{p}{2(B+1)}(B+1)} = 2B.$$

Thus

$$\frac{1}{k^{p_1}} \sum_{\tau=1}^{k} w(\tau) \geq \frac{1}{2B} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^{p_1}}, \tag{53}$$

that is $w \in \mathcal{B}_{p_1}(2B)$. For $s \geq k$, we see that

$$\begin{aligned}\sum_{\tau=1}^{s} w(\tau) &= \sum_{\tau=1}^{k} w(\tau) + \sum_{\tau=k+1}^{s} w(\tau) \leq \sum_{\tau=1}^{k} w(\tau) + \sum_{\tau=k}^{s} \frac{w(\tau)}{\tau^{p_1}} \tau^{p_1} \\ &\leq \left(\frac{s}{k}\right)^{p_1} \sum_{\tau=1}^{k} w(\tau) + s^{p_1} \sum_{\tau=k}^{s} \frac{w(\tau)}{\tau^{p_1}} \\ &\leq \left(\frac{s}{k}\right)^{p_1} \sum_{\tau=1}^{k} w(\tau) + s^{p_1} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^{p_1}} \\ &\leq \left(\frac{s}{k}\right)^{p_1} \sum_{\tau=1}^{k} w(\tau) + s^{p_1} \frac{2B}{k^{p_1}} \sum_{\tau=1}^{k} w(\tau) = (1+2B) \left(\frac{s}{k}\right)^{p_1} \sum_{\tau=1}^{k} w(\tau).\end{aligned}$$

That is

$$\sum_{\tau=1}^{k} w(\tau) \geq C_{p_1}(w) \left(\frac{k}{s}\right)^{p_1} \sum_{\tau=1}^{s} w(\tau), \text{ for } s > k,$$

which is the desired inequality (51). This proves the necessity of the condition (51) and the first inequality between constants in (52). To prove the sufficiency, we assume that $p_1 < p$ and (51) holds. Multiplying (51) by $k^{-p_1} s^{p_1 - 1 - p}$, we get for $s \geq k$ that

$$\frac{1}{C_{p_1}(w) k^{p_1} s^{1+p-p_1}} \sum_{\tau=1}^{k} w(\tau) \geq \frac{1}{s^{p+1}} \sum_{\tau=1}^{s} w(\tau).$$

Summing with respect to s from k to ∞, we have that

$$\sum_{s=k}^{\infty} \frac{1}{C_{p_1}(w) k^{p_1} s^{1+p-p_1}} \sum_{\tau=1}^{k} w(\tau) \geq \sum_{s=k}^{\infty} \frac{1}{s^{p+1}} \sum_{\tau=1}^{s} w(\tau),$$

and then

$$\frac{1}{C_{p_1}(w) k^{p_1}} \sum_{\tau=1}^{k} w(\tau) \sum_{s=k}^{\infty} s^{p_1-p-1} \geq \sum_{s=k}^{\infty} \frac{1}{s^{p+1}} \sum_{\tau=1}^{s} w(\tau). \tag{54}$$

By employing the inequality (44) with $\gamma = p_1 - p < 0$, we have that

$$(p_1 - p) \sum_{s=k}^{\infty} s^{p_1-p-1} \leq \sum_{s=k}^{\infty} \Delta s^{p_1-p} = -\frac{k^{p_1}}{k^{p}}.$$

This and (54) imply that

$$\frac{1}{C_{p_1}(w)\,(p-p_1)\,k^{p}} \sum_{\tau=1}^{k} w(\tau) \geq \sum_{s=k}^{\infty} \frac{1}{s^{p+1}} \sum_{\tau=1}^{s} w(\tau). \tag{55}$$

By setting

$$v(s) = \sum_{\tau=1}^{s} w(\tau), \text{ and } u(s) = -\sum_{\tau=s}^{\infty} \frac{1}{\tau^{p+1}},$$

we see that

$$\sum_{s=k}^{\infty}\frac{1}{s^{p+1}}\sum_{\tau=1}^{s}w(\tau) = -u(k)v(k)+\sum_{s=k}^{\infty}(-u(k))\,w(k)$$
$$\geq \sum_{s=k}^{\infty}(-u(k))\,w(k) = \sum_{s=k}^{\infty}\left(\sum_{\tau=k}^{\infty}\frac{1}{\tau^{p+1}}\right)w(k).$$

