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Characterization of the spectra of the Hill's equation coupled to different boundary value conditions and application to nonlinear boundary problems

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Abstract. In this paper we will characterize the spectrum of the second order Hill's equation coupled to several boundary value conditions. More concisely, the idea consists of study the spectrum of the second-order differential Hill's equation coupled to Initial, Final, Neumann, Dirichlet, Periodic and Mixed boundary conditions, by applying the equality (10) proved by the authors in [5] and expressing the Green's function of the Hill's equation coupled to a given boundary condition as a combination of the Green's function related to another different boundary condition. These spectra are characterized as suitable sets of real values that verify an equality that depends on the Green's function of each case. We will also deduce some properties of these spectra and identities between Green's functions. The work continuous on the lines initiated on [6] and [3]. It is important to remark that the ideas and arguments used to deduce the comparison between the corresponding spectrum of the considered problems, and their characterization in many cases, are completely different to the ones used in [3].

1. Introduction

The topic of nonlinear boundary value problems has been widely considered since long time in the literature, with special attention to the existence of solutions for such problems. It is in this context where the corresponding spectral theory arises, due to the fact that topological methods, such as degree theory, fixed point index in cones or lower and upper functions method, are mainly based on the invertibility of certain linear operators, for which the eigenvalues of such operators need to be considered.

Moreover, these eigenvalues also appear in the search of constant sign solutions as they usually define the limits of the regions in which the corresponding Green's functions have negative or positive sign (see [7] for details). This constant sign for the Green's function related to the considered problem is an usual hypothesis when looking for positive solutions of the problem or when monotone iterative techniques are used (see [2, 23] and references therein)

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The reader can find some classical and recent references, in which the spectral theory is considered in, among others, the following ones [1, 3, 8–10, 13, 14, 16, 17, 19–21, 24–30].

In this paper, we will consider the Hill's equation

$$u''(t) + a(t)\,u(t) = 0, \quad t \in [0, T],$$

and show some relations between the spectra of the usual two-point boundary value problems: Neumann, Dirichlet, mixed and periodic conditions.

As it has been noticed in [3] and it is very well know in the literature, under some suitable regularity assumptions on the coefficients of the equation, the general second order differential equation can be, after a simple change of variables, rewritten as a Hill's equation. Such equation models the linear approach of the pendulum movement, the classical spring mass system equation or, among many others, the Airy's and the Mathieu's equation.

The main idea to prove the aforementioned relations is based on rewriting each of the considered problems as a combination of the other ones, by means of considering an homogeneous problem of a certain type as a non homogeneous one of a different type, involving some functional conditions on the boundaries. This technique, as it has been showed in [6], allows us to rewrite the expression of a certain Green's function in terms of another one. From these relations, we will deduce now some connections between various spectra and some other sets characterized as the zeros of certain function. More precisely, for each pair of boundary conditions, we will prove a relation of the following form

$$S_{1,2} \subset \Lambda_1 \subset \Lambda_2 \sqcup S_{1,2},$$

where Λ_1 and Λ_2 will be the spectra of the Hill's equation related two different boundary conditions and $S_{1,2}$ will be some appropriate set, characterized as the zeros of a suitable function and by $\sqcup$ we denote the disjoint union of two sets. In many of the considered cases, when the spectra Λ_1 and Λ_2 are disjoint, we immediately characterize $\Lambda_1 = S_{1,2}$. This last characterization shows to be very useful from a numerical point of view, as it is simple to compute the zeros of the function which defines the set $S_{1,2}$. Some of the characterizations proved using this technique were very previously known (some of them can be found, for instance, in [24]) although some others are, as far as we are concerned, new in the literature.

It is important to point out that the arguments used here are completely different to the ones used in [3], where the ideas are based in the extension (even or odd, depending of the boundary condition) of the potential $a(t)$ to the intervals $[0, 2\,T]$ and $[0, 4\,T]$.

The paper is divided in four sections: After the introductory section, we compile, in Section 2, some preliminaries regarding the decomposition of some Green's functions in terms of another one. In Section 3 is deduced the decomposition and characterization of all the aforementioned spectra. Finally, Section 4 shows and application of the obtained results to prove the existence of solution of nonlinear problems related to the Hill's equation coupled to different boundary conditions.

2. Preliminaries

Let us define

$$T_n[\lambda]\,u(t) := u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + (a_n(t) + \lambda)u(t), \quad t \in I, \quad n \geq 1, \quad \lambda \in \mathbb{R},$$

with $I \equiv [0, T]$, $a_i : I \to \mathbb{R}$, $a_i \in L^\alpha(I)$, $\alpha \geq 1$ and

$$B_i(u) := \sum_{j=0}^{n-1} \left(\alpha^i_j u^{(j)}(0) + \beta^i_j u^{(j)}(T)\right), \quad i = 1, \ldots, n,$$

being α^i_j, β^i_j real constants for all $i = 1, \ldots, n$, and $j = 0, \ldots, n-1$.

Consider the space

$$W^{n,1}(I) = \{u \in C^{n-1}(I) : u^{(n-1)} \in AC(I)\},$$

where $AC(I)$ denotes the set of absolutely continuous functions on I. In particular, we will consider $X \subset W^{n,1}(I)$ a Banach space such that the following definition is satisfied.

Definition 2.1. *Given a Banach space X, operator $T_n[\lambda]$ is said to be nonresonant in X if and only if the homogeneous equation*

$$T_n[\lambda]\, u(t) = 0 \ \text{ a.e. } \ t \in I, \quad u \in X,$$

has only the trivial solution.

It is very well-known, see for instance [11, 15], that, if operator $T_n[\lambda]$ is nonresonant in X, then, for any $\sigma \in L^1(I)$, the non-homogeneous linear problem

$$T_n[\lambda]\, u(t) = \sigma(t) \ \text{ a.e. } \ t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t) = \int_0^T G_\lambda(t,s)\, \sigma(s)\, ds, \quad \forall t \in I.$$

Here G_λ denotes the Green's function related to operator $T_n[\lambda]$ on X, whose definition can be found in [2, Definition 1.4.1].

We compile now some properties of Green's functions related to operator $T_n[\lambda]$. The following result is provided in [5, Lemma 1].

Lemma 2.2. *There exists a unique Green's function g_λ related to problem*

$$\begin{cases} T_n[\lambda]\, u(t) = 0, & \text{a.e. } t \in I, \\ B_i(u) = 0, & i = 1, \dots n, \end{cases} \tag{2.1}$$

if and only if for any $i \in \{1, \cdots, n\}$, the following problem

$$\begin{cases} T_n[\lambda]\, u(t) = 0, & \text{a.e. } t \in I, \\ B_j(u) = 0, & j \neq i, \ j = 1, \dots n, \\ B_i(u) = 1, & \end{cases} \tag{2.2}$$

has a unique solution, that we denote as $\omega_i(t)$, $t \in I$.

Let us consider the following problem

$$\begin{cases} T_n[\lambda]\, u(t) = \sigma(t), & \text{a.e. } t \in I, \\ B_i(u) = 0, & i = 1, \dots n, \end{cases} \tag{2.3}$$

with $\sigma \in L^1(I)$.

Here, by considering $C_i : C^{n-1}(I) \to \mathbb{R}$, $\quad i = 1, \dots n$, n linear and continuous operators, we formulate the following result for general n-th order non-local boundary value problems.

Theorem 2.3. *[5, Theorem 2] Assume that Problem (2.1) has $u = 0$ as its unique solution and let g_λ be its related Green's function. Let $\sigma \in L^1(I)$, and δ_i, $i = 1, \dots, n$, be such that*

$$\det(I_n - A) \neq 0,$$

with I_n the identity matrix of order n and $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}$ given by

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1, \dots, n\}.$$

Then Problem

$$\begin{cases} T_n[\lambda]\, u(t) = \sigma(t), \quad a.e. \;\; t \in I, \\ \qquad B_i(u) = \delta_i\, C_i(u), \quad i = 1, \dots, n, \end{cases} \tag{2.4}$$

has a unique solution $u \in W^{n,1}(I)$, given by the expression

$$u(t) = \int_0^T G_\lambda(t, s, \delta_1, \dots, \delta_n)\, \sigma(s) ds,$$

where

$$G_\lambda(t, s, \delta_1, \dots, \delta_n) := g_\lambda(t, s) + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j(g_\lambda(\cdot, s)), \quad t,\ s \in I, \tag{2.5}$$

with ω_i defined in (2.2) *and $B = (b_{ij})_{n\times n} = (I_n - A)^{-1}$.*

For any $\lambda \in \mathbb{R}$, consider operator $L[\lambda]$ defined as follows for any $u \in W^{2,1}(I)$

$$L[\lambda]\, u(t) := u''(t) + (a(t) + \lambda)\, u(t), \quad t \in I.$$

For this operator, to indicate the dependence of the Green's function on the parameter λ, we will denote by $G[\lambda]$ the Green's function related to $L[\lambda]$.

