Filomat 38:1 (2024), 171–187
https://doi.org/10.2298/FIL2401171G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation results for the operators involving beta function and the Boas-Buck-Sheffer polynomials

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Abstract. In this study, we consider a sequence of linear positive operators involving the beta function and the Boas-Buck-Sheffer polynomials, and compute the convergence error of these operators using the first and second modulus of continuities. We give approximation properties in weighted space and we give a global error estimate in Lipschitz type space. We also construct a sequence of bivariate extensions of these operators and give the rate of convergence using the partial and full modulus of continuities. In addition, some examples, including graphs, are given for one- and two-variable functions to visually illustrate convergence to a function.

1. Introduction

One of the most important examples of the use of special polynomials in the sequences of linear positive operators in approximation theory was given by Jakimovski and Leviatan in 1969 [1]. They introduced a generalization of the Favard-Szász operators [2] by using the Appell polynomials. Later, in 1974, Ismail presented a generalization of the Jakimovski and Leviatan operators by using the Sheffer polynomials, which are more general than the Appell polynomials [3]. Thereby, examining the approximation properties of operators defined with the help of special polynomials with well-known properties (see, for example, [4],[5]), which have improved approximation properties than the operators existing in the literature, is an interesting problem.

In [6], the authors obtained quantitative estimate by the Jakimovski-Leviatan operators. In [7], Ciupa introduced integral type generalization of the Jakimovski-Leviatan operators and gave the degree of approximation of these operators. For operators including some special polynomials and beta function we refer the papers [8–20].

Recently, in [21], Wani and Nisar introduced and studied approximation properties of the operators given by

$$S_n\left(f;x\right)=\frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty}{}_{p}s_k(nx)f\left(\frac{k}{n}\right),x\geq 0,n\in\mathbb{N} \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 41A35, 41A36, 33B15
Keywords. Boas-Buck-Sheffer polynomials, linear positive operators, modulus of continuity, beta function, bivariate operators.
Received: 15 December 2022; Revised: 13 June 2023; Accepted: 03 July 2023
Communicated by Hari M. Srivastava
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where ${}_ps_k$ is a mixed family of polynomials, called Boas-Buck-Sheffer polynomials defined by the generating function

$$\mathcal{A}(w)A(J(w))\Psi(xH(J(w))) = \sum_{n=0}^{\infty} {}_ps_n(x)w^n \tag{2}$$

where

$$\begin{aligned}
\mathcal{A}(w) &= \sum_{k=0}^{\infty} \alpha_k \frac{w^k}{k!},\ \alpha_0 \neq 0 \\
A(w) &= \sum_{j=0}^{\infty} a_j \frac{w^j}{j!},\ a_0 \neq 0 \\
J(w) &= \sum_{\eta=0}^{\infty} c_\eta \frac{w^\eta}{\eta!},\ c_0 \neq 0 \\
\Psi(w) &= \sum_{l=0}^{\infty} \gamma_\imath \frac{w^l}{l!},\ \gamma_\imath \neq 0,\ \forall \imath \\
H(w) &= \sum_{m=1}^{\infty} h_m \frac{w^m}{m!},\ h_1 \neq 0.
\end{aligned} \tag{3}$$

The investigation has been conducted under the following assumption.

1. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, ${}_ps_k(x) \geq 0$,
2. $\mathcal{A}(1) \neq 0,\ A(J(1)) \neq 0,\ J'(1) = 1,\ H'(J(1)) = 1$,
3. $\Psi : \mathbb{R} \to (0, \infty)$,
4. The power series given by (3) converge for $|w| < R$ $(R > 1)$.

Inspiring by above studies, we introduce a sequence of operators, which involves beta function and the Boas-Buck-Sheffer polynomials for $f \in C_\gamma[0, \infty) = \{f \in C[0, \infty) : f(t) = O(t^\gamma)$ as $t \to \infty\}$ and $\gamma > n$,

$$\Omega_n(f;x) = \frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))} \sum_{k=0}^{\infty} {}_ps_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f(t)dt \tag{4}$$

under the above hypothesis ((1)-(4)) on functions $\mathcal{A}, A, J, \Psi, H$.

Remark 1.1. *Since Boas-Buck-Sheffer polynomials are a mixed family of polynomials, we obtain different operators with the help of this polynomial family for some special cases:*

1. *For $J(w) = w$, the operators (4) reduce to the Durrmeyer type operators containing Boas-Buck-Appell polynomials and the beta function.*
2. *For $\mathcal{A}(w) = 1$ and $J(w) = w$, one can get from the operators (4) the beta-Durrmeyer variant of the Boas-Buck operators defined in [22].*
3. *For $\mathcal{A}(w) = 1, J(w) = w$ and $H(w) = w$, the operators (4) reduce to the operators given in [23].*
4. *For $\mathcal{A}(w) = 1, J(w) = w$ and $\Psi(w) = exp(w)$, the operators (4) reduce to the Durrmeyer variant of the operators given in [3].*
5. *For $\mathcal{A}(w) = 1, J(w) = w, H(w) = w$ and $\Psi(w) = exp(w)$, the operators (4) reduce to the operators given in [24].*

6. *For $\mathcal{A}(w) = 1, A(w) = 1, J(w) = w, H(w) = w$ and $\Psi(w) = exp(w)$, the operators (4) reduce to the Szász-beta-Durrmeyer operators given in [25].*

It should be noticed that, different choices of the functions $\mathcal{A}, A, J, \Psi, H$ will give rise to many potentially interesting and useful new operators as a special case of this general family.

The remainder of this paper is structured as follows: In section 2, we consider the quantitative estimate of $\Omega_n(f;x)$ by using the first and second order modulus of continuity of f in the space of continuous functions and we gave some approximation results in weighted space and in Lipschitz type space. In section 3, we construct a sequence of bivariate operators Ω_{n_1,n_2}, which involves beta function and the Boas-Buck-Sheffer polynomials. Then the degree of approximation is obtained with the help of full modulus of continuity and partial modulus of continuity of f. Also we illustrate convergence of the sequence of operators Ω_n and Ω_{n_1,n_2} to certain functions with special choices of analytical functions. Furthermore, in future studies, A-statistical approximation theorem and the improved error estimate problem can be examined ([26],[27]).

2. Approximation Properties of the Operators $\Omega_n(.;.)$

Let us give some lemmas and theorems to study the approximation properties of Ω_n.