Since $-p\tau^{-p-1} \leq \Delta\tau^{-p} \leq -p(\tau+1)^{-p-1}$, we get that

$$\sum_{\tau=k}^{\infty}\frac{1}{\tau^{p+1}} \geq -\frac{1}{p}\sum_{\tau=k}^{\infty}\Delta(\tau)^{p} = \frac{1}{p}k^{-p},$$

and thus

$$\sum_{s=k}^{\infty}\frac{1}{s^{p+1}}\sum_{\tau=1}^{s}w(\tau) \geq \frac{1}{p}k^{-p}\sum_{\tau=k}^{\infty}w(\tau).$$

This and (55) lead to

$$\sum_{k=m}^{\infty}\frac{w(k)}{k^{p}} \leq \frac{p}{C_{p_1}(w)\,(p-p_1)}\frac{1}{k^{p}}\sum_{\tau=1}^{k}w(\tau).$$

By choosing $\epsilon = p/2(B+1)$, we see that

$$p_1 > p-\epsilon = p-\frac{p}{2(B+1)} = p\frac{2B+1}{2B+2},$$

and $\mathcal{B}_p(w) \leq \left(p/C_{p_1}(w)\,(p-p_1)\right)$. This completes the proof of the sufficiency and the second inequality between the constants in (52). The proof is complete. □

To show the similarity of the definitions of the ordinary Muckenhoupt A^p- classes (see [19]), we introduce a class $\mathcal{B}_\infty$ which will soon become evident that the corresponding to the definition of A_∞ would be to define $\mathcal{B}_\infty$ as the class of weights (see [13]), for which there exist two constants $\alpha < 1$ and $\beta > 0$ such that

$$1 > \frac{k}{n} \geq \alpha \Rightarrow \frac{\sum_{\tau=1}^{k}w(k)}{\sum_{\tau=1}^{n}w(k)} > \beta, \text{ for } 0 < k < n.$$

This is equivalent to the following definition which is more easy to grasp. $\mathcal{B}_\infty$ is the nonnegative discrete weights with the property that there exists a constant $C > 0$ such that

$$C\sum_{\tau=1}^{k}w(\tau) \geq \sum_{\tau=1}^{n}w(\tau), \quad \text{for all } 0 < k < n. \tag{56}$$

This in fact can be written in the form

$$C\sum_{\tau=1}^{n}w(\tau) \geq \sum_{\tau=1}^{2n}w(\tau), \quad \text{for all } n \geq 1. \tag{57}$$

The doubling constant $\mathcal{B}_\infty(w)$ is the minimum of all C such that (57) is valid. If $\mathcal{B}_\infty(w)$ is finite, we will say that w has the doubling property. A weight in $\mathcal{B}_p$ has the doubling property. Just relax the summation in the left hand side of (35) by reducing the summation to become $(n, 2n)$ and much work should be done to obtain a better estimate of C in (57) and can be used an alternative characterization of $\mathcal{B}_p$. Since $\mathcal{B}_p \subset \mathcal{B}_q$ for every $0 < p \leq q < \infty$, we can prove that the class $\mathcal{B}_\infty$ is the collection of weights belonging to some $\mathcal{B}_p$ and this will be proved in the next theorem.

Theorem 2.11. $\mathcal{B}_\infty = \cup_{p>0}\mathcal{B}_p$.

Proof. Suppose $w \in \mathcal{B}_p$ for some $p > 0$. It is immediate clear from Theorem 2.10 that w satisfies the requirements for being $\mathcal{B}_\infty$ (see (51)). Thus $\cup_{p>0}\mathcal{B}_p \subset \mathcal{B}_\infty$. Suppose on the other hand that $w \in \mathcal{B}_\infty$ with $\mathcal{B}_\infty(w) = C$, i.e.,

$$\sum_{\tau=1}^{k} w(\tau) \geq C_1 \sum_{\tau=1}^{n} w(\tau), \text{ for all } 0 \leq k \leq n, \quad \text{and } C_1 > 0.$$

Now, since $k < n$, we see for $p > 0$ that

$$\left(\frac{1}{k}\right)^p \sum_{\tau=1}^{k} w(\tau) \geq C_1 \left(\frac{1}{k}\right)^p \sum_{\tau=1}^{n} w(\tau) \geq C_1 \left(\frac{1}{n}\right)^p \sum_{\tau=1}^{n} w(\tau).$$