In this paper, we will deal with some problems related to operator $L[\lambda]$, which we describe in the sequel:

- Initial problem:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_I = \{u \in W^{2,1}(I) : u(0) = u'(0) = 0\}. \tag{2.6}$$

- Final problem:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_F = \{u \in W^{2,1}(I) : u(T) = u'(T) = 0\}. \tag{2.7}$$

- Neumann problem:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_N = \{u \in W^{2,1}(I) : u'(0) = u'(T) = 0\}. \tag{2.8}$$

- Dirichlet problem:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_D = \{u \in W^{2,1}(I) : u(0) = u(T) = 0\}. \tag{2.9}$$

- Mixed problem 1:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_{M_1} = \{u \in W^{2,1}(I) : u'(0) = u(T) = 0\}. \tag{2.10}$$

- Mixed problem 2:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e.} \;\; t \in I, \; u \in X_{M_2} = \{u \in W^{2,1}(I) : u(0) = u'(T) = 0\}. \tag{2.11}$$

- Periodic problem:

$$L[\lambda]\, u(t) \; = \sigma(t), \quad \text{a.e.} \;\; t \in I, \quad u \in X_P = \{u \in W^{2,1}(I) : u(0) = u(T),\ u'(0) = u'(T)\}. \tag{2.12}$$

We denote by $G_I[\lambda]$, $G_F[\lambda]$, $G_D[\lambda]$, $G_P[\lambda]$, $G_N[\lambda]$, $G_{M_1}[\lambda]$ and $G_{M_2}[\lambda]$ the Green's function related to Initial, Final, Dirichlet, Periodic, Neumann, Mixed 1 and Mixed 2 problems, respectively. Moreover, we denote by u_D, u_P, u_N, u_{M_1} and u_{M_2} the solutions of the corresponding problems and by λ_0^D, λ_0^P, λ_0^N, $\lambda_0^{M_1}$ and $\lambda_0^{M_2}$ the first eigenvalues of each problem.

3. Decomposition of the Spectra

In this section,using Theorem 2.3 we will show the relation between the spectra of the problems (2.3) and (2.4).

We will denote by $\Lambda_{(3)}$, $\Lambda_{(4)}$, Λ_D, Λ_P, Λ_N, Λ_{M_1} and Λ_{M_2}, the set of eigenvalues of problems (2.3), (2.4), (2.9), (2.12), (2.8), (2.10) and (2.11), respectively.

Note that the spectra of problems (2.6) and (2.7) (denoted by Λ_I and Λ_F) are empty, i.e., $\Lambda_I = \Lambda_F = \emptyset$.

Moreover, we denote by $\omega_I^i[\lambda]$, $\omega_F^i[\lambda]$, $\omega_D^i[\lambda]$, $\omega_P^i[\lambda]$, $\omega_N^i[\lambda]$, $\omega_{M_1}^i[\lambda]$ and $\omega_{M_2}^i[\lambda]$, $i = 1,2$, the functions ω_i, $i = 1,2$ of Lemma 2.2 of the Initial, Final, Dirichlet, Periodic, Neumann, Mixed 1 and Mixed 2 problems, respectively.

Note that, as it has been shown in [6, page 9], that the expressions of functions $\omega_D^i[\lambda]$, $\omega_P^i[\lambda]$, $\omega_N^i[\lambda]$, $\omega_{M_1}^i[\lambda]$ and $\omega_{M_2}^i[\lambda]$, $i = 1,2$ are given by

$$\begin{array}{lll} \omega_D^1[\lambda](t) = -\frac{\partial}{\partial s} G_D[\lambda](t,0), & \omega_D^2[\lambda](t) = \frac{\partial}{\partial s} G_D[\lambda](t,T), & \omega_P^1[\lambda](t) = -\frac{\partial}{\partial s} G_P[\lambda](t,0), \\ \omega_P^2[\lambda](t) = G_P[\lambda](t,0), & \omega_N^1[\lambda](t) = G_N[\lambda](t,0), & \omega_N^2[\lambda](t) = -G_N[\lambda](t,T), \\ \omega_{M_2}^1[\lambda](t) = -\frac{\partial}{\partial s} G_{M_2}[\lambda](t,0), & \omega_{M_2}^2[\lambda](t) = -G_{M_2}[\lambda](t,T), & \omega_{M_1}^1[\lambda](t) = G_{M_1}[\lambda](t,0), \\ \omega_{M_1}^2[\lambda](t) = -\frac{\partial}{\partial s} G_{M_1}[\lambda](t,T). & & \end{array}$$

Let us define

$$S_{(4)} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{(3)} \;/\; \det(I_n - A) = 0\Big\},$$

with A the matrix defined on Theorem 2.3.

Next, we give some results that relate the spectra of problems (2.3) and (2.4) through an inclusion relationship.

Theorem 3.1. *It holds that*

$$S_{(4)} \subset \Lambda_{(4)} \subset \Lambda_{(3)} \sqcup S_{(4)} \quad (\sqcup \textit{ denotes the disjoint union}).$$

Proof. Suppose that $\lambda \in S_{(4)}$. Let's see that $\lambda \in \Lambda_{(4)}$. Since $\lambda \notin \Lambda_{(3)}$ we have that for $\sigma = 0$ the solution of problem (2.4) is given by the following expression

$$u(t) = \sum_{i=1}^{n} \omega_i(t) \delta_i C_i(u), \quad t \in I. \tag{3.1}$$

Applying linear continuous operators C_j on both sides of (3.1) we infer that

$$C_j(u) = C_j\left(\sum_{i=1}^{n} \delta_i\, \omega_i(t)\, C_i(u)\right) = \sum_{i=1}^{n} \delta_i\, C_j(\omega_i)\, C_i(u), \quad j = 1,\ldots,n,$$

from which we deduce that

$$C_j(u) - \sum_{i=1}^{n} \delta_i\, C_j(\omega_i)\, C_i(u) = 0, \quad j = 1,\ldots,n.$$

Therefore, we arrive at the following systems of equations

$$(I_n - A)\begin{pmatrix} C_1(u) \\ C_2(u) \\ \vdots \\ C_n(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.2}$$

Since $\lambda \in S_{(4)}$ the previous system has infinite solutions. Then, problem (2.4) has infinite solutions for $\sigma = 0$ and so $\lambda \in \Lambda_{(4)}$. Therefore, we have that $S_{(4)} \subset \Lambda_{(4)}$.

In the other hand, Theorem 2.3 states that if $\lambda \notin \Lambda_{(3)}$ and $\lambda \notin S_{(4)}$ then $\lambda \notin \Lambda_{(4)}$, that is,

$$\Lambda_{(4)} \subset \Lambda_{(3)} \cup S_{(4)},$$

being the union disjoint since $S_{(4)} \cap \Lambda_{(3)} = \emptyset$. □

Note that, in general, the inclusions given in previous result are not equalities. In particular, it is easy to prove the following results.

Corollary 3.2. *Assume that $\lambda \notin \Lambda_{(3)}$, then it holds that*

$$\lambda \in \Lambda_{(4)} \text{ if and only if } \lambda \in S_{(4)}.$$

Corollary 3.3. *The following properties hold:*

1. $S_{(4)} = \Lambda_{(4)}$ *if and only if* $\Lambda_{(3)} \cap \Lambda_{(4)} = \emptyset$.
2. $\Lambda_{(4)} = \Lambda_{(3)} \sqcup S_{(4)}$ *if and only if* $\Lambda_{(3)} \subset \Lambda_{(4)}$.
3. $\Lambda_{(4)} \subset \Lambda_{(3)}$ *if and only if* $S_{(4)} = \emptyset$.

Later we will show an example with strict inclusions.

3.1. Spectral characterization of the Periodic problem

In this subsection, using the inclusions proved in Theorem 3.1 and the identity obtained in Corollary 3.3 we will obtain the expression for spectrum of Periodic problem (2.12) as the set of real values that satisfy an equality involving the Green's function of another problem.

We will do the proof of the characterization of the spectrum of the Periodic problem by means of the Initial one. The idea consists on the fact that we can think the periodic boundary conditions as the two dimensional initial condition $(u(0), u'(0))$ equals to $(u(T), u'(T)) \in \mathbb{R}^2$, where this last value has the role of a given position and velocity.

The rest of the situations are proved in a similar way and we will omit them. The proofs are on the basis that we can write any boundary condition as a function of any other one.

Thus, for the Periodic and Initial problems we arrive at the next result.