Lemma 2.1. *For all $x \geq 0$, we have*

$$\Omega_n(1;x) = 1,$$

$$\Omega_n(t;x) = \frac{n}{n-1}\left[\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}x + \frac{1}{n}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))}\right)\right] + \frac{1}{n-1}, \ n > 1,$$

$$\begin{aligned}\Omega_n(t^2;x) &= \frac{n^2}{(n-1)(n-2)}\left\{\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))}x^2 + \frac{1}{n}\left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\right. \\ &\times \frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}x + \frac{1}{n^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A''(J(1))}{A(J(1))} + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(J(1))}{A(J(1))}\right. \\ &\left.\left. + \frac{A'(J(1))(J''(1)+5)}{A(J(1))}\right)\right\} + \frac{2}{(n-1)(n-2)}, \ n > 2.\end{aligned}$$

Proof. If we take $w = 1$ and replace x by nx in (2), we get

$$\sum_{k=0}^{\infty} {}_p s_k(nx) = \mathcal{A}(1)A(J(1))\Psi(nxH(J(1))) \tag{5}$$

which yields $\Omega_n(1;x) = 1$. Differentiating (2) with respect to w, we get

$$\begin{aligned}\sum_{k=0}^{\infty} {}_p s_k(x) k w^{k-1} &= \mathcal{A}'(w)A(J(w))\Psi(xH(J(w))) + \mathcal{A}(w)A'(J(w))J'(w)\Psi(xH(J(w))) \\ &+ \mathcal{A}(w)A(J(w))\Psi'(xH(J(w)))xH'(J(w))J'(w).\end{aligned}$$

Taking $w = 1$ and replacing x by nx, it follows

$$\sum_{k=0}^{\infty} k_p s_k(nx) = \mathcal{A}'(1)A(J(1))\Psi(nxH(J(1))) + \mathcal{A}(1)A'(J(1))\Psi(nxH(J(1))) + \mathcal{A}(1)A(J(1))\Psi'(nxH(J(1)))nx. \tag{6}$$

Thus, considering that the beta function has the properties $B(p,q+1) = B(p,q)\frac{q}{p+q}$ and $B(p+1,q) = B(p,q)\frac{p}{p+q}$, we obtain

$$\Omega_n(t;x) = \frac{n}{n-1}\left[\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}x + \frac{1}{n}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))}\right)\right] + \frac{1}{n-1}.$$

Differentiating (2) with respect to w twice, we get

$$\begin{aligned}\sum_{k=0}^{\infty} {}_p s_k(x)k(k-1)w^{k-2} &= \mathcal{A}''(w)A(J(w))\Psi(xH(J(w))) + 2\mathcal{A}'(w)A'(J(w))\Psi(xH(J(w)))J'(w)\\ &+ 2\mathcal{A}'(w)A(J(w))\Psi'(xH(J(w)))xH'(J(w))J'(w) + \mathcal{A}(w)A''(J(w))J'(w)J'(w)\Psi(xH(J(w)))\\ &+ \mathcal{A}(w)A'(J(w))J''(w)\Psi(xH(J(w))) + 2\mathcal{A}(w)A'(J(w))J'(w)\Psi'(xH(J(w)))xH'(J(w))J'(w)\\ &+ \mathcal{A}(w)A(J(w))\Psi''(xH(J(w)))xH'(J(w))J'(w)xH'(J(w))J'(w)\\ &+ \mathcal{A}(w)A(J(w))\Psi'(xH(J(w)))xH''(J(w))\left(J'(w)\right)^2\\ &+ \mathcal{A}(w)A(J(w))\Psi'(xH(J(w)))xH'(J(w))J''(w).\end{aligned}$$

Taking $w = 1$ and replacing x by nx, it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} {}_p s_k(nx)k(k-1) &= \left[\mathcal{A}''(1)A(J(1)) + 2\mathcal{A}'(1)A'(J(1)) + \mathcal{A}(1)A''(J(1)) + \mathcal{A}(1)A'(J(1))J''(1)\right]\Psi(nxH(J(1)))\\ &+ [2\mathcal{A}'(1)A(J(1)) + 2\mathcal{A}(1)A'(J(1)) + \mathcal{A}(1)A(J(1))J''(1)\\ &+ \mathcal{A}(1)A(J(1))H''(J(1))]\, nx\Psi'(nxH(J(1))) + \mathcal{A}(1)A(J(1))\Psi''(nxH(J(1)))(nx)^2.\end{aligned}$$

We obtain the following equality by using (6) and the above equation

$$\begin{aligned}\sum_{k=0}^{\infty} k^2 {}_p s_k(nx) &= \left[2\mathcal{A}'(1)A'(J(1)) + \mathcal{A}''(1)A(J(1)) + \mathcal{A}(1)A''(J(1)) + \mathcal{A}'(1)A(J(1))\right.\\ &+ \mathcal{A}(1)A'(J(1))(J''(1) + 1)]\,\Psi(nxH(J(1)))\\ &+ [\mathcal{A}(1)A(J(1))(J''(1) + H''(J(1)) + 1) + 2\mathcal{A}'(1)A(J(1))\\ &+ 2\mathcal{A}(1)A'(J(1))]\Psi'(nxH(J(1))nx + \mathcal{A}(1)A(J(1))\Psi''(nxH(J(1)))\,(nx)^2.\end{aligned}$$

Hence,

$$\begin{aligned}\Omega_n(t^2;x) &= \frac{n^2}{(n-1)(n-2)}\left\{\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))}x^2 + \frac{1}{n}\left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\right.\\ &\times\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}x + \frac{1}{n^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A''(J(1))}{A(J(1))} + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(J(1))}{A(J(1))}\right.\\ &\left.\left.+\frac{A'(J(1))(J''(1)+5)}{A(J(1))}\right)\right\} + \frac{2}{(n-1)(n-2)},\ n > 2.\end{aligned}$$

□

For the central moments, we state the following lemma.

Lemma 2.2. *Let Ω_n be the sequence of linear positive operators defined in (4). Then we have*

$$\Omega_n(t-x;x) = \left(\frac{n}{n-1}\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} - 1\right)x + \frac{1}{n-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right),\ n > 1,$$

$$\begin{aligned}\Omega_n((t-x)^2;x) &= \left(\frac{n^2}{(n-1)(n-2)}\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} - \frac{2n}{n-1}\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} + 1\right)x^2\\ &+\left(\frac{n}{(n-1)(n-2)}\left(2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\frac{A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}\right.\\ &\left.-\frac{2}{n-1}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right)\right)x\\ &+\frac{1}{(n-1)(n-2)}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))} + \frac{A''(J(1))}{A(J(1))} + \frac{A'(J(1))(J''(1)+5)}{A(J(1))}\right.\\ &\left.+\frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\right),\ n > 2.\end{aligned}$$

Proof. From the linearity of the operators Ω_n and the lemma given above, one can find the desired results. □

For the convergence problem of Ω_n, throughout this paper, we need the following assumptions:

$$\lim_{n\to\infty}\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))}=1 \text{ and } \lim_{n\to\infty}\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))}=1,$$

uniformly on the subinterval of $[0,\infty)$.

Theorem 2.3. *For each $f\in C[0,\infty)\cap E$ and $n>2$, the uniform convergence of the operators defined in (4) is satisfied on each compact subsets of $[0,\infty)$, i.e.,*

$$\lim_{n\to\infty}\Omega_n(f;x)=f(x)$$

where $E:=\left\{f:\dfrac{f(x)}{1+x^2}\text{ is convergent as } x\to\infty\right\}.$

Proof. From Lemma 2.1, we have

$$\lim_{n\to\infty}\Omega_n(e^i;x)=x^i,\ i=0,1,2$$

uniformly on each compact subset of $[0,\infty)$. Hence, the result follows from the well-known Korovkin theorem [28]. □

Theorem 2.4. *For all $f\in\tilde{C}[0,\infty)\cap E$ and $n>2$, we have*

$$\left|\Omega_n(f;x)-f(x)\right|\le 2\omega_1\left(f,\sqrt{\Omega_n((t-x)^2;x)}\right).$$

Here $\omega_1(f,\delta)$ is the modulus of continuity of a function $f\in\tilde{C}[0,\infty)$ defined by the relation

$$\omega_1(f,\delta)=\sup_{\substack{|x-y|\le\delta\\ x,y\in[0,\infty)}}\left|f(x)-f(y)\right|$$

and $\tilde{C}[0,\infty)$ is the space of all uniformly continuous functions on $[0,\infty)$ [29].