This implies

$$\sum_{\tau=1}^{k} w(\tau) \geq C_1 \left(\frac{k}{n}\right)^p \sum_{\tau=1}^{n} w(\tau), \text{ for } n \geq k \geq 1,$$

which is the condition (51) in Theorem 2.10. Now from Theorem 2.10, we deduce that $w \in \mathcal{B}_p$ for some $p > 0$ and thus $\mathcal{B}_\infty \subset \cup_{p>0}\mathcal{B}_p$ and the proof is complete. □

3. Discrete Extrapolation Theorems

In this section, we will prove the discrete extrapolation theorems for pairs of nonincreasing sequences and then apply to get some extrapolation theorems for nonincreasing operators. The results in this section give the answer of the first question that has been posed in the introduction.

Theorem 3.1. *Let φ be an increasing function on $(0, \infty)$ and f and g are positive nonincreasing sequences. Let $0 < p_0 < \infty$ and suppose for every $w \in \mathcal{B}_{p_0}$ that*

$$\sum_{s=1}^{\infty} f(s)w(s) \leq \varphi(\mathcal{B}_{p_0}(w)) \sum_{s=1}^{\infty} g(s)w(s). \tag{58}$$

Then for every ϵ such that $0 < \epsilon < p_0$

$$\sum_{s=1}^{k} f(s)s^{p_0-1-\epsilon} \leq \varphi\left(\frac{p_0+1}{\epsilon}\right) \sum_{s=1}^{k} g(s)s^{p_0-1-\epsilon}. \tag{59}$$

Proof. Let $w(k) = v(k)k^{p_0-1-\epsilon}$ with v is a nonincreasing sequence. Then

$$k^{p_0} \sum_{\tau=k}^{\infty} \frac{w(\tau)}{\tau^{p_0}} = k^{p_0} \sum_{\tau=k}^{\infty} \frac{v(\tau)}{\tau^{1+\epsilon}} \leq k^{p_0} v(k) \sum_{\tau=k}^{\infty} \frac{1}{\tau^{1+\epsilon}}. \tag{60}$$

By employing the inequality (44) with $\gamma = -\epsilon < 0$, we see that

$$-\epsilon(\tau-1)^{-\epsilon-1} \leq \tau^{-\epsilon} - (\tau-1)^{-\epsilon} \leq -\epsilon\tau^{-\epsilon-1},$$

and then, we have that

$$\sum_{\tau=k}^{\infty} \frac{1}{\tau^{1+\epsilon}} \leq -\frac{1}{\epsilon} \sum_{\tau=k+1}^{\infty} \Delta(\tau-1)^{-\epsilon} = \frac{1}{\epsilon k^{\epsilon}}.$$

Then we have from (60) that

$$k^{p_0}\sum_{\tau=k}^{\infty}\frac{w(\tau)}{\tau^{p_0}} \leq \frac{1}{\epsilon}k^{p_0}v(k)\frac{1}{k^{\epsilon}} \leq \frac{1}{\epsilon}v(k)(k+1)^{p_0-\epsilon}. \tag{61}$$

Case 1). If $0 < p_0 - \epsilon < 1$, we obtain by employing the inequality (41), that

$$(k+1)^{p_0-\epsilon} - 1 = \sum_{\tau=1}^{k}\Delta\tau^{p_0-\epsilon} \leq (p_0-\epsilon)\sum_{\tau=1}^{k}\tau^{p_0-\epsilon-1},$$

and thus

$$(k+1)^{p_0-\epsilon} \leq (p_0-\epsilon)\sum_{\tau=1}^{k}\tau^{p_0-\epsilon-1} + 1. \tag{62}$$

By combining (61) and (62), we have that

$$k^{p_0}\sum_{\tau=k}^{\infty}\frac{w(\tau)}{\tau^{p_0}} \leq \frac{1}{\epsilon}v(k)(p_0+1-\epsilon)\sum_{\tau=1}^{k}\tau^{p_0-\epsilon-1}. \tag{63}$$

Since v is decreasing, we have that

$$k^{p_0}\sum_{\tau=k}^{\infty}\frac{w(\tau)}{\tau^{p_0}} \leq \frac{(p_0+1)}{\epsilon}\sum_{\tau=1}^{k}v(\tau)\tau^{p_0-\epsilon-1},$$

and hence $w \in \mathcal{B}_{p_0}$ with a constant less or equal to $(p_0+1)/\epsilon$. Now, by taking $v(k) = \chi_{[0,s]}(k)$ and applying (58), we obtain that

$$\sup_{s>0}\frac{\sum_{k=1}^{s} f(k)(s)^{p_0-1-\epsilon}}{\sum_{k=1}^{s} g(k)(s)^{p_0-1-\epsilon}} \leq \varphi(\mathcal{B}_{p_0}(w)) = \varphi\left(\frac{p_0+1}{\epsilon}\right) < \infty,$$

which is the desired inequality (59).