Theorem 3.4. *If*

$$|I - A_I^1[\lambda]| := \left(1 + \frac{\partial}{\partial s} G_I[\lambda](T,0)\right)\left(1 - \frac{\partial}{\partial t} G_I[\lambda](T,0)\right) + G_I[\lambda](T,0)\,\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0) \neq 0,$$

then the following equality holds

$$\begin{aligned} G_P[\lambda](t,s) =& G_I[\lambda](t,s) - \frac{G_I[\lambda](T,s)}{|I - A_I^1[\lambda]|}\left(\left(1 - \frac{\partial}{\partial t} G_I[\lambda](T,0)\right)\frac{\partial}{\partial s} G_I[\lambda](t,0) + G_I[\lambda](t,0)\,\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0)\right) \\ &+ \frac{\frac{\partial}{\partial t} G_I[\lambda](T,s)}{|I - A_I^1[\lambda]|}\left(G_I[\lambda](t,0)\left(1 + \frac{\partial}{\partial s} G_I[\lambda](T,0)\right) - G_I[\lambda](T,0)\,\frac{\partial}{\partial s} G_I[\lambda](t,0)\right), \quad \forall (t,s) \in I \times I. \end{aligned}$$

Proof. Let us rewrite Periodic problem as a non homogeneous Initial Value Problem:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in I, \quad u(0) = u(T), \quad u'(0) = u'(T).$$

Using the notation of Theorem 2.3, we have that in this case $C_1(u) = u(T)$, $C_2(u) = u'(T)$ and $\delta_1 = \delta_2 = 1$. Using the matrix argument developed in [6] it is immediate to verify that $\omega_I^1[\lambda](t) = -\frac{\partial}{\partial s} G_I[\lambda](t,0)$, $\omega_I^2[\lambda](t) = G_I[\lambda](t,0)$ and the matrix $A_I^1[\lambda]$ in this case is given by

$$A_I^1[\lambda] = \begin{pmatrix} -\frac{\partial}{\partial s} G_I[\lambda](T,0) & G_I[\lambda](T,0) \\ -\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0) & \frac{\partial}{\partial t} G_I[\lambda](T,0) \end{pmatrix}$$

and $|I - A_I^1[\lambda]| = \left(1 + \frac{\partial}{\partial s} G_I[\lambda](T,0)\right)\left(1 - \frac{\partial}{\partial t} G_I[\lambda](T,0)\right) + G_I[\lambda](T,0)\,\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0) \neq 0$.

Therefore, by applying formula (2.5) and taking into account that

$$(I - A_I^1[\lambda])^{-1} = \begin{pmatrix} \frac{1 - \frac{\partial}{\partial t} G_I[\lambda](T,0)}{|I - A_I^1[\lambda]|} & \frac{G_I[\lambda](T,0)}{|I - A_I^1[\lambda]|} \\ -\frac{\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0)}{|I - A_I^1[\lambda]|} & \frac{1 + \frac{\partial}{\partial s} G_I[\lambda](T,0)}{|I - A_I^1[\lambda]|} \end{pmatrix}$$

we obtain the result. □

Now, we will consider the Periodic and Final problems. Following the same argument than before, we arrive at the following result.

Theorem 3.5. *Assume that*

$$|I - A_F^1[\lambda]| := \left(1 - \frac{\partial}{\partial s} G_F[\lambda](0,T)\right)\left(1 + \frac{\partial}{\partial t} G_F[\lambda](0,T)\right) + G_F[\lambda](0,T)\,\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T) \neq 0,$$

then it holds that

$$\begin{aligned} G_P[\lambda](t,s) =& G_F[\lambda](t,s) + \frac{G_F[\lambda](0,s)}{|I - A_F^1[\lambda]|}\left(\left(1 + \frac{\partial}{\partial t} G_F[\lambda](0,T)\right)\frac{\partial}{\partial s} G_F[\lambda](t,T) - G_F[\lambda](t,T)\,\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T)\right) \\ & - \frac{\frac{\partial}{\partial t} G_F[\lambda](0,s)}{|I - A_F^1[\lambda]|}\left(G_F[\lambda](t,T)\left(1 - \frac{\partial}{\partial s} G_F[\lambda](0,T)\right) + G_F[\lambda](0,T)\,\frac{\partial}{\partial s} G_F[\lambda](t,T)\right), \quad \forall (t,s) \in I \times I. \end{aligned}$$

Proof. We write the Periodic problem based on the Final problem as the follows

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in I, \quad u(T) = u(0), \quad u'(T) = u'(0).$$

In this case, we have that $C_1(u) = u(0)$, $C_2(u) = u'(0)$ and $\delta_1 = \delta_2 = 1$. Moreover, using the matrix argument developed in [6] we have that $\omega_F^1[\lambda](t) = \frac{\partial}{\partial s} G_F[\lambda](t,T)$, $\omega_F^2[\lambda](t) = -G_F[\lambda](t,T)$ and the matrix $A_F^1[\lambda]$ is

$$A_F^1[\lambda] = \begin{pmatrix} \frac{\partial}{\partial s} G_F[\lambda](0,T) & -G_F[\lambda](0,T) \\ \frac{\partial}{\partial t \partial s} G_F[\lambda](0,T) & -\frac{\partial}{\partial t} G_F[\lambda](0,T) \end{pmatrix}$$

and

$$|I - A_F^1[\lambda]| = \left(1 - \frac{\partial}{\partial s} G_F[\lambda](0,T)\right)\left(1 + \frac{\partial}{\partial t} G_F[\lambda](0,T)\right) + G_F[\lambda](0,T)\,\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T).$$

So,

$$(I - A_F^1[\lambda])^{-1} = \begin{pmatrix} \frac{1 + \frac{\partial}{\partial t} G_F[\lambda](0,T)}{|I - A_F^1[\lambda]|} & \frac{-G_F[\lambda](0,T)}{|I - A_F^1[\lambda]|} \\ \frac{\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T)}{|I - A_F^1[\lambda]|} & \frac{(1 - \frac{\partial}{\partial s} G_F[\lambda](0,T)}{|I - A_F^1[\lambda]|} \end{pmatrix}.$$

In consequence, as a direct application of the equality (2.5) we obtain the result. □

Remark 3.6. *It should be noted that the relations obtained in Theorems 3.4 and 3.5 complement the decompositions obtained in [6].*

Theorem 3.7. *The spectrum of the periodic problem* (2.12) *has the following properties:*

1)
$$\begin{aligned} \Lambda_P &= \left\{\lambda \in \mathbb{R} \;/\; \left(1 + \frac{\partial}{\partial s} G_I[\lambda](T,0)\right)\left(1 - \frac{\partial}{\partial t} G_I[\lambda](T,0)\right) + G_I[\lambda](T,0)\,\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0) = 0\right\} \\ &= \left\{\lambda \in \mathbb{R} \;/\; 2 - \omega_I^1[\lambda](T) - (\omega_I^2)'[\lambda](T) = 0\right\}. \end{aligned}$$

2) $$\Lambda_P = \left\{\lambda \in \mathbb{R} \ / \ \left(1 - \frac{\partial}{\partial s} G_F[\lambda](0,T)\right)\left(1 + \frac{\partial}{\partial t} G_F[\lambda](0,T)\right) + G_F[\lambda](0,T)\,\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T) = 0\right\}$$
$$= \left\{\lambda \in \mathbb{R} \ / \ 2 - \omega_F^1[\lambda](0) - (\omega_F^2)'[\lambda](0) = 0\right\}.$$

3) $S_N^1 \subset \Lambda_P \subset \Lambda_N \sqcup S_N^1$ *where*

$$S_N^1 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ G_N[\lambda](T,0) - G_N[\lambda](0,0) - G_N[\lambda](T,T) + G_N[\lambda](0,T) = 0\right\}$$
$$= \left\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ \omega_N^1[\lambda](T) - \omega_N^1[\lambda](0) + \omega_N^2[\lambda](T) - \omega_N^2[\lambda](0) = 0\right\}.$$

4) $S_D^1 \subset \Lambda_P \subset \Lambda_D \sqcup S_D^1$ *where*

$$S_D^1 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ \frac{\partial^2}{\partial s \partial t} G_D[\lambda](T,T) + \frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0) - 2\frac{\partial^2}{\partial s \partial t} G_D[\lambda](T,0) = 0\right\}$$
$$= \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ 2(\omega_D^1)'[\lambda](T) + (\omega_D^2)'[\lambda](T) - (\omega_D^1)'[\lambda](0) = 0\right\}.$$

5) $S_{M_1}^1 \subset \Lambda_P \subset \Lambda_{M_1} \sqcup S_{M_1}^1$ *where*

$$S_{M_1}^1 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ \left(1 - \frac{\partial}{\partial t} G_{M_1}[\lambda](T,0)\right)\left(1 + \frac{\partial}{\partial s} G_{M_1}[\lambda](0,T)\right) + G_{M_1}[\lambda](0,0)\,\frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](T,T) = 0\right\}$$
$$= \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ \left(1 - (\omega_{M_1}^1)'[\lambda](T)\right)\left(1 - \omega_{M_1}^2[\lambda](0)\right) - \omega_{M_1}^1[\lambda](0)\,(\omega_{M_1}^2)'[\lambda](T) = 0\right\}.$$

6) $S_{M_2}^1 \subset \Lambda_P \subset \Lambda_{M_2} \sqcup S_{M_2}^1$ *where*

$$S_{M_2}^1 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ \left(1 + \frac{\partial}{\partial s} G_{M_2}[\lambda](T,0)\right)\left(1 + \frac{\partial}{\partial t} G_{M_2}(0,T)\right) - G_{M_2}[\lambda](T,T)\,\frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0) = 0\right\}$$
$$= \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ \left(1 - \omega_{M_2}^1[\lambda](T)\right)\left(1 - (\omega_{M_2}^2)'(0)\right) - \omega_{M_2}^2[\lambda](T)\,(\omega_{M_2}^1)'[\lambda](0) = 0\right\}.$$

Proof. It follows from Theorem 2.3 together with the relationships obtained in Theorem 3.4, Theorem 3.5, [6, Theorem 4.20], [6, Theorem 4.10], [6, Theorem 4.24] and [6, Theorem 4.23]. Equalities 1) and 2) follow from the fact that $\Lambda_I = \emptyset$ and $\Lambda_F = \emptyset$. □