Proof. By using $\Omega_n(1;x)=1$, linearity of the sequence Ω_n and applying the property of $\omega_1(f,\delta)$, we can write

$$\begin{aligned}
\left|\Omega_n(f;x)-f(x)\right| &= \left|\frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty}{}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}\left(f(t)-f(x)\right)dt\right|\\
&\le \frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty}{}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}\left|f(t)-f(x)\right|dt\\
&\le \frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty}{}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k}{(1+t)^{n+k+1}}\left(1+\frac{|t-x|}{\delta}\right)\omega_1(f,\delta)\,dt\\
&= \left(1+\frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\frac{1}{\delta}\sum_{k=0}^{\infty}{}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^\infty\frac{t^k|t-x|}{(1+t)^{n+k+1}}dt\right)\omega_1(f,\delta).
\end{aligned}$$

By applying the Cauchy-Schwarz inequality for the integral term, we get

$$\left|\Omega_n(f;x)-f(x)\right|\le\left(1+\frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\frac{1}{\delta}\sum_{k=0}^{\infty}{}_p s_k(nx)\frac{\sqrt{B(k+1,n)}}{B(k+1,n)}\sqrt{\int_0^\infty\frac{t^k(t-x)^2}{(1+t)^{n+k+1}}dt}\right)\omega_1(f,\delta).$$

Now, if we use the Cauchy-Schwarz inequality again, we have

$$\begin{aligned}\left|\Omega_n(f;x)-f(x)\right| &\leq \left(1+\frac{1}{A(1)A(J(1))\Psi(nxH(J(1)))}\frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty}{}_p s_k(nx)}\sqrt{\sum_{k=0}^{\infty}{}_p s_k(nx)\int_0^{\infty}\frac{t^k(t-x)^2}{(1+t)^{n+k+1}}dt}\right)\omega_1(f,\delta)\\ &=\left(1+\frac{1}{\delta}\sqrt{\Omega_n((t-x)^2;x)}\right)\omega_1(f,\delta).\end{aligned}$$

Choosing $\delta=\sqrt{\Omega_n((t-x)^2;x)}$, the desired result is obtained. □

Now, we consider a quantitative estimate of $\Omega_n(f;x)$ by using the second order modulus of continuity of f [29]. For this purpose, we give some definitions and theorems.

The Peetre's K-functional is defined by

$$\kappa(f;\delta):=\inf_{g\in C_B^2}\left\{\left\|f-g\right\|_{C_B}+\delta\left\|g\right\|_{C_B^2}\right\}$$

where $C_B^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$, with the norm

$$\left\|f\right\|_{C_B^2}=\left\|f\right\|_{C_B}+\left\|f'\right\|_{C_B}+\left\|f''\right\|_{C_B}$$

and C_B is the space of all bounded and uniformly continuous functions on $[0,\infty)$ with the norm

$$\left\|f\right\|_{C_B}=\sup_{x\in[0,\infty)}\left|f(x)\right|.$$

The second order modulus of continuity of $f\in C[0,\infty)$ is defined by the formula

$$\omega_2(f,\delta)=\sup_{0<t\leq\delta}\left\|f(.+2t)-2f(.+t)+f(.)\right\|.$$

The inequality that gives the relationship between the second order modulus of continuity and the Peetre's K-functional is as follows:

$$\kappa(f;\delta)\leq C\left\{\omega_2\left(f,\sqrt{\delta}\right)+\min(1,\delta)\left\|f\right\|_{C_B}\right\} \tag{7}$$

where C is a positive constant [7].

Theorem 2.5. *For all $f\in C_B^2[0,\infty)$ and $n>2$, we have*

$$\left|\Omega_n(f;x)-f(x)\right|\leq\frac{1}{2}\delta_n\left(2+\delta_n\right)\left\|f\right\|_{C_B^2},$$

where $\delta_n=\Omega_n\left((t-x)^2;x\right)^{1/2}$.

Proof. Let $f\in C_B^2[0,\infty)$. From the Taylor series expansion we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(\xi)(t-x)^2,$$

where ξ is between x and t. Using the definition of $\|.\|_{C_B^2}$, we can write

$$\left|f(t)-f(x)\right|\leq K_1\left|t-x\right|+\frac{1}{2}K_2(t-x)^2,$$

where

$$K_1 = \sup_{x\in[0,\infty)} \left|f'(x)\right| = \left\|f'\right\|_{C_B[0,\infty)} \leq \left\|f\right\|_{C_B^2[0,\infty)},$$
$$K_2 = \sup_{x\in[0,\infty)} \left|f''(x)\right| = \left\|f''\right\|_{C_B[0,\infty)} \leq \left\|f\right\|_{C_B^2[0,\infty)}.$$

Thus,

$$\left|f(t) - f(x)\right| \leq \left(|t-x| + \frac{1}{2}(t-x)^2\right)\left\|f\right\|_{C_B^2}.$$

Now, by using linearity of the operators Ω_n, we get

$$\begin{aligned}\left|\Omega_n(f;x) - f(x)\right| &\leq \left(\Omega_n\left(|t-x|;x\right) + \frac{1}{2}\Omega_n\left((t-x)^2;x\right)\right)\left\|f\right\|_{C_B^2[0,\infty)}\\ &\leq \frac{1}{2}\delta_n\left(2+\delta_n\right)\left\|f\right\|_{C_B^2[0,\infty)}\end{aligned}$$

since $\Omega_n\left(|t-x|;x\right) \leq \Omega_n\left((t-x)^2;x\right)^{1/2} = \delta_n$. □

Theorem 2.6. *Let Ω_n be the sequence of linear positive operators defined in (4). For all $f \in C_B[0,\infty)$ and $n > 2$, we have*

$$\left|\Omega_n(f;x) - f(x)\right| \leq 2C\left\{\omega_2\left(f, \sqrt{\frac{1}{4}\delta_n\left(2+\delta_n\right)}\right) + \min(1, \frac{1}{4}\delta_n\left(2+\delta_n\right))\left\|f\right\|_{C_B}\right\}$$

where C is a positive constant and δ_n is the same as the expression given in Theorem 2.5.