Case 2). If $1 < p_0 - \epsilon$, we get by employing the inequality (44) that

$$(p_0-\epsilon)(k-1)^{p_0-\epsilon-1} \leq \Delta(k-1)^{p_0-\epsilon} \leq (p_0-\epsilon)(k)^{p_0-\epsilon-1}.$$

That is

$$k^{p_0-\epsilon} = \sum_{\tau=1}^{k}\Delta(\tau-1)^{p_0-\epsilon} \leq (p_0-\epsilon)\sum_{\tau=1}^{k}\tau^{p_0-\epsilon-1}. \tag{64}$$

By combining (61) and (64), we have that

$$k^{p_0}\sum_{\tau=k}^{\infty}\frac{w(\tau)}{\tau^{p_0}} \leq \frac{1}{\epsilon}v(k)(p_0-\epsilon)\sum_{\tau=1}^{k}\tau^{p_0-\epsilon-1}$$

Since v is nonincreasing, we have

$$k^{p_0}\sum_{\tau=k}^{\infty}\frac{w(\tau)}{\tau^{p_0}} \leq \frac{(p_0-\epsilon)}{\epsilon}\sum_{\tau=1}^{k}v(\tau)\tau^{p_0-\epsilon-1} \leq \frac{p_0}{\epsilon}\sum_{\tau=1}^{k}v(\tau)\tau^{p_0-\epsilon-1}.$$

and hence $w \in \mathcal{B}_{p_0}$ with constant less or equal to $(p_0)/\epsilon$. Now, by taking $v(k) = \chi_{[1,s]}(k)$ and applying (58) and the fact that φ is an increasing function, we obtain that

$$\sup_{s>1} \frac{\sum_{k=1}^{s} f(k) s^{p_0-1-\epsilon}}{\sum_{k=1}^{s} g(k) s^{p_0-1-\epsilon}} \leq \varphi(\mathcal{B}_{p_0}(w)) \leq \varphi\left(\frac{p_0}{\epsilon}\right) \leq \varphi\left(\frac{p_0+1}{\epsilon}\right) < \infty.$$

which is again the desired inequality (59). The proof is complete. □

Our next result is striking application of the class of $\mathcal{B}_p$ weights. It says that an estimate on ℓ^{p_0} for a single $p_0 > 0$ and all $\mathcal{B}_{p_0}$ weights implies a similar ℓ^p estimate for all $p > 0$. This property is referred to as extrapolation.

Theorem 3.2. *Let φ be an increasing function on $(0, \infty)$, f and g are positive nonincreasing sequences and let $0 < p_0 < \infty$. Suppose that for every $w \in \mathcal{B}_{p_0}$,*

$$\sum_{k=1}^{\infty} f^{p_0}(k) w(k) \leq \varphi(\mathcal{B}_{p_0}(w)) \sum_{k=1}^{\infty} g^{p_0}(k) w(k). \tag{65}$$

Then, for every $0 < p < \infty$ and $w \in \mathcal{B}_p$,

$$\sum_{k=1}^{\infty} f^{p}(k) w(k) \leq \varphi^*(\mathcal{B}_{p}(w)) \sum_{k=1}^{\infty} g^{p}(k) w(k), \tag{66}$$

where

$$\varphi^*(\mathcal{B}_p(w)) = \inf_{0<\epsilon} \left(\frac{p_0+1-\epsilon}{p_0-\epsilon} \varphi\left(\frac{p_0-\epsilon}{\epsilon}\right)\right)^{p/p_0} \frac{C\frac{(p_0-\epsilon)p}{p_0} \mathcal{B}_p(w)}{\frac{(p_0-\epsilon)p_0}{p} - \epsilon(\mathcal{B}_p(w)+1)}.$$