Remark 3.8. *The Characterization* 1) *given in previous result is very well known and has been widely used in the literature, see for instance [4, 24, 29].*

Remark 3.9. *In cases* 3) *to* 6) *in general there is no characterization of Λ_P since $\Lambda_P \cap \Lambda_N$, $\Lambda_P \cap \Lambda_D$, $\Lambda_P \cap \Lambda_{M_1}$ and $\Lambda_P \cap \Lambda_{M_2}$ can be non-empty (in which case it would not be true that $S_N^1 = \Lambda_P$, $S_D^1 = \Lambda_P$, $S_{M_1}^1 = \Lambda_P$ and $S_{M_2}^1 = \Lambda_P$). The following examples show some cases where $\Lambda_P \cap \Lambda_N$ and $\Lambda_P \cap \Lambda_D$ are nonempty. In the case of intersections $\Lambda_P \cap \Lambda_{M_1}$ and $\Lambda_P \cap \Lambda_{M_2}$ will be see later in Examples 3.34 and 3.35 a case of non-empty intersections once we have characterized the spectrum of Mixed problems.*

Corollary 3.10. *The following inclusions hold:*

1) $S_N^1 \cup S_D^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_D \sqcup S_D^1)$.
2) $S_N^1 \cup S_{M_1}^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_{M_1} \sqcup S_{M_1}^1)$.
3) $S_N^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.
4) $S_D^1 \cup S_{M_1}^1 \subset \Lambda_P \subset (\Lambda_D \sqcup S_D^1) \cap (\Lambda_{M_1} \sqcup S_{M_1}^1)$.
5) $S_D^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_D \sqcup S_D^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.
6) $S_{M_1}^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_{M_1} \sqcup S_{M_1}^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.

Example 3.11. *We consider the differential equation $u''(t) + \lambda\, u(t) = 0$, $t \in [0,1]$. In this case, $a(t) = 0$, $t \in [0,1]$.*

It is very well known that $\Lambda_P = \left\{(2k\pi)^2 : k = 0, 1 \dots\right\}$ and $\Lambda_N = \left\{(k\pi)^2 : k = 0, 1, \dots\right\}$. In this case, it is evident that the intersection $\Lambda_P \cap \Lambda_N$ is nonempty.

Example 3.12. *We use in this example the same equation as in Example 3.11 ($a(t) = 0$, $t \in [0, 1]$).*

Since $\Lambda_P = \left\{(2k\pi)^2 : k = 0, 1 \ldots\right\}$ and $\Lambda_D = \left\{(k\pi)^2 : k = 1, 2, \ldots\right\}$ we have that $\Lambda_P \cap \Lambda_D$ can be a not empty set, and we cannot affirm a stronger information for the spectral relationship of Dirichlet and periodic problem, without any additional assumption on function a.

3.2. *Spectral characterization of the Neumann problem*

The same arguments of the previous subsections are applicable to characterize the spectrum of Neumann problem (2.8).

We start this subsection by relating the spectra of the Neumann and Initial problems. We obtain the following result.

Theorem 3.13. *If $\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T, 0) \neq 0$, then the next equality holds*

$$G_N[\lambda](t,s) = G_I[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_I[\lambda](t,0)}{\frac{\partial^2}{\partial s\partial t} G_I[\lambda](T,0)} \frac{\partial}{\partial t} G_I[\lambda](T,s), \quad \forall (t,s) \in I \times I.$$

Similarly, we may study Neumann problem as a function of the Final one.

Theorem 3.14. *If $\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0, T) \neq 0$, then the following equality holds*

$$G_N[\lambda](t,s) = G_F[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_F[\lambda](t,T)}{\frac{\partial^2}{\partial s\partial t} G_F[\lambda](0,T)} \frac{\partial}{\partial t} G_F[\lambda](0,s), \quad \forall (t,s) \in I \times I.$$

Theorem 3.15. *The spectrum of the Neumann problem* (2.8) *has the following properties:*

1) $\Lambda_N = \left\{\lambda \in \mathbb{R} \ / \ \frac{\partial^2}{\partial s\partial t} G_I[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \ / \ (\omega_I^1)'[\lambda](T) = 0\right\}$.

2) $\Lambda_N = \left\{\lambda \in \mathbb{R} \ / \ \frac{\partial^2}{\partial s\partial t} G_F[\lambda](0,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \ / \ (\omega_F^1)'[\lambda](0) = 0\right\}$.

3) $S_D^2 \subset \Lambda_N \subset \Lambda_D \sqcup S_D^2$ *where*

$$\begin{aligned} S_D^2 &= \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ \frac{\partial^2}{\partial s\partial t} G_D[\lambda](0,0) \frac{\partial^2}{\partial s\partial t} G_D[\lambda](T,T) + \frac{\partial^2}{\partial s\partial t} G_D[\lambda](0,T) \frac{\partial^2}{\partial s\partial t} G_D[\lambda](T,0) = 0\right\} \\ &= \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ (\omega_D^1)'[\lambda](0)\, (\omega_D^2)'[\lambda](T) - (\omega_D^2)'[\lambda](0)\, (\omega_D^1)'[\lambda](T) = 0\right\}. \end{aligned}$$

4) $S_P^1 \subset \Lambda_N \subset \Lambda_P \sqcup S_P^1$ *where*

$$S_P^1 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_P \ / \ \frac{\partial^2}{\partial s\partial t} G_P[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_P \ / \ (\omega_P^1)'[\lambda](T) = 0\right\}.$$

5) $\Lambda_N = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ \frac{\partial^2}{\partial s\partial t} G_{M_1}[\lambda](T,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ (\omega_{M_1}^2)'[\lambda](T) = 0\right\}$.

6) $\Lambda_N = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ \frac{\partial^2}{\partial s\partial t} G_{M_2}[\lambda](0,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ (\omega_{M_2}^1)'[\lambda](0) = 0\right\}$.

Proof. It follows from Theorem 2.3 together with the relationships obtained in Theorem 3.13, Theorem 3.14, [6, Theorem 4.8], [6, Theorem 4.19], [6, Theorem 4.17] and [6, Theorem 4.18].

As in the previous case, equalities 1) and 2) follow from the fact that $\Lambda_I = \emptyset$ and $\Lambda_F = \emptyset$ while equality in 5) and 6) is held by the fact that $\Lambda_{M_1} \cap \Lambda_N = \emptyset$ and $\Lambda_{M_1} \cap \Lambda_N = \emptyset$ (see [3] and [4, Corollary 4.20]) and Corollary 3.3. □

Remark 3.16. *In cases* 3) *and* 4) *in general it is not true that* $\Lambda_D \cap \Lambda_N = \emptyset$ *and* $\Lambda_P \cap \Lambda_N = \emptyset$*, so we can not obtain a characterization of* Λ_N *as a function of the Green's functions of the Dirichlet and the Periodic problems, in which case it would not meet that* $S_D^2 = \Lambda_N$ *and* $S_P^1 = \Lambda_N$*.*

As a consequence of the previous theorem, we deduce the following result.

Corollary 3.17. *The following inclusions hold*

$$S_D^2 \cup S_P^1 \subset \Lambda_N \subset (\Lambda_D \sqcup S_D^2) \cap (\Lambda_P \sqcup S_P^1).$$

Example 3.18. *Considering* $a(t) = \ln(1+t)$, $t \in [0, 6]$, *we obtain by numerical approach the following approximations of eigenvalues of the Neumann problem:*

$$\lambda_0^N \approx -0.80533, \quad \lambda_1^N \approx -0.02394 \quad \lambda_2^N \approx 1.25492 \quad and \quad \lambda_3^N \approx 3.14451.$$

These values have been obtained by numerical approximation as the zeros of the function $(\omega_{M_2}^1)'[\lambda](0)$ *for* $\lambda \in [-1, 4]$ *according the characterization* 6)*. Figure 1 represents the functions* $(\omega_I^1)'[\lambda](6)$, $(\omega_F^1)'[\lambda](0)$, $(\omega_{M_1}^2)'[\lambda](6)$ *and* $(\omega_{M_2}^1)'[\lambda](0)$ *that have the same zeros in the interval* $[-1, 4]$*.*

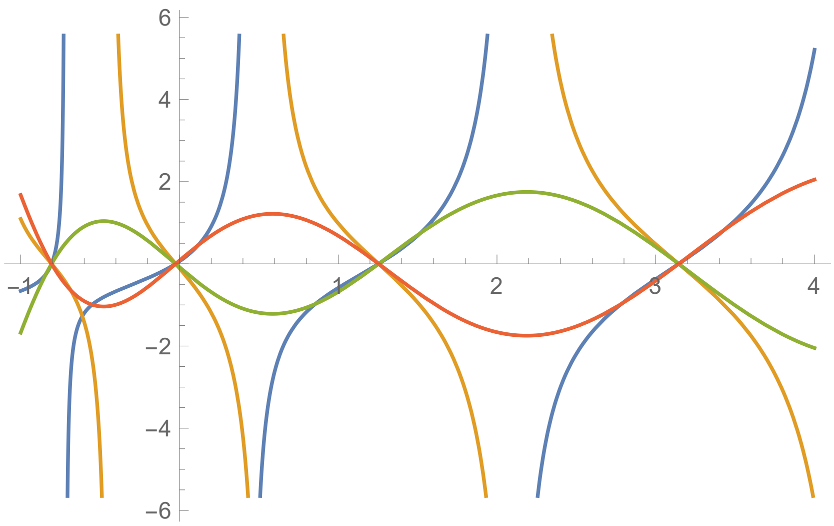

Figure 1: The blue graph represents the function $(\omega_{M_2}^1)'[\lambda](0)$, the orange graph represents the function $(\omega_{M_1}^2)'[\lambda](6)$, the green graph represents the function $(\omega_F^1)'[\lambda](0)$ and the red graph represents the $(\omega_I^1)'[\lambda](6)$ for λ in $[-1, 4]$.