Proof. Let $f \in C_B[0,\infty)$. From the linearity of Ω_n and Theorem 2.5, we have

$$\begin{aligned}\left|\Omega_n(f;x) - f(x)\right| &\leq \left|\Omega_n(f-g;x) - f(x)\right| + \left|\Omega_n(g;x) - g(x)\right| + \left|f(x) - g(x)\right|\\ &\leq 2\left\|f-g\right\|_{C_B[0,\infty)} + \frac{1}{2}\delta_n\left(2+\delta_n\right)\left\|g\right\|_{C_B^2}\\ &\leq 2\left(\left\|f-g\right\|_{C_B[0,\infty)} + \frac{1}{4}\delta_n\left(2+\delta_n\right)\left\|g\right\|_{C_B^2}\right).\end{aligned}$$

If we use the definition of Peetre's K-functional, we can write

$$\left|\Omega_n(f;x) - f(x)\right| \leq 2\kappa(f; \frac{1}{4}\delta_n\left(2+\delta_n\right)).$$

Now, using the inequality (7), we have

$$\left|\Omega_n(f;x) - f(x)\right| \leq 2C\left\{\omega_2\left(f, \sqrt{\frac{1}{4}\delta_n\left(2+\delta_n\right)}\right) + \min(1, \frac{1}{4}\delta_n\left(2+\delta_n\right))\left\|f\right\|_{C_B}\right\}$$

which completes the proof. □

Let us state the approximation properties of Ω_n in the weighted space (see [30]-[32]).

Since the well-known Korovkin theorem does not hold in the unbounded intervals, it is necessary to put some restrictions given below.

Let $B_\rho[0,\infty)$ be the space of all functions defined on $[0,\infty)$ satisfying the inequality $\left|f(x)\right| \leq M_f(1+x^2)$. Here, M_f is a constant only depending on f and $\rho(x) = 1+x^2$ is a weight function. Furthermore $C_\rho[0,\infty)$ is the space of all continuous functions belonging to $B_\rho[0,\infty)$ and $C^*_\rho[0,\infty) = \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = k_f < \infty \right\}$. These function spaces are endowed with the $\rho-$norm:

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Theorem 2.7. *If the sequence of positive linear operators L_n from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ satisfies the conditions*

$$\lim_{n\to\infty} \|L_n(t^\nu;x) - x^\nu\|_\rho = 0,\ \nu = 0,1,2,$$

then for any function $f \in C^*_\rho[0,\infty)$

$$\lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Theorem 2.8. *For given $f \in C^*_\rho[0,\infty)$,*

$$\lim_{n\to\infty} \|\Omega_n(f;x) - f(x)\|_\rho = 0.$$

Proof. It is clear from the equality $\Omega_n(1;x) = 1$ and the definition of the $\rho-$norm that $\lim_{n\to\infty} \|\Omega_n(1;x) - 1\|_\rho = 0$. Now, we may write

$$\sup_{x\in[0,\infty)} \frac{|\Omega_n(t;x) - x|}{1+x^2} \le \left(\frac{nc_n}{n-1} + \frac{1}{n-1}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1}{n-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2}$$

where c_n is a number sequence that satisfies the inequality $\left|\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))} - 1\right| \le c_n$ with $c_n \to 0$ as $n\to\infty$, since $\lim_{n\to\infty} \frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} = 1$. Thus, we get $\lim_{n\to\infty} \|\Omega_n(t;x) - x\|_\rho = 0$. Likewise, since $\lim_{n\to\infty} \frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} = 1$, we can find a sequence d_n and a bound e such that $\left|\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} - 1\right| \le d_n$ with $d_n \to 0$ as $n\to\infty$ and $\left|\frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))}\right| \le e$.

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{|\Omega_n(t^2;x) - x^2|}{1+x^2} \le\ & \frac{n^2dn}{(n-1)(n-2)} \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
& + \frac{n.e}{(n-1)(n-2)}\left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
& + \frac{1}{(n-1)(n-2)}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A''(J(1))}{A(J(1))} + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(J(1))}{A(J(1))}\right. \\
& \left. + \frac{A'(J(1))(J''(1)+5)}{A(J(1))} + 2\right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2}.
\end{aligned}$$

Thus, we have $\lim_{n\to\infty} \|\Omega_n(t^2;x) - x^2\|_\rho = 0$. Hence, we get the desired result from the Theorem 2.7. □

Now we compute the rate of convergence of the sequence of operators Ω_n in Lipschitz type space which is defined by

$$Lip_M(\alpha) = \left\{ f \in C_B[0,\infty) : |f(y) - f(x)| \le M \frac{|y-x|^\alpha}{(y+x^2+1)^{\frac{\alpha}{2}}}, x, y \in (0,\infty) \right\},$$

where $0 < \alpha \leq 1$ and M is a positive constant [33]. Here, $C_B[0,\infty)$ denote the space of bounded continuous functions on $[0,\infty)$.

Theorem 2.9. *If $f \in Lip_M(\alpha)$, $\alpha \in (0,1]$, then the following estimate holds for $x \in [0,\infty)$*

$$\left|\Omega_n(f;x) - f(x)\right| \leq M\left\{a_n + b_n + \frac{1}{(n-1)(n-2)}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))} + \frac{A''(J(1))}{A(J(1))} + \frac{A'(J(1))(J''(1)+5)}{A(J(1))} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\right)\right\}^{\alpha/2}$$

where (a_n) and (b_n) are number sequences that satisfy the inequalities

$$a_n(\Psi, H, J) = \sup_{x\in[0,\infty)}\left|\frac{n^2}{(n-1)(n-2)}\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} - \frac{2n}{n-1}\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} + 1\right|$$

with $a_n \to 0$ as $n \to \infty$ and

$$b_n(\Psi, H, J) = \sup_{x\in[0,\infty)}\left|\frac{n}{(n-1)(n-2)}\left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} - \frac{2}{n-1}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right)\right|$$

with $b_n \to 0$ as $n \to \infty$.

Proof. *Let $x \in [0,\infty)$. For $\alpha = 1$ and so $f \in Lip_M(1)$, we get*

$$\begin{aligned}\left|\Omega_n(f;x) - f(x)\right| &\leq \frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty} {}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\left|f(t) - f(x)\right|dt \\ &\leq \frac{M}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))}\sum_{k=0}^{\infty} {}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\frac{|t-x|}{(t+x^2+1)^{\frac{1}{2}}}dt \\ &\leq \frac{M}{(x^2+1)^{1/2}\left(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))\right)}\sum_{k=0}^{\infty} {}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k|t-x|}{(1+t)^{n+k+1}}dt.\end{aligned}$$

Applying Cauchy-Schwarz inequality for the last inequality we have

$$\left|\Omega_n(f;x) - f(x)\right| \leq \frac{M}{(x^2+1)^{1/2}\left(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))\right)}\left(\sum_{k=0}^{\infty} {}_p s_k(nx)\sqrt{\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k(t-x)^2}{(1+t)^{n+k+1}}dt}\right)$$

and by using Cauchy-Schwarz inequality again and using Lemma 2.2, we have

$$\begin{aligned}\left|\Omega_n(f;x) - f(x)\right| &\leq \frac{M}{(x^2+1)^{1/2}}\left(\sqrt{\frac{1}{(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1))))}\sum_{k=0}^{\infty} {}_p s_k(nx)\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k(t-x)^2}{(1+t)^{n+k+1}}dt}\right) \\ &\leq \frac{M}{(x^2+1)^{1/2}}\left\{\left|\frac{n^2}{(n-1)(n-2)}\frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} - \frac{2n}{n-1}\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} + 1\right|x^2\right. \\ &+ \left|\frac{n}{(n-1)(n-2)}\left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))}\right. \\ &\left. - \frac{2}{n-1}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right)\right|x \\ &\left. + \frac{1}{(n-1)(n-2)}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))} + \frac{A''(J(1))}{A(J(1))} + \frac{A'(J(1))(J''(1)+5)}{A(J(1))} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\right)\right\}^{1/2} \\ &\leq M\left\{a_n + b_n + \frac{1}{(n-1)(n-2)}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))} + \frac{A''(J(1))}{A(J(1))}\right.\right. \\ &\left.\left. + \frac{A'(J(1))(J''(1)+5)}{A(J(1))} + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\right)\right\}^{1/2}\end{aligned}$$

Here (a_n) and (b_n) are number sequences in the statement of the theorem.