Proof. Let $p > 0$, $w \in \mathcal{B}_p$ and $0 < \epsilon < p_0$. We will consider the case when $p_0 - \epsilon < 1$ and since the proof of the case when $p_0 - \epsilon > 1$ is similar we omitted it. Using the fact f is nonincreasing, we see that

$$f^p(k) \leq \left(\frac{k^{p_0-\epsilon}}{k^{p_0-\epsilon}} f^{p_0}(\tau)\right)^{p/p_0} \leq \left(\frac{1}{k^{p_0-\epsilon}} f^{p_0}(\tau)\,(k+1)^{p_0-\epsilon}\right)^{p/p_0}. \tag{67}$$

Since $0 < p_0 - \epsilon < 1$, we have from the inequality (41) that

$$(p_0-\epsilon)\,(k+1)^{p_0-\epsilon-1} \leq \Delta\,(k)^{p_0-\epsilon} \leq (p_0-\epsilon)\,(k)^{p_0-\epsilon-1},$$

and then

$$(k+1)^{p_0-\epsilon} - 1 = \sum_{\tau=1}^{k} \Delta \tau^{p_0-\epsilon} \leq (p_0-\epsilon) \sum_{\tau=1}^{k} \tau^{p_0-\epsilon-1},$$

that is

$$(k+1)^{p_0-\epsilon} \leq (p_0-\epsilon) \sum_{\tau=1}^{k} \tau^{p_0-\epsilon-1} + 1 \leq (p_0+1-\epsilon) \sum_{\tau=1}^{k} \tau^{p_0-\epsilon-1}. \tag{68}$$

By combining (67) and (68), we have that

$$w(k) f^p(k) \leq \left(\frac{p_0+1-\epsilon}{k^{p_0-\epsilon}} \sum_{\tau=1}^{k} f^{p_0}(\tau) \tau^{p_0-1-\epsilon}\right)^{p/p_0} w(k).$$

Thus

$$\sum_{k=1}^{\infty} w(k) f^{p}(k) \leq \sum_{k=1}^{\infty} \left(\frac{p_0 + 1 - \epsilon}{k^{p_0 - \epsilon}} \sum_{\tau=1}^{k} f^{p_0}(\tau) \tau^{p_0 - 1 - \epsilon} \right)^{p/p_0} w(k).$$

By applying Theorem 3.1, we have that

$$\begin{aligned} &\sum_{k=1}^{\infty} w(k) f^{p}(k) \\ \leq\ & \left(\frac{p_0 + 1 - \epsilon}{p_0 - \epsilon} \right)^{p/p_0} \sum_{k=1}^{\infty} \left(\frac{p_0 - \epsilon}{k^{p_0 - \epsilon}} \sum_{\tau=1}^{k} f^{p_0}(\tau) \tau^{p_0 - 1 - \epsilon} \right)^{p/p_0} w(k) \\ \leq\ & \left(\frac{p_0 + 1 - \epsilon}{p_0 - \epsilon} \varphi \left(\frac{p_0 + 1}{\epsilon} \right) \right)^{p/p_0} \\ & \times \sum_{k=1}^{\infty} \left(\frac{p_0 - \epsilon}{k^{p_0 - \epsilon}} \sum_{\tau=1}^{k} g^{p_0}(\tau) \tau^{p_0 - 1 - \epsilon} \right)^{p/p_0} w(k). \end{aligned}$$

Since $0 < p_0 - \epsilon < 1$, we have from the inequality (41) that

$$(p_0 - \epsilon) \sum_{\tau=1}^{k} \tau^{p_0 - \epsilon - 1} \leq \sum_{\tau=1}^{k} \Delta (\tau - 1)^{p_0 - \epsilon} = k^{p_0 - \epsilon},$$

and then

$$\frac{p_0 - \epsilon}{k^{p_0 - \epsilon}} \leq \left(\sum_{\tau=1}^{k} \tau^{p_0 - \epsilon - 1} \right)^{-1}.$$

Now, by setting $\lambda(\tau) = \tau^{p_0 - \epsilon - 1}$, and $\Lambda(k) = \sum_{\tau=1}^{k} \tau^{p_0 - \epsilon - 1}$, we see that

$$\frac{p_0 - \epsilon}{k^{p_0 - \epsilon}} \leq \frac{1}{\Lambda(k)}.$$

Thus, we have that

$$\sum_{k=1}^{\infty} w(k) f^{p}(k) \leq \left(\frac{p_0 + 1 - \epsilon}{(p_0 - \epsilon)} \varphi \left(\frac{p_0 - \epsilon}{\epsilon} \right) \right)^{p/p_0} \sum_{k=1}^{\infty} \left(S_{\lambda} g^{p_0}(k) \right)^{p/p_0} w(k),$$

where

$$S_{\lambda} g^{p_0}(k) = \frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} g^{p_0}(\tau) \lambda(\tau).$$