3.3. Spectral characterization of the Dirichlet problem

In this section, reasoning as in the previous case, we can characterize the spectrum of Dirichlet problem (2.9).

First, we will consider the Dirichlet and Initial problems. Following the same steps than in the previous subsection, we attain the following result.

Theorem 3.19. *If* $G_I[\lambda](T, 0) \neq 0$*, then the following equality holds*

$$G_D[\lambda](t,s) = G_I[\lambda](t,s) - \frac{G_I[\lambda](t,0)}{G_I[\lambda](T,0)} G_I[\lambda](T,s), \quad \forall (t,s) \in I \times I.$$

Similarly, we arrive at the following result for the Final problem.

Theorem 3.20. *If $G_F[\lambda](0,T) \neq 0$, then the next equality is fulfilled*

$$G_D[\lambda](t,s) = G_F[\lambda](t,s) - \frac{G_F[\lambda](t,T)}{G_F[\lambda](0,T)}\, G_F[\lambda](0,s), \quad \forall (t,s) \in I \times I.$$

Theorem 3.21. *The spectrum of the Dirichlet problem* (2.9) *satisfies the following properties:*

1) $\Lambda_D = \left\{\lambda \in \mathbb{R} \;/\; G_I[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \;/\; \omega_I^2[\lambda](T) = 0\right\}$.

2) $\Lambda_D = \left\{\lambda \in \mathbb{R} \;/\; G_F[\lambda](0,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \;/\; \omega_F^2[\lambda](0) = 0\right\}$.

3) $S_N^2 \subset \Lambda_D \subset \Lambda_N \sqcup S_N^2$ *where*

$$\begin{aligned} S_N^2 &= \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \;/\; G_N[\lambda](0,T)\, G_N[\lambda](T,0) - G_N[\lambda](0,0)\, G_N[\lambda](T,T) = 0\right\} \\ &= \left\{\lambda \in \mathbb{R} \setminus \Lambda_D \;/\; \omega_N^1[\lambda](0)\, \omega_N^2[\lambda](T) - \omega_N^2[\lambda](0)\, \omega_N^1[\lambda](T) = 0\right\}. \end{aligned}$$

4) $S_P^2 \subset \Lambda_D \subset \Lambda_P \sqcup S_P^2$ *where*

$$S_P^2 = \left\{\lambda \in \mathbb{R} \setminus \Lambda_P \;/\; G_P[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_P \;/\; \omega_P^2[\lambda](T) = 0\right\}.$$

5) $\Lambda_D = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \;/\; G_{M_1}[\lambda](0,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \;/\; \omega_{M_1}^1[\lambda](0) = 0\right\}$.

6) $\Lambda_D = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \;/\; G_{M_2}[\lambda](T,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \;/\; \omega_{M_2}^2[\lambda](T) = 0\right\}$.

Proof. It is proved using Theorem 2.3 and the relationships obtained in Theorem 3.19, Theorem 3.20, [6, Theorem 4.9], [6, Theorem 4.12], [6, Theorem 4.7] and [6, Theorem 4.6].

As we have noticed, equalities 1) and 2) follow from the fact that $\Lambda_I = \Lambda_F = \emptyset$ while the equalities 5) and 6) hold by the fact that $\Lambda_{M_1} \cap \Lambda_D = \emptyset$ and $\Lambda_{M_1} \cap \Lambda_D = \emptyset$ (see [3] and [4, Corollary 4.20]) and Corollary 3.3. □

Remark 3.22. *In general, the intersection $\Lambda_D \cap \Lambda_N$ can be nonempty, so we cannot ensure that $S_N^2 = \Lambda_D$.*

Let us consider the differential equation $u''(t) + (a(t) + \lambda)\, u(t) = 0$, $t \in [0,1]$ with $a(t) = 0$, $t \in [0,1]$.

It is very well known that $\Lambda_D = \left\{(k\pi)^2 : k = 1,2,\dots\right\}$ and $\Lambda_N = \left\{(k\pi)^2 : k = 0,1,\dots\right\}$. In this case, we have that $\Lambda_N = \{0\} \cup \Lambda_D$ and the intersection $\Lambda_D \cap \Lambda_N$ is nonempty and, in particular, $S_N^2 \neq \Lambda_D$.

Remark 3.23. *In Example 3.12 it has been seen that, in general, the intersection $\Lambda_P \cap \Lambda_D$ can be nonempty, so that the spectrum Λ_D cannot be characterized as the set S_P^2.*

Example 3.24. *Consider the following problem with constant coefficients on* $[0,1]$ *related to the operator*

$$L\, u(t) \equiv u''(t) + \lambda\, u(t), \quad t \in [0,1] \;\text{ and }\; \lambda > 0.$$

It is easy to see that $\Lambda_{M_1} = \Lambda_{M_2} = \left\{(k\pi + \frac{1}{2})^2 : k = 0,1,\dots\right\}$ and $G_{M_1}[\lambda](0,0) = G_{M_2}[\lambda](1,1) = -\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}$. Therefore,

$$\Lambda_D = \left\{\lambda \in \mathbb{R} / \;\tan\sqrt{\lambda} = 0\right\} = \left\{(k\pi)^2 : k = 1,\dots\right\}.$$

Next example shows that the inclusions in previous theorem might be strict.

Example 3.25. *Consider*

$$Lu(t) \equiv u''(t) + (a(t) + \lambda)\, u(t), \quad t \in [0, 2\pi],$$

with $a(t) = \cos(2t)$*. In this case, it holds that*

$$S_P^2 \subset \Lambda_D \subset \Lambda_P \sqcup S_P^2$$

and

$$S_N^2 \subset \Lambda_D \subset \Lambda_N \sqcup S_N^2$$

and all the inclusions are strict, as it can be seen in Figures 2 and 3. Indeed, the fourth zero of Λ_D *is not in* S_P^2 *and the second zero of* Λ_P *is neither in* S_P^2 *nor in* Λ_D*. Similarly, the fifth zero of* Λ_D *is not in* S_N^2 *and the first zero of* Λ_N *is neither in* S_N^2 *nor in* Λ_D*.*

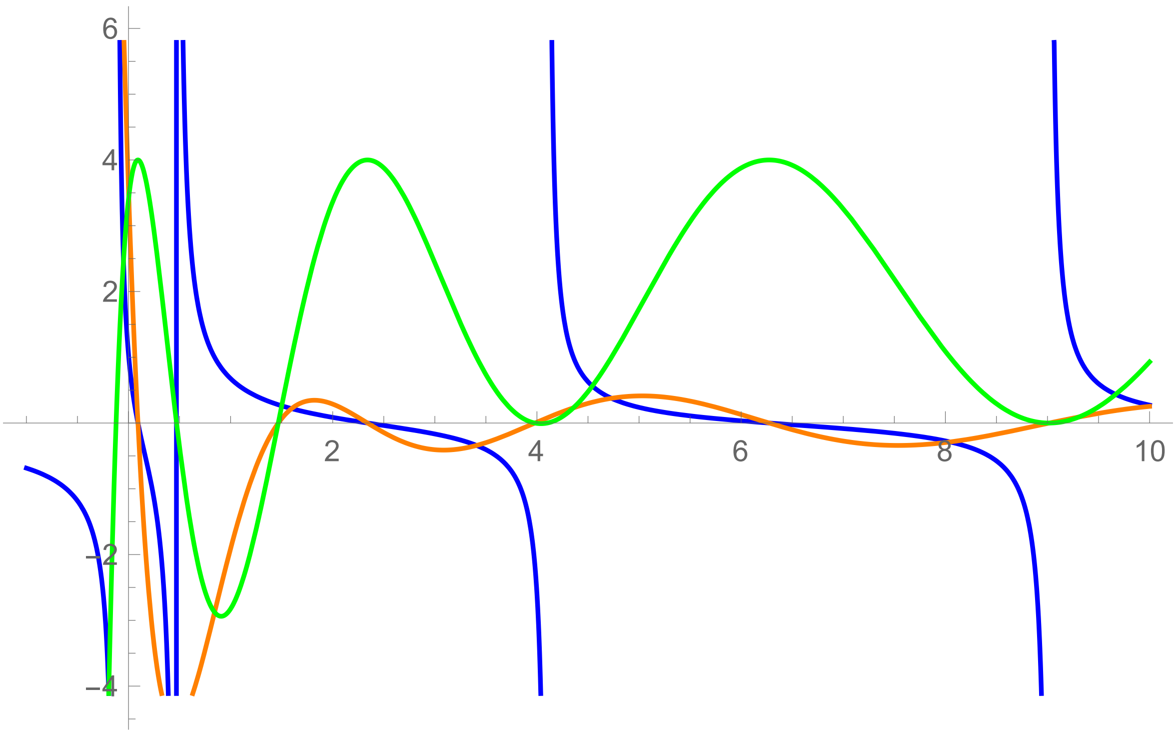

Figure 2: S_P^2 is the set of zeros of the blue graphic, Λ_D is the set of zeros of the orange graphic and Λ_P is the set of zeros of the green graphic.