For $\alpha \in (0,1)$ and $f \in Lip_M(\alpha)$, we have

$$\begin{aligned}
\left|\Omega_n(f;x) - f(x)\right| &\leq \frac{1}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))} \sum_{k=0}^{\infty} {}_p s_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} \left|f(t) - f(x)\right| dt \\
&\leq \frac{M}{\mathcal{A}(1)A(J(1))\Psi(nxH(J(1)))} \sum_{k=0}^{\infty} {}_p s_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} \frac{|t-x|^{\alpha}}{(t+x^2+1)^{\frac{\alpha}{2}}} dt \\
&\leq \frac{M}{(x^2+1)^{\alpha/2}(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1))))} \sum_{k=0}^{\infty} {}_p s_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} |t-x|^{\alpha} dt.
\end{aligned}$$

Applying the Hölder inequality by taking $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, we get

$$\left|\Omega_n(f;x) - f(x)\right| \leq \frac{M}{(x^2+1)^{\alpha/2}(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1))))} \sum_{k=0}^{\infty} {}_p s_k(nx) \left(\frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k |t-x|}{(1+t)^{n+k+1}} dt \right)^{\alpha}.$$

Now, by taking $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ and by using the Hölder inequality one more time, we have

$$\left|\Omega_n(f;x) - f(x)\right| \leq \frac{M}{(x^2+1)^{\alpha/2}} \left(\frac{1}{(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1))))} \sum_{k=0}^{\infty} {}_p s_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k |t-x|}{(1+t)^{n+k+1}} dt \right)^{\alpha}.$$

Finally, applying the Cauchy-Schwarz inequality twice in a row, we get

$$\begin{aligned}
\left|\Omega_n(f;x) - f(x)\right| &\leq \frac{M}{(x^2+1)^{\alpha/2}} \left(\frac{1}{(\mathcal{A}(1)A(J(1))\Psi(nxH(J(1))))} \sum_{k=0}^{\infty} {}_p s_k(nx) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k (t-x)^2}{(1+t)^{n+k+1}} dt \right)^{\alpha/2} \\
&\leq \frac{M}{(x^2+1)^{\alpha/2}} \left\{ \left| \frac{n^2}{(n-1)(n-2)} \frac{\Psi''(nxH(J(1)))}{\Psi(nxH(J(1)))} - \frac{2n}{n-1} \frac{\Psi'(nxH(J(1)))}{\Psi(nxH(J(1)))} + 1 \right| x^2 \right. \\
&\quad + \left| \frac{n}{(n-1)(n-2)} \left(\frac{2\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{2A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4 \right) \frac{\Psi'(nxH(J(1))}{\Psi(nxH(J(1))} \right. \\
&\quad \left. - \frac{2}{n-1} \left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1 \right) \right| x \\
&\quad + \frac{1}{(n-1)(n-2)} \left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))} + \frac{A''(J(1))}{A(J(1))} + \frac{A'(J(1))(J''(1)+5)}{A(J(1))} \right. \\
&\quad \left. \left. + \frac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + 2 \right) \right\}^{\alpha/2}.
\end{aligned}$$

Thus, the desired result is obtained similar to the first case. □

Example 2.10. *In order to make the proof of the theorem more clear, let's give an example of the (a_n) and (b_n) sequences mentioned in the proof of Theorem 2.9. If we choose $\mathcal{A}(w) = 1, A(w) = 1, J(w) = w, H(w) = w$ and $\Psi(w) = exp(w)$, we get $a_n = \frac{n+2}{(n-1)(n-2)}$ and $b_n = \frac{2(n+2)}{(n-1)(n-2)}$. Thus, we see that the right-hand term in the theorem's statement approaches to 0, considering the fact that the terms $\frac{x^2}{1+x^2}$ and $\frac{x}{1+x^2}$ are both less than 1, since $x \in [0,\infty)$.*

Let $\mathcal{A}(w) = 1, A(w) = 1, J(w) = w, H(w) = w$ and $\Psi(w) = exp(w)$. Thus, we obtain ${}_p s_k(nx) = (nx)^k/k!$. In the following examples, we examine the convergence of the sequence of operators Ω_n to certain functions for these special choices of $\mathcal{A}, A, J, H$ and Ψ. It is observed that as n gets larger, $\Omega_n(f;x)$ converges to f.

Example 2.11. *For* $x \in [0,1]$, $n = 30$ *(red) and* $n = 50$ *(yellow), the approximation of* $\Omega_n(f;x)$ *to* $f(x) = \sqrt{x}$ *(blue) is shown in Figure 1.*

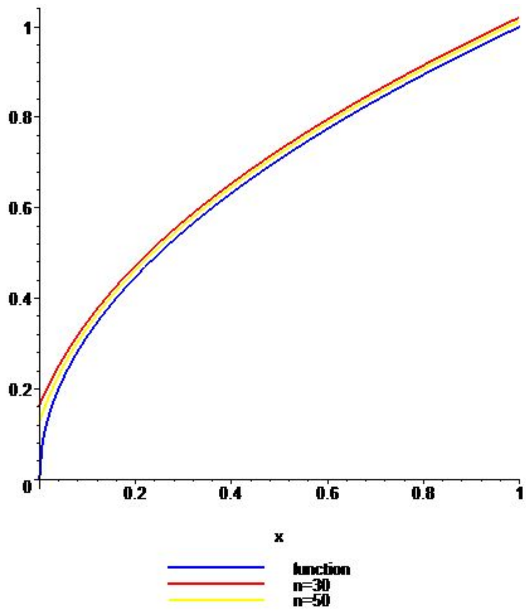

Figure 1: The convergenge of $\Omega_n(f;x)$ to $f(x) = \sqrt{x}$ for Ω_{30} and Ω_{50}

Example 2.12. *For* $x \in [0,1]$, $n = 5$ *(red) and* $n = 10$ *(yellow), the approximation of* $\Omega_n(f;x)$ *to* $f(x) = 1 + exp(-x)$ *(blue) is shown in Figure 2.*