From Theorems 2.6 and 2.7, we have that

$$\sum_{k=1}^{\infty} \left(S_{\lambda} g^{p_0}(k) \right)^{p/p_0} w(k) \leq C \sum_{k=1}^{\infty} w(k) g^{p}(k),$$

if and only if

$$\sum_{k=n}^{\infty} \frac{w(k)}{\Lambda^{p/p_0}(k)} \leq \frac{B}{\Lambda^{p/p_0}(n)} \sum_{k=1}^{n} w(k), \tag{69}$$

and $B > 0$, where $\lambda(\tau) = (\tau)^{p_0-\epsilon-1}$ and $\Lambda(k) = \sum_{\tau=1}^{k} \lambda(\tau) = \sum_{\tau=1}^{k} (\tau)^{p_0-\epsilon-1}$. Moreover, if C and B are chosen best-possible then, we have

$$C \leq \begin{cases} \left(\frac{p}{p_0}\right)^{p_0} (B+1), & \text{when} \quad p/p_0 < 1 \\ \left(\frac{p}{p_0}\right)^{p_0} (B+1)^{p/p_0}, & \text{when} \quad p/p_0 > 1 \end{cases}.$$

That is

$$\sum_{k=1}^{\infty} w(k) f^p(k) \leq C \left(\frac{p_0 + 1 - \epsilon}{p_0 - \epsilon} \varphi\left(\frac{p_0 - \epsilon}{\epsilon}\right)\right)^{p/p_0} \sum_{k=1}^{\infty} w(k) g^p(k),$$

if and only if (69) holds, where $\Lambda(k) \approx (k)^{p_0-\epsilon}$. This is equivalent to say, by Theorem 2.8, that

$$w \in \mathcal{B}_{\frac{(p_0-\epsilon)p}{p_0}}, \quad \text{with } B = \mathcal{B}_{\frac{(p_0-\epsilon)p}{p_0}}(w).$$

Since $w \in B_p$ then by Theorem 2.8, there exists $\varepsilon > 0$ so that $w \in \mathcal{B}_{p-\varepsilon}$. To complete the proof it is suffices to take ϵ small enough so that $p - \varepsilon = (p_0 - \epsilon) p/p_0$ to get the result. Moreover, by (35), we see that

$$B = \mathcal{B}_{\frac{(p_0-\epsilon)p}{p_0}}(w) = \mathcal{B}_{p-\varepsilon}(w) \leq \frac{\frac{(p_0-\epsilon)p}{p_0} \mathcal{B}_p(w)}{\frac{(p_0-\epsilon)p}{p_0} - \epsilon(\mathcal{B}_p(w) + 1)}.$$

Consequently, for every $\epsilon < 1/\left(p(\mathcal{B}_p(w) + 1) + p_0\right)$, we have that

$$\sum_{k=1}^{\infty} w(k) f^p(k) \leq \varphi^*(\mathcal{B}_p(w)) \sum_{k=1}^{\infty} w(k) g^p(k),$$

where

$$\varphi^*(\mathcal{B}_p(w)) = \inf_{0<\epsilon} \left(\frac{p_0 + 1 - \epsilon}{p_0 - \epsilon} \varphi\left(\frac{p_0 - \epsilon}{\epsilon}\right)\right)^{p/p_0} \frac{C \frac{(p_0-\epsilon)p}{p_0} \mathcal{B}_p(w)}{\frac{(p_0-\epsilon)p}{p_0} - \epsilon(\mathcal{B}_p(w) + 1)}.$$

The proof is complete. □

Theorem 3.3. *Let φ be an increasing function on $(0, \infty)$ and $0 < p_0 < \infty$ and f and g are positive nonincreasing sequences. Suppose for every $w \in \mathcal{B}_\infty$ that*

$$\sum_{k=1}^{\infty} f^{p_0}(k) w(k) \leq \varphi(\mathcal{B}_\infty(w)) \sum_{k=1}^{\infty} g^{p_0}(k) w(k).$$

Then, for every $0 < p < \infty$ and $w \in \mathcal{B}_\infty$,

$$\sum_{k=1}^{\infty} f^{p}(k) w(k) \leq \varphi^*(\mathcal{B}_\infty(w)) \sum_{k=1}^{\infty} g^{p}(k) w(k),$$

where $\varphi^*(\mathcal{B}_\infty(w)) = C\varphi^{p/p_0}(1)\mathcal{B}_\infty(w)$.