As a consequence of the previous theorem, we arrive at the following result.

Corollary 3.26. *The following inclusions hold*

$$S_P^2 \cup S_N^2 \subset \Lambda_D \subset (\Lambda_P \sqcup S_P^2) \cap (\Lambda_N \sqcup S_N^2).$$

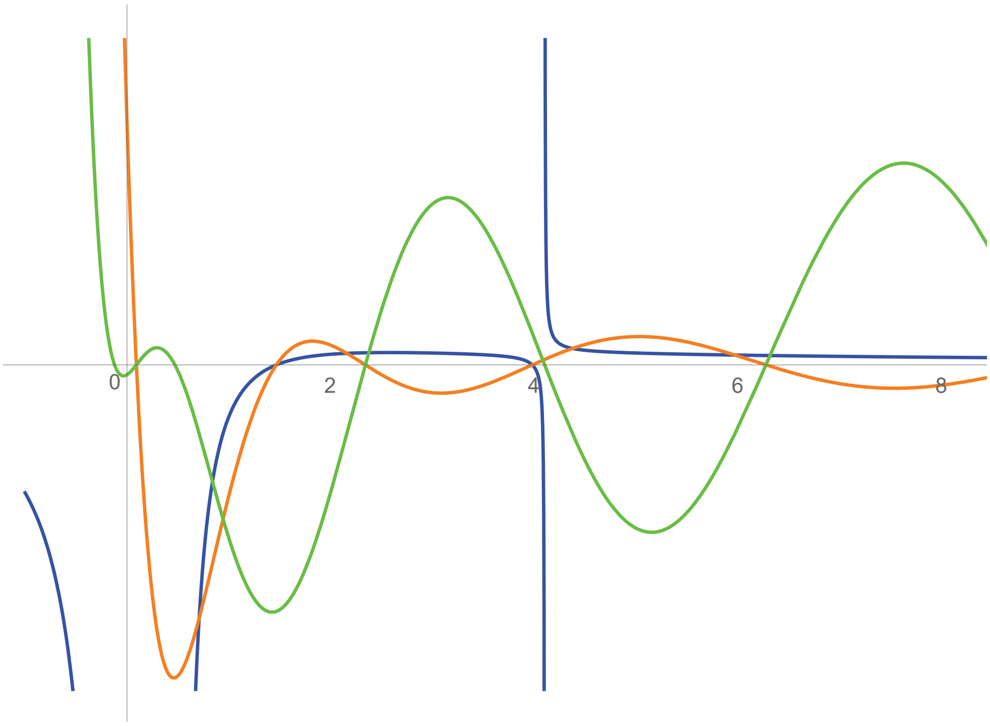

Figure 3: S_N^2 is the set of zeros of the blue graphic, Λ_D is the set of zeros of the orange graphic and Λ_N is the set of zeros of the green graphic.

Remark 3.27. *The relationship of inclusions in the above result can be strict. For example, for the previous example taking into account Figures 2 and 3 and using numerical calculus we have that $\lambda \approx 9.0176$ belongs to Λ_D but it is neither in S_P^2 nor S_N^2 and so $S_P^2 \cup S_N^2 \subsetneq \Lambda_D$ (note that this value of λ is a vertical asymptote of the blue curves whose zeros are S_P^2 and S_N^2 respectively).*

On the other hand, using [3, pages 94-96] we know that

$$\Lambda_N[a,T] \subset \Lambda_N[\tilde{a},2T] \cap \Lambda_P[\tilde{a},2T],$$

for all $a \in L^\alpha(I)$, $\alpha \geq 1$ where $\Lambda_N[a,T]$ and $\Lambda_N[\tilde{a},2T]$ denote the spectrum of Neumann's problem at the intervals I and $[0,2T]$ respectively, $\Lambda_P[\tilde{a},2T]$ denotes the spectrum of the Periodic problem in the interval $[0,2T]$ and function $\tilde{a}$ is the even extension of a to $[0,2T]$.

In our case, since the function $a(t) = \cos(2t)$, $t \in [0,\pi]$ is even we have that it coincides with its even extension on $[0,2\pi]$. Therefore,

$$\Lambda_N[a,\pi] \subset \Lambda_N[a,2\pi] \cap \Lambda_P[a,2\pi] = \Lambda_N \cap \Lambda_P.$$

By numerical approach, we have that $\lambda^* \approx 4.1009$ belongs to $\Lambda_N[a,\pi]$ and so is in $\Lambda_N \cap \Lambda_P$, as can be seen in Figure 4.

Therefore, Λ_D is strictly contained in $(\Lambda_P \sqcup S_P^2) \cap (\Lambda_N \sqcup S_N^2)$ because we have that Λ_P and Λ_N have in common λ^*, which does not belong to Λ_D.

3.4. Spectral characterization of the Mixed problems

In this case, doing an analogous study to the previous sections we can characterize the spectrum of Mixed problems.

As for the Mixed 1 and Initial problems reasoning as in the previous sections one arrives at the following result.

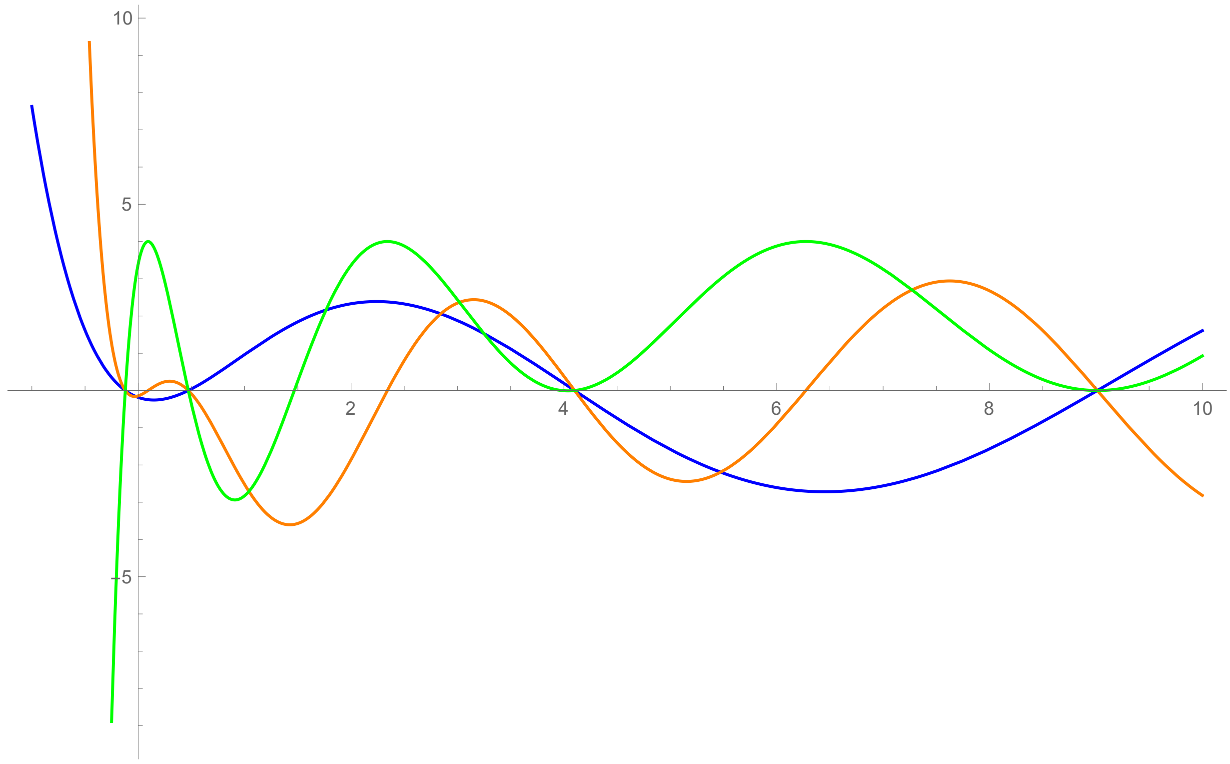

Figure 4: $\Lambda_N[a,\pi]$ is the set of zeros of the blue graphic, Λ_N is the set of zeros of the orange graphic and Λ_P is the set of zeros of the green graphic.

Theorem 3.28. *If* $\frac{\partial}{\partial s}G_I[\lambda](T,0) \neq 0$, *then the following equality is fulfilled*

$$G_{M_1}[\lambda](t,s) = G_I[\lambda](t,s) - \frac{\frac{\partial}{\partial s}G_I[\lambda](t,0)}{\frac{\partial}{\partial s}G_I[\lambda](T,0)}\, G_I[\lambda](T,s), \quad \forall (t,s) \in I \times I.$$

For the Mixed 1 and Final problems we obtain the following result that relates $G_{M_1}[\lambda]$ and $G_F[\lambda]$.