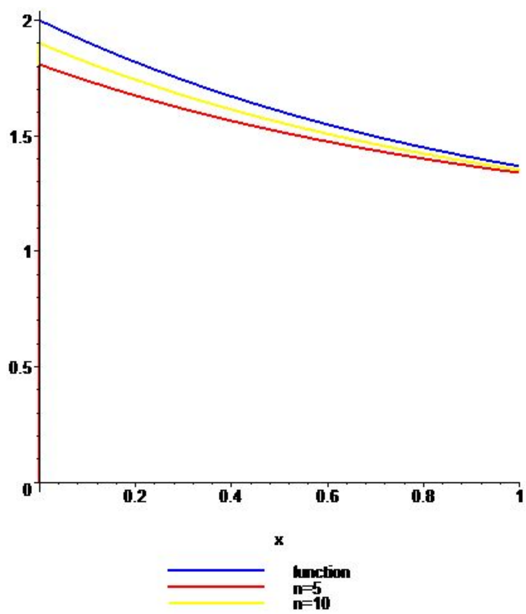

Figure 2: The convergenge of $\Omega_n(f;x)$ to $f(x) = 1 + exp(-x)$ for Ω_5 and Ω_{10}

3. Bivariate $\Omega_{n_1,n_2}(.;.)$ Operators

Let $I^2 = \{(x_1, x_2) : 0 \leq x_1 < \infty, 0 \leq x_2 < \infty\}$ and $C\left(I^2\right)$ is the space of all continuous functions on I^2 endowed with the sup norm. We introduce the bivariate variant of the operators involving beta function and Boas-Buck-Sheffer polynomials as follows:

$$\Omega_{n_1,n_2}(f; x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} R_1(n_1, x_1) R_2(n_2, x_2) \int_0^{\infty} \int_0^{\infty} U_1(n_1, x_1) U_2(n_2, x_2) f(t_1, t_2) dt_1 dt_2 \quad (8)$$

where $R_i(n_i, x_i) = \dfrac{1}{\mathcal{A}(1)A(J(1))\Psi(n_i x_i H(J(1)))}{}_p s_{k_i}(n_i x_i) \dfrac{1}{B(k_i+1, n_i)}$ and $U_i(n_i, xi) = \dfrac{(t_i)^{k_i}}{(1+t_i)^{n_i+k_i+1}}$, $i = 1, 2$.

In order to examine the approximation properties, we give the following lemmas using the test functions $e_{i,,j}(x_1,x_2) = x_1^i x_2^j$, $i, j \in \{0, 1, 2\}$.

Lemma 3.1. *Let Ω_{n_1,n_2} be the operators defined by (8). For all $n_1, n_2 \in \mathbb{N}$, we have*

$$\Omega_{n_1,n_2}(e_{0,0}; x_1, x_2) = 1,$$

$$\Omega_{n_1,n_2}(e_{1,0}; x_1, x_2) = \frac{n_1}{n_1-1}\left[\frac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}x_1 + \frac{1}{n_1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))}\right)\right] + \frac{1}{n_1-1},\ n_1 > 1$$

$$\Omega_{n_1,n_2}(e_{0,1}; x_1, x_2) = \frac{n_2}{n_2-1}\left[\frac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}x_2 + \frac{1}{n_2}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))}\right)\right] + \frac{1}{n_2-1},\ n_2 > 1$$

$$\begin{aligned}\Omega_{n_1,n_2}(e_{2,0}; x_1, x_2) &= \tfrac{n_1^2}{(n_1-1)(n_1-2)}\left\{\tfrac{\Psi''(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}x_1^2 + \tfrac{1}{n_1}\left(2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\tfrac{A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\right.\\ &\times \tfrac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}x_1 + \tfrac{1}{n_1^2}\left.\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \tfrac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + \tfrac{A''(J(1))}{A(J(1))} + 2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)}\tfrac{A'(J(1))}{A(J(1))} + \tfrac{A'(J(1))(J''(1)+5)}{A(J(1))}\right)\right\}\\ &+ \tfrac{2}{(n_1-1)(n_1-2)},\ n_1, n_2 > 2\end{aligned}$$

$$\begin{aligned}\Omega_{n_1,n_2}(e_{0,2}; x_1, x_2) &= \tfrac{n_2^2}{(n_2-1)(n_2-2)}\left\{\tfrac{\Psi''(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}x_2^2 + \tfrac{1}{n_2}\left(2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\tfrac{A'(J(1))}{A(J(1))} + J''(1) + H''(J(1)) + 4\right)\right.\\ &\times \tfrac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}x_2 + \tfrac{1}{n_2^2}\left.\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \tfrac{5\mathcal{A}'(1)}{\mathcal{A}(1)} + \tfrac{A''(J(1))}{A(J(1))} + 2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)}\tfrac{A'(J(1))}{A(J(1))} + \tfrac{A'(J(1))(J''(1)+5)}{A(J(1))}\right)\right\}\\ &+ \tfrac{2}{(n_2-1)(n_2-2)}\ n_1, n_2 > 2.\end{aligned}$$

Proof. Taking the definition (8) into account, one can obtain the following identities

$$\begin{aligned}\Omega_{n_1,n_2}(e_{0,0}; x_1, x_2) &= \Omega_{n_1}(e_0; x_1)\Omega_{n_2}(e_0; x_2),\\ \Omega_{n_1,n_2}(e_{1,0}; x_1, x_2) &= \Omega_{n_1}(e_1; x_1)\Omega_{n_2}(e_0; x_2),\\ \Omega_{n_1,n_2}(e_{0,1}; x_1, x_2) &= \Omega_{n_1}(e_0; x_1)\Omega_{n_2}(e_1; x_2),\\ \Omega_{n_1,n_2}(e_{2,0}; x_1, x_2) &= \Omega_{n_1}(e_2; x_1)\Omega_{n_2}(e_0; x_2),\\ \Omega_{n_1,n_2}(e_{0,2}; x_1, x_2) &= \Omega_{n_1}(e_0; x_1)\Omega_{n_2}(e_2; x_2).\end{aligned}$$

which prove Lemma 3.1. □

Lemma 3.2. *Let $\eta_{i,,j}^{x_1,x_2}(t_1, t_2) = (t_{1-}x_1)^i (t_{2-}x_2)^j$, $i, j \in \{0, 1, 2\}$ be the central moments. Then the operators Ω_{n_1,n_2} satisfy the following identities*

$$\Omega_{n_1,n_2}(\eta_{0,0}^{x_1,x_2}; x_1, x_2) = 1,$$

$$\Omega_{n_1,n_2}(\eta_{1,0}^{x_1,x_2}; x_1, x_2) = \left(\frac{n_1}{n_1-1}\frac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))} - 1\right)x_1 + \frac{1}{n_1-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(J(1))}{A(J(1))} + 1\right),$$

$$\Omega_{n_1,n_2}(\eta_{0,1}^{x_1,x_2};x_1,x_2)=\left(\frac{n_2}{n_2-1}\frac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}-1\right)x_2+\frac{1}{n_2-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(J(1))}{A(J(1))}+1\right),$$

$$\begin{aligned}\Omega_{n_1,n_2}(\eta_{1,1}^{x_1,x_2};x_1,x_2) &= \left(\frac{n_1}{n_1-1}\frac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}-1\right)x_1+\frac{1}{n_1-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(J(1))}{A(J(1))}+1\right)\\ &\times\left(\frac{n_2}{n_2-1}\frac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}-1\right)x_2+\frac{1}{n_2-1}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}+\frac{A'(J(1))}{A(J(1))}+1\right),\end{aligned}$$