Proof. By the assumptions, we have that

$$\sum_{k=1}^{\infty} f^{p_0}(k) w(k) \leq \varphi(\mathcal{B}_\infty(w)) \sum_{k=1}^{\infty} g^{p_0}(k) w(k),$$

for every $w \in \mathcal{B}_\infty$. Then taking $w(k) = \left[\chi_{[0,s]}k\right](k)^\beta$ with $s \geq 1$ and $-1 < \beta < 0$, we have that $w \in \mathcal{B}_\infty$ and $\mathcal{B}_\infty(w) = 1$. Hence

$$\sum_{k=1}^{s} f^{p_0}(k)(k)^\beta \leq \varphi(1) \sum_{k=1}^{s} g^{p_0}(k)(k)^\beta, \text{ for } s \geq 1. \tag{70}$$

Now let $p > 0$ and let $w \in \mathcal{B}_\infty$ be an arbitrary. Then from the definition of $\mathcal{B}_\infty$ there exists $q > 0$ such that $w \in \mathcal{B}_q$. Using again the fact that f is nonincreasing, we see that

$$f^p(k) \leq \left(\frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} f^{p_0}(\tau)\tau^\beta \right)^{p/p_0}, \tag{71}$$

where $\Lambda(k) = \sum_{\tau=1}^{k-1}(\tau)^\beta$. So we have that

$$w(k)f^p(k) \leq \left(\frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} f^{p_0}(\tau)\,(\tau)^\beta \right)^{p/p_0} w(k),$$

and thus

$$\sum_{k=1}^{\infty} w(k)f^p(k) \leq \sum_{k=1}^{\infty} \left(\frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} f^{p_0}(\tau)(\tau)^\beta \right)^{p/p_0} w(k).$$

Then by using (70), we have that

$$\begin{aligned} \sum_{k=1}^{\infty} w(k)f^p(k) &\leq \varphi^{p/p_0}(1) \sum_{k=1}^{\infty} \left(\frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} g^{p_0}(\tau)(\tau)^\beta \right)^{p/p_0} w(k) \\ &= \varphi^{p/p_0}(1) \sum_{k=1}^{\infty} \left(S_\lambda g^{p_0}(\tau)\right)^{p/p_0} w(k), \end{aligned}$$

with $\lambda(k) = k^\beta$ is a decreasing sequence since $-1 < \beta < 0$ and

$$S_\lambda g^{p_0}(k) = \frac{1}{\Lambda(k)} \sum_{\tau=1}^{k} g^{p_0}(\tau)\lambda(\tau).$$

From Lemma 2.6, we have that

$$\sum_{k=1}^{\infty} \left(S_\lambda g^{p_0}(k)\right)^{p/p_0} w(k) \leq C \sum_{k=1}^{\infty} w(k)g^p(k),$$

if and only if

$$\sum_{k=n}^{\infty} \frac{w(k)}{\Lambda^{p/p_0}(k)} \leq \frac{B}{\Lambda^{p/p_0}(n)} \sum_{k=1}^{n} w(k), \tag{72}$$

and $B > 0$, where $\Lambda(k) = \sum_{\tau=1}^{k-1} \lambda(\tau) = \sum_{\tau=1}^{k-1} (\tau)^\beta$ where $\lambda(\tau) = (\tau)^\beta$. Moreover, if C and B are chosen best-possible then, we have

$$C \leq \begin{cases} \frac{p}{p_0}(B+1), & \text{when } p/p_0 \leq 1, \\ \left(\frac{p}{p_0}\right)^{p_0} (B+1)^{p/p_0}, & \text{when } \; p/p_0 > 1. \end{cases}.$$

That is

$$\sum_{k=1}^{\infty} w(k)f^p(k) \leq C\varphi^{p/p_0}(1)\sum_{k=1}^{\infty} w(k)g^p(k),$$

if and only if (72) holds where $\Lambda(k) \approx (k)^{\beta+1}$. This is equivalent to say, by Theorem 2.8, that

$$w \in \mathcal{B}_{\frac{(1+\beta)p}{p_0}}, \quad \text{with } B = \mathcal{B}_{\frac{(1+\beta)p}{p_0}}(w),$$