Theorem 3.29. *If* $\frac{\partial}{\partial t}G_F[\lambda](0,T) \neq 0$, *then it holds that*

$$G_{M_1}[\lambda](t,s) = G_F[\lambda](t,s) - \frac{G_F[\lambda](t,T)}{\frac{\partial}{\partial t}G_F[\lambda](0,T)}\, \frac{\partial}{\partial t}G_F[\lambda](0,s), \quad \forall (t,s) \in I \times I.$$

Theorem 3.30. *The spectrum of the Mixed 1 problem* (2.10) *satisfies the following properties:*

1) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R} \;/\; \frac{\partial}{\partial s}G_I[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \;/\; \omega_I^1[\lambda](T) = 0\right\}$.

2) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R} \;/\; \frac{\partial}{\partial t}G_F[\lambda](0,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \;/\; (\omega_F^2)'[\lambda](0) = 0\right\}$.

3) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \;/\; \frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,0) = 0\right\} = \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \;/\; (\omega_D^1)'[\lambda](0) = 0\right\}$.

4) $S_P^3 \subset \Lambda_{M_1} \subset \Lambda_P \sqcup S_P^3$ *where*

$$\begin{aligned} S_P^3 &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_P \;/\; \left(1+\frac{\partial}{\partial s}G_P[\lambda](0,0)\right)\left(1+\frac{\partial}{\partial t}G_P[\lambda](T,0)\right) - G_P[\lambda](0,0)\,\frac{\partial^2}{\partial s\partial t}G_P[\lambda](T,0) = 0\right\} \\ &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_P \;/\; \left(1-\omega_P^1[\lambda](0)\right)\left(1+(\omega_P^2)'[\lambda](T)\right) + \omega_P^2[\lambda](0)\,(\omega_P^1)'[\lambda](T) = 0\right\}. \end{aligned}$$

5) $\Lambda_{M_1} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ G_N[\lambda](T,T) = 0\Big\}\Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ \omega_N^2[\lambda](T) = 0\Big\}$.

Proof. Using Theorem 2.3 with the relations obtained in Theorem 3.28, Theorem 3.29, [6, Theorem 4.3], [6, Theorem 4.21] and [6, Theorem 4.16] yields the result.

As we know, equalities 1) and 2) are a direct consequence of the fact that $\Lambda_I = \emptyset$ and $\Lambda_F = \emptyset$ while the equalities 3) and 5) follow by the fact that $\Lambda_{M_1} \cap \Lambda_D = \emptyset$ and $\Lambda_{M_1} \cap \Lambda_N = \emptyset$ (see [3] and [4, Corollary 4.20]) and Corollary 3.3. □

Similarly, for Mixed 2 problem (2.11) we arrive at the following results.

Let us consider the Mixed 2 and Initial problems and arguing as before, we get at the following theorem.

Theorem 3.31. *If* $\frac{\partial}{\partial t} G_I[\lambda](T,0) \neq 0$, *then the following equality is fulfilled*

$$G_{M_2}[\lambda](t,s) = G_I[\lambda](t,s) - \frac{G_I[\lambda](t,0)}{\frac{\partial}{\partial t} G_I[\lambda](T,0)} \frac{\partial}{\partial t} G_I[\lambda](T,s), \quad \forall (t,s) \in I \times I.$$

Doing the calculations analogously for the Mixed 2 problem as a function of Final one we get at the next result.

Theorem 3.32. *If* $\frac{\partial}{\partial s} G_F[\lambda](0,T) \neq 0$, *then the following equality is fulfilled*

$$G_{M_2}[\lambda](t,s) = G_F[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_F[\lambda](t,T)}{\frac{\partial}{\partial s} G_F[\lambda](0,T)} G_F[\lambda](0,s), \quad \forall (t,s) \in I \times I.$$

Theorem 3.33. *The spectrum of the Mixed 2 problem* (2.11) *satisfies the following properties:*

1) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \ / \ \frac{\partial}{\partial t} G_I[\lambda](T,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \ / \ (\omega_I^2)'[\lambda](T) = 0\Big\}$.

2) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \ / \ \frac{\partial}{\partial s} G_F[\lambda](0,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \ / \ \omega_F^1[\lambda](0) = 0\Big\}$.

3) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ \frac{\partial^2}{\partial s \partial t} G_D[\lambda](T,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \ / \ (\omega_D^2)'[\lambda](T) = 0\Big\}$.

4) $S_P^4 \subset \Lambda_{M_2} \subset \Lambda_P \sqcup S_P^4$ *where*

$$\begin{aligned} S_P^4 &= \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \ / \ \Big(1 - \frac{\partial}{\partial s} G_P[\lambda](T,0)\Big)\Big(1 - \frac{\partial}{\partial t} G_P[\lambda](0,0)\Big) - G_P[\lambda](T,0) \frac{\partial^2}{\partial s \partial t} G_P[\lambda](0,0) = 0\Big\} \\ &= \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \ / \ \Big(1 + \omega_P^1[\lambda](T)\Big)\Big(1 - (\omega_P^2)'[\lambda](0)\Big) + (\omega_P^1)'[\lambda](0)\, \omega_P^2[\lambda](T) = 0\Big\}. \end{aligned}$$

5) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ G_N[\lambda](0,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \ / \ \omega_N^1[\lambda](0) = 0\Big\}$.

Proof. It is proved using Theorem 2.3 with the relations obtained in Theorem 3.31, Theorem 3.32, [6, Theorem 4.1], [6, Theorem 4.22] and [6, Theorem 4.14].

Equalities 1) and 2) are fulfilled because $\Lambda_I = \emptyset$ and $\Lambda_F = \emptyset$. In other hand, the equalities 3) and 5) follow from the fact that $\Lambda_{M_1} \cap \Lambda_D = \emptyset$ and $\Lambda_{M_1} \cap \Lambda_N = \emptyset$ (see [3] and [4, Corollary 4.20]) and Corollary 3.3. □

Example 3.34. *Consider the Mixed problem 1 and periodic problems on the same interval* $[0,1]$ *related to operator*

$$L\, u(t) \equiv u''(t) + (a(t) + \lambda)\, u(t), \ t \in [0,1],$$

where

$$a(t) = \begin{cases} t, & t < \frac{1}{2}, \\ p\,t, & t \geq \frac{1}{2}, \end{cases}$$

with $p \in \mathbb{R}$.

By numerical approach one can verify that the first eigenvalues of both problems coincide for $p \approx -9.6809$ *(See Figure 5).*

As a consequence we have that Λ_P *may intersects* Λ_{M_1} *and so, we cannot ensure that in general,* $S_{M_1}^1 = \Lambda_P$.

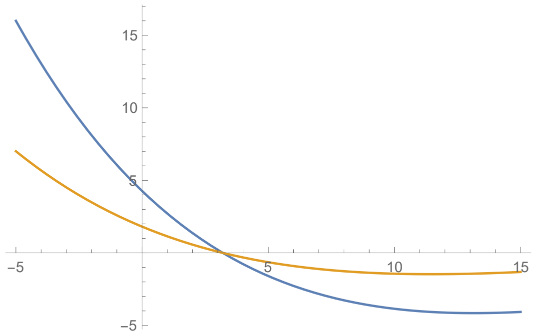

Figure 5: The blue graph corresponds to the curve of the function $\omega_I^1[\lambda](1) + (\omega_I^2)'[\lambda](1) - 2$ and the orange graph represents the curve of the function $\omega_I^1[\lambda](1)$ for λ in $[-5, 15]$.

Example 3.35. *Let us consider in this example the same equation as in Example 3.34.*

As in Example 3.12, one can show that $\lambda_0^{M_1} = \lambda_0^P$ *for* $p_0 \approx 9.32$. *So, we cannot ensure that, in general, both spectrum are disjoint.*

Graphically, this situation would be represented in Figure 6.

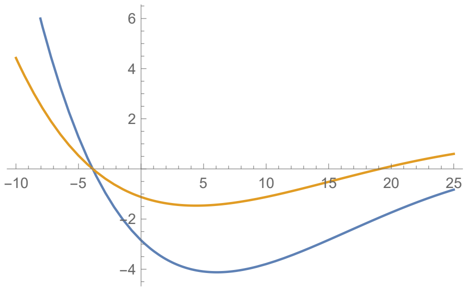

Figure 6: The blue graph corresponds to the curve of the function $\omega_I^1[\lambda](1) + (\omega_I^2)'[\lambda](1) - 2$ and the orange graph represents the curve of the function $(\omega_I^2)'[\lambda](1)$ for λ in $[-10, 25]$.

4. Application to nonlinear problems

In this section we apply the results obtained in the previous one to study the existence of solutions of the nonlinear problems $u''(t) + a(t)\,u(t) = f(t, u(t))$, a.e., $t \in I$ with different types of boundary conditions, where $f : I \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies regularity conditions that we give later.