$$\begin{aligned}\Omega_{n_1,n_2}(\eta_{2,0}^{x_1,x_2};x_1,x_2) &= \left(\tfrac{n_1^2}{(n_1-1)(n_1-2)}\tfrac{\Psi''(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}-\frac{2n_1}{n_1-1}\tfrac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}+1\right)x_1^2\\ &+ \left(\tfrac{n_1}{(n_1-1)(n_1-2)}\left(2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)}+2\tfrac{A'(J(1))}{A(J(1))}+J''(1)+H''(J(1))+4\right)\tfrac{\Psi'(n_1x_1H(J(1)))}{\Psi(n_1x_1H(J(1)))}\right.\\ &\left.-\tfrac{2}{n_1-1}\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\tfrac{A'(J(1))}{A(J(1))}+1\right)\right)x_1\\ &+ \tfrac{1}{(n_1-1)(n_1-2)}\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\tfrac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))}+\tfrac{A''(J(1))}{A(J(1))}+\tfrac{A'(J(1))(J''(1)+5)}{A(J(1))}+\tfrac{5\mathcal{A}'(1)}{\mathcal{A}(1)}+2\right),\end{aligned}$$

$$\begin{aligned}\Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2) &= \left(\tfrac{n_2^2}{(n_2-1)(n_2-2)}\tfrac{\Psi''(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}-\frac{2n_2}{n_2-1}\tfrac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}+1\right)x_2^2\\ &+ \left(\tfrac{n_2}{(n_2-1)(n_2-2)}\left(2\tfrac{\mathcal{A}'(1)}{\mathcal{A}(1)}+2\tfrac{A'(J(1))}{A(J(1))}+J''(1)+H''(J(1))+4\right)\tfrac{\Psi'(n_2x_2H(J(1)))}{\Psi(n_2x_2H(J(1)))}\right.\\ &\left.-\tfrac{2}{n_2-1}\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\tfrac{A'(J(1))}{A(J(1))}+1\right)\right)x_2\\ &+ \tfrac{1}{(n_2-1)(n_2-2)}\left(\tfrac{\mathcal{A}''(1)}{\mathcal{A}(1)}+\tfrac{2\mathcal{A}'(1)A'(J(1))}{\mathcal{A}(1)A(J(1))}+\tfrac{A''(J(1))}{A(J(1))}+\tfrac{A'(J(1))(J''(1)+5)}{A(J(1))}+\tfrac{5\mathcal{A}'(1)}{\mathcal{A}(1)}+2\right).\end{aligned}$$

Proof. Let $\eta_i^{x_1}(t_1)=(t_1-x_1)^i,\eta_i^{x_2}(t_2)=(t_2-x_2)^i,\ i=0,1,2$. Using the definition (8) and Lemma 2.2, we have the following identities

$$\begin{aligned}\Omega_{n_1,n_2}(\eta_{0,0}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_0^{x_1};x_1)\Omega_{n_2}(\eta_0^{x_2};x_2),\\ \Omega_{n_1,n_2}(\eta_{1,0}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_1^{x_1};x_1)\Omega_{n_2}(\eta_0^{x_2};x_2),\\ \Omega_{n_1,n_2}(\eta_{0,1}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_0^{x_1};x_1)\Omega_{n_2}(\eta_1^{x_2};x_2),\\ \Omega_{n_1,n_2}(\eta_{1,1}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_1^{x_1};x_1)\Omega_{n_2}(\eta_1^{x_2};x_2),\\ \Omega_{n_1,n_2}(\eta_{2,0}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_2^{x_1};x_1)\Omega_{n_2}(\eta_0^{x_2};x_2),\\ \Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2) &= \Omega_{n_1}(\eta_0^{x_1};x_1)\Omega_{n_2}(\eta_2^{x_2};x_2).\end{aligned}$$

which prove Lemma 3.2. □

Lemma 3.3. *Taking into account the Lemmas 3.1 and 3.2, we are led to*

(i)

$$\Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2)\le C_1(x_1^2+x_1+1),\ \textit{as } n_1\to\infty$$

(ii)

$$\Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2)\le C_2(x_2^2+x_2+1),\ \textit{as } n_2\to\infty$$

where C_1 and C_2 are constants.

Let f be a continuous function defined on I^2. The full modulus of continuity for bivariate case is defined as follows:

$$\omega(f;\delta) = sup\left|f(t_1,t_2) - f(x_1,x_2)\right| : (t_1,t_2),(x_1,x_2) \in I^2$$

and

$$\sqrt{(t_1-x_1)^2+(t_2-x_2)^2} \leq \delta.$$

Moreover,

$$\begin{aligned} \omega^1(f;\delta) &= \sup_{\substack{y\in[0,\infty)\\ |x_1-x_2|\leq\delta}} \left|f(x_1,y)-f(x_2,y)\right|, \\ \omega^2(f;\delta) &= \sup_{\substack{x\in[0,\infty)\\ |y_1-y_2|\leq\delta}} \left|f(x,y_1)-f(x,y_2)\right|, \end{aligned}$$

are the partial moduli of continuities of f with respect to x and y, respectively. It is clear that they satisfy the basic properties of the usual modulus of continuity [34].

Now we give the error of approximation of the operators Ω_{n_1,n_2} defined by (8) in the space of continuous functions on compact subset Δ of I^2.

Theorem 3.4. *For all $f \in C(\Delta)$ the following inequalities holds true:*
(i)

$$\left|\Omega_{n_1,n_2}(f;x_1,x_2) - f(x_1,x_2)\right| \leq 2\omega(f;\delta_{n_1,n_2}),$$

where $\delta_{n_1,n_2} = \left(C_1(x_1^2+x_1+1)+C_2(x_2^2+x_2+1)\right)^{1/2}$.
(ii)

$$\left|\Omega_{n_1,n_2}(f;x_1,x_2) - f(x_1,x_2)\right| \leq 2\left(\omega^1(f;\delta)+\omega^2(f;\delta)\right),$$

where $\delta_{n_1}^2 = \Omega_{n_1,n_2}(\eta_{2,0}^{x_1,x_2};x_1,x_2)$ *and* $\delta_{n_2}^2 = \Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2)$. *Here* $\eta_{2,0}^{x_1,x_2}$ *and* $\eta_{0,2}^{x_1,x_2}$ *are defined as in Lemma 3.2,* $n_1,n_2>2$.