To complete the proof it is suffices to choose $\beta > -1$ such that $(1+\beta)\,p/p_0 = q$, i.e., $\beta = (qp_0/p) - 1$ to get that

$$\sum_{k=1}^{\infty} w(k)f^p(k) \leq C\varphi^{p/p_0}(1)\mathcal{B}_q(w)\sum_{k=1}^{\infty} w(k)g^p(k),$$

Taking the infimum of such q^s we get the required result with

$$\varphi^*(\mathcal{B}_\infty(w)) = C\varphi^{p/p_0}(1)\mathcal{B}_\infty(w).$$

The proof is complete. □

Remark 3.4. *In Theorem 3.1 it has been implicitly proved for a given $0 < p < \infty$ fixed and a pair of decreasing sequences (f, g)*

$$\sum_{k=1}^{\infty} f^p(k)w(k) \leq C\sum_{k=1}^{\infty} g^p(k)w(k),$$

holds for every $w \in \mathcal{B}_p$ with a constant C depending only on $\mathcal{B}_p(w)$, if and only if for every $n > 0$ and every $-1 < \beta < p - 1$,

$$\sum_{k=1}^{n} f^p(k)k^\beta \leq C_1\sum_{k=1}^{n} g^p(k)k^\beta,$$

with C_1 independent of n and depending only on $\mathcal{B}_p(w)$. This observation is especially useful for characterizing the boundedness on the space $\ell^p_{decr}(w)$ of certain decreasing operators.

Similarly, in the case of two linear operators, we have the following result.

Theorem 3.5. *Let T_1, T_2 are two linear operators such that for every decreasing sequence f and T_1f, T_2f are also decreasing whenever they are well defined. Then, we have*

$$\sum_{k=1}^{\infty} (T_1f(k))^p\, w(k) \leq C\sum_{k=1}^{\infty} (T_2f(k))^p\, w(k),$$

for every $w \in \mathcal{B}_p$ and every decreasing sequence f with C depending only on $\mathcal{B}_p(w)$ if and only if for every $r, s > 1$ and every $-1 < \alpha < 0$, such that

$$\sum_{k=1}^{s} T_1\chi_{[1,r]}(k)(k)^\alpha \leq C_1\sum_{k=1}^{s} T_2\chi_{[1,r]}(k)(k)^\alpha,$$

where C_1 independent of r and s.

In the following, we shall present mainly two theorems which interesting consequences. Both of them are consequences of Theorems 3.2 and 3.3 and give the extrapolation theorems of decreasing operators by considering the decreasing pair (Tf, f) instead of (f, g).

Theorem 3.6. *Let T be an operator such that for every decreasing f, Tf is also decreasing when it is well defined. Suppose for some p_0, $0 < p_0 < \infty$, and every $w \in \mathcal{B}_{p_0}$, there exists a constant C depending only on $\mathcal{B}_{p_0}(w)$ such that*

$$\sum_{k=1}^{\infty} (Tf(k))^{p_0} w(k) \leq C \sum_{k=1}^{\infty} f^{p_0}(k) w(k).$$

Then for every $0 < p < \infty$ and every $w \in \mathcal{B}_p$ there exists a constant C_1 depending only on $\mathcal{B}_p(w)$ such that

$$\sum_{k=1}^{\infty} (Tf(k))^{p} w(k) \leq C_1 \sum_{k=1}^{\infty} f^{p}(k) w(k).$$

Theorem 3.7. *Let T be an operator such that for every decreasing f, Tf is also decreasing when it is well defined. Suppose for some p_0, $0 < p_0 < \infty$, and every $w \in \mathcal{B}_{\infty}$, there exists a constant C depending only on $\mathcal{B}_{\infty}(w)$ such that*

$$\sum_{k=1}^{\infty} (Tf(k))^{p_0} w(k) \leq C \sum_{k=1}^{\infty} f^{p_0}(k) w(k).$$

Then for every $0 < p < \infty$ and every $w \in \mathcal{B}_{\infty}$ there exists a constant C_1 depending only on $\mathcal{B}_{\infty}(w)$ such that

$$\sum_{k=1}^{\infty} (Tf(k))^{p} w(k) \leq C_1 \sum_{k=1}^{\infty} f^{p}(k) w(k).$$

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