Proof. (i) With the help of the definition of the full modulus of continuity of $f(x,y)$, we can write

$$\begin{aligned} \left|\Omega_{n_1,n_2}(f;x_1,x_2) - f(x_1,x_2)\right| &\leq \Omega_{n_1,n_2}\left(\left|f(t_1,t_2)-f(x_1,x_2)\right|;x_1,x_2\right) \\ &\leq \Omega_{n_1,n_2}\left(\omega\left(f;\sqrt{(t_1-x_1)^2+(t_2-x_2)^2}\right);x_1,x_2\right) \\ &\leq \omega(f;\delta_{n_1,n_2})\left[1+\frac{1}{\delta_{n_1,n_2}}\Omega_{n_1,n_2}\left(\sqrt{(t_1-x_1)^2+(t_2-x_2)^2};x_1,x_2\right)\right]. \end{aligned}$$

Applying Cauchy-Schwarz inequality and Lemma 3.3, we have

$$\begin{aligned} \left|\Omega_{n_1,n_2}(f;x_1,x_2) - f(x_1,x_2)\right| &\leq \omega(f;\delta_{n_1,n_2})\left[1+\frac{1}{\delta_{n_1,n_2}}\left\{\Omega_{n_1,n_2}\left((e_{1,0}-x_1)^2+(e_{0,1}-x_2)^2;x_1,x_2\right)\right\}^{1/2}\right] \\ &\leq \omega(f;\delta_{n_1,n_2})\left[1+\frac{1}{\delta_{n_1,n_2}}\left\{\Omega_{n_1,n_2}\left((e_{1,0}-x_1)^2;x_1,x_2\right)\right.\right. \\ &\qquad \left.\left.+\Omega_{n_1,n_2}\left((e_{0,1}-x_2)^2;x_1,x_2\right)\right\}^{1/2}\right] \\ &\leq \omega(f;\delta_{n_1,n_2})\left[1+\frac{1}{\delta_{n_1,n_2}}\left\{C_1(x_1^2+x_1+1)+C_2(x_2^2+x_2+1)\right\}^{1/2}\right]. \end{aligned}$$

This proves the result.

(ii) Using the definition of the partial moduli of continuity, Lemma 3.3, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|\Omega_{n_1,n_2}(f;x_1,x_2)-f(x_1,x_2)\right| &\leq \Omega_{n_1,n_2}\left(\left|f(t_1,t_2)-f(x_1,x_2)\right|;x_1,x_2\right)\\
&\leq \Omega_{n_1,n_2}\left(\left|f(t_1,t_2)-f(x_1,t_2)\right|;x_1,x_2\right)+\Omega_{n_1,n_2}\left(\left|f(x_1,t_2)-f(x_1,x_2)\right|;x_1,x_2\right)\\
&\leq \Omega_{n_1,n_2}\left(\omega^1(f;|t_1-x_1|);x_1,x_2\right)+\Omega_{n_1,n_2}\left(\omega^2(f;|t_2-x_2|);x_1,x_2\right)\\
&\leq \omega^1(f;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}\Omega_{n_1,n_2}\left(|t_1-x_1|;x_1,x_2\right)\right]\\
&+ \omega^2(f;\delta_{n_2})\left[1+\frac{1}{\delta_{n_1}}\Omega_{n_1,n_2}\left(|t_2-x_2|;x_1,x_2\right)\right]\\
&\leq \omega^1(f;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}\left(\Omega_{n_1,n_2}\left((t_1-x_1)^2;x_1,x_2\right)\right)^{1/2}\right]\\
&+ \omega^2(f;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}\left(\Omega_{n_1,n_2}\left((t_2-x_2)^2;x_1,x_2\right)\right)^{1/2}\right].
\end{aligned}$$

Choosing $\delta_{n_1}=\left(\Omega_{n_1,n_2}(\eta_{2,0}^{x_1,x_2};x_1,x_2)\right)$ and $\delta_{n_2}=\Omega_{n_1,n_2}(\eta_{0,2}^{x_1,x_2};x_1,x_2)$, for all $(x_1,x_2)\in I^2$, we reach the desired result. □

Let $\mathcal{A}(w)=1, A(w)=1, J(w)=w, H(w)=w$ and $\Psi(w)=exp(w)$. Thus, we obtain ${}_p s_{k_1}(n_1x_1)=(n_1x_1)^{k_1}/k_1!$ and ${}_p s_{k_2}(n_2x_2)=(n_2x_2)^{k_2}/k_2!$. In the following examples, we observe that, for $n_1=n_2=20$, the convergence of the operator Ω_{n_1,n_2} to the function f is better than the case $n_1=n_2=10$.

Example 3.5. *Approximation of $\Omega_{n_1,n_2}(f;x_1,x_2)$ to $f(x_1,x_2)=x_1^2x_2^3+x_1^3x_2^2$ (blue) for $n_1=n_2=10$ (yellow) and $n_1=n_2=20$ (red) is illustrated in Figure 3.*

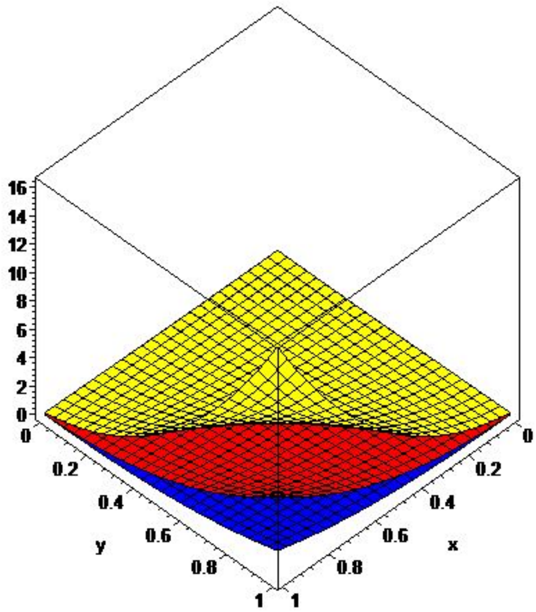

Figure 3: The convergenge of $\Omega_{n_1,n_2}(f;x_1,x_2)$ to $f(x_1,x_2)=x_1^2x_2^3+x_1^3x_2^2$ for $\Omega_{10,10}$ and $\Omega_{20,20}$

Example 3.6. *Approximation of* $\Omega_{n_1,n_2}(f;x_1,x_2)$ *to* $f(x_1,x_2) = \sqrt{x_1x_2}$ *(blue) for* $n_1 = n_2 = 10$ *(yellow) and* $n_1 = n_2 = 20$ *(red) is illustrated in Figure 4.*

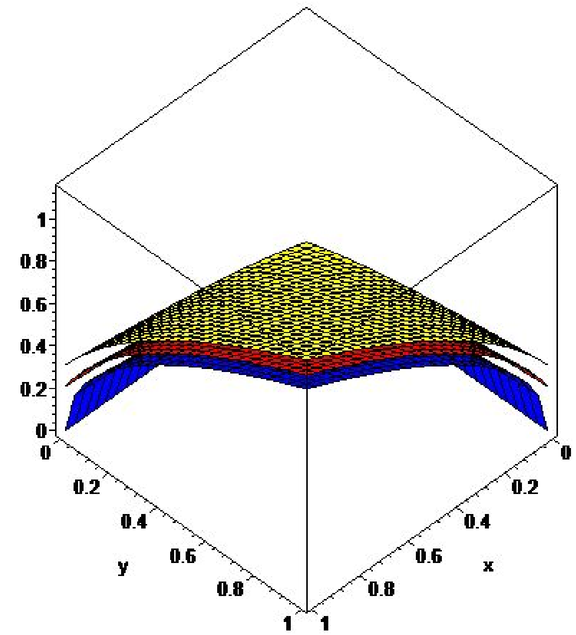

Figure 4: The convergenge of $\Omega_{n_1,n_2}(f;x_1,x_2)$ to $f(x_1,x_2) = \sqrt{x_1x_2}$ for $\Omega_{10,10}$ and $\Omega_{20,20}$

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