In particular, we will deal with the following nonlinear problems:

- Dirichlet problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in I, \quad u \in X_D. \tag{4.1}$$

- Periodic problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in I, \quad u \in X_P. \tag{4.2}$$

- Neumann problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in I, \quad u \in X_N. \tag{4.3}$$

- Mixed problem 1:

$$L\,u(t) = f(t,u(t)), \quad \text{a.e. } t \in I, \quad u \in X_{M_1}. \tag{4.4}$$

- Mixed problem 2:

$$L\,u(t) = f(t,u(t)), \quad \text{a.e. } t \in I, \; u \in X_{M_2}. \tag{4.5}$$

where $L\,u(t) := u''(t) + a(t)\,u(t)$ is nonresonant in X_D, X_P, X_N, X_{M_1} and X_{M_2}.

For $\lambda = 0$, we denote by G_D, G_P, G_N, G_{M_1} and G_{M_2} the Green's function related to problems (2.9), (2.12), (2.10) and (2.11), respectively.

We will begin to study the existence of solutions of the nonlinear second order Dirichlet problem (4.1).

The existence of nontrivial solutions is obtained by applying the Krasnosel'skki-Zabreiko fixed-point theorem (see [22]) of the integral operator defined in Banach spaces that we will state below.

Theorem 4.1. *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. If there exist a bounded linear operator $A : X \to X$ such that 1 is not an eigenvalue and*

$$\lim_{\|u\|\to\infty} \frac{\|T(u) - A(u)\|}{\|u\|} = 0,$$

then T has a fixed point in X.

We assume that the following conditions are met:

(H_1) $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^∞ Carathédory function, that is,

- $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a.e. $t \in I$.
- For every $r > 0$ there exists $h_r \in L^\infty(\mathbb{R})$ such that

$$\left| f(t,x) \right| \leq h_r(t),$$

for all $x \in [-r, r]$ and a.e. $t \in I$.

(H_2) $\displaystyle\lim_{|u|\to\infty} \frac{f(t,u)}{u} = -\lambda, \quad \forall t \in I,$

(H_3) $\omega_I^2[\lambda](T) \neq 0$.

Remark 4.2. *Note that the condition (H_3) is equivalent to the fact that $\lambda \notin \Lambda_D$ by the equality* 1) *of Theorem 3.21. The condition (H_3) is also equivalent to either $\omega_F^2[\lambda](0) \neq 0$, $\omega_{M_1}^1[\lambda](0) \neq 0$ or $\omega_{M_2}^2[\lambda](T) \neq 0$ according to equalities* 2), 5) *and* 6) *of Theorem 3.21.*

In our case consider $X \equiv (C(I), \|\cdot\|_\infty)$ the real Banach space endowed with the supremum norm

$$\|u\|_\infty = \sup_{t\in I} \; |u(t)|, \quad \text{for all } \; u \in X,$$

and define the operator $T_D : X \to X$ as follows:

$$T_D\,u(t) := \int_0^T G_D(t,s)\,f(s,u(s))\,ds, \quad t \in I. \tag{4.6}$$

Note that it is very known that the fixed points of operator T_D coincide with the solutions of problem (4.1).

Next, following the line of [18], we will use Theorem 4.1 to guarantee the existence of solutions of the nonlinear problem (4.1).

Theorem 4.3. *Suppose that* (H_1), (H_2) *and* (H_3) *hold. Then the Dirichlet Problem* (4.1) *has a least one solution* $u \in X$.

Proof. We will consider the Banach space $(X, \|.\|_\infty)$ and the operator T_D defined in (4.6).

By assumption (H_1) and using standard techniques one can prove that the operator T_D is completely continuous.

Let us consider the following linear problem

$$u''(t) + a(t)\,u(t) = -\lambda\, u(t), \quad \text{a.e. } t \in I, \quad u \in X_D, \tag{4.7}$$

and we define a completely continuous operator $A : X \to X$ by

$$A\,u(t) := -\lambda \int_0^T G_D(t,s)\,u(s)\,ds, \quad t \in I.$$

Again, the fixed points of the operator A coincide with the solutions of problem (4.7).

From the assumption (H_3), we know that $\lambda \notin \Lambda_D$ and so problem (4.7) has only the trivial solution $u = 0$. Thus, we have that 1 is not an eigenvalue of A.

Finally, we will prove that

$$\lim_{\|u\|\to\infty} \frac{\|T(u) - A(u)\|}{\|u\|} = 0.$$

Given $\varepsilon > 0$ arbitrarily fixed. By assumption (H_2) we have that there exists $M > 0$ such that if $|u| > M$ then

$$\Big| f(t,u) + \lambda\, u \Big| < \varepsilon\, |u|, \quad \forall t \in I. \tag{4.8}$$

Take

$$\tilde{M} = \max_{|u|\le M, t\in I} |f(t,u)|$$

and let $r > M$ be large enough such that $\frac{M|\lambda| + \tilde{M}}{r} < \varepsilon$.

Let be $u \in X$ such that $\|u\| > r$. Thus, if $|u(s)| \le M$ we have that

$$\Big| f(s,u(s)) + \lambda\, u(s) \Big| \le |f(s,u(s))| + |\lambda|\,|u(s)| \le \tilde{M} + M\,|\lambda| < \varepsilon\, r \le \varepsilon\, \|u\|.$$

On the other hand, if $|u(s)| > M$ then, from (4.8), we deduce that

$$\Big| f(s,u(s)) + \lambda\, u(s) \Big| < \varepsilon\, |u(s)| \le \varepsilon\, \|u\|.$$

Therefore, for all $s \in I$, the following inequality holds

$$\Big| f(s,u(s)) + \lambda\, u(s) \Big| \le \varepsilon\, \|u\|. \tag{4.9}$$

Using the regularity of the Green's function $G_D(t,s)$ we deduce that there exists $N \in \mathbb{R}$, $N > 0$ such that $\sup_{t\in I} \int_0^T |G_D(t,s)|\,ds \le N$.

Now, let $u \in X$ with $\|u\| > r$. From (4.9) we have that

$$\begin{aligned}\|T_D(u) - A(u)\| &= \sup_{t\in I} \Big| \int_0^T G_D(t,s)\,(f(s,u(s)) + \lambda\, u(s))\,ds \Big| \le \sup_{t\in I} \int_0^T |G_D(t,s)| \Big| f(s,u(s)) + \lambda\, u(s) \Big|\,ds \\ &\le \varepsilon\, \|u\| \sup_{t\in I} \int_0^T |G_D(t,s)|\,ds \le \varepsilon\, N\, \|u\|.\end{aligned}$$

Thus

$$\lim_{\|u\|\to\infty} \frac{\|T_D(u) - A(u)\|}{\|u\|} = 0.$$

Therefore, applying Theorem 4.1, we conclude that T_D has a fixed point $u \in X$, that is, problem (4.1) has at least one solution $u \in X$. □

As a consequence of previous Theorem, we obtain the following result.

Corollary 4.4. *Assume that* $f : I \times [0,\infty) \to (-\infty, 0]$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda, \quad \forall t \in I$ *and suppose that* (H_3) *holds. Then for all* $\lambda < \lambda_0^D$ *the Dirichlet Problem* (4.1) *has a nonnegative solution* u.

Proof. Let us define $\tilde{f} : I \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t,u) = \begin{cases} f(t,u), & u \geq 0, \\ f(t,-u), & u < 0. \end{cases}$$

It is clear that $\tilde{f}$ is continuous on $I \times \mathbb{R}$ and $\lim_{|u|\to\infty} \frac{\tilde{f}(t,u)}{u} = -\lambda$.

Note that $G_D[\lambda] < 0$ on $(0,T)\times(0,T)$ if and only if $\lambda < \lambda_0^D$ [4, Lemma 36]. From Theorem (4.3) we deduce that problem

$$L\,u(t) = \tilde{f}(t, u(t)), \quad \text{a.e. } t \in I, \quad u \in X_D,$$

has a solution u. In such case, the solution u is given by

$$u(t) = \int_a^b G_D(t,s)\, \tilde{f}(s, u(s))\, ds,$$

and so $u \geq 0$ on I. Since $\tilde{f}(s,u(s)) = f(s,u(s))$, $s \in I$ we have that u is solution of (4.1), that is, problem (4.1) has at least one nonnegative solution u. □

Now, arguing as in the previous case, we can obtain similar results to prove the existence of solutions of Periodic, Neumann, Mixed 1 and Mixed 2 problems.

Theorem 4.5. *Suppose that* (H_1) *and* (H_2) *holds. Moreover, assume that*

$$\omega_I^1[\lambda](T) + (\omega_I^2)'[\lambda](T) \neq 2. \tag{4.10}$$

Then the Periodic Problem (4.2) *has a least one solution* $u \in X$.

Corollary 4.6. *Assume that* $f : I \times [0,\infty) \to (-\infty, 0]$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda, \quad \forall t \in I$ *and suppose that* (4.10) *holds. Then for all* $\lambda < \lambda_0^P$ *the Periodic Problem* (4.2) *has a nonnegative solution* u.

Theorem 4.7. *Suppose that* (H_1) *and* (H_2) *holds. Moreover, assume that*

$$(\omega_{M_1}^2)'[\lambda](T) \neq 0. \tag{4.11}$$

Then the Neumann Problem (4.3) *has a least one solution* $u \in X$.

Corollary 4.8. *Assume that* $f : I \times [0,\infty) \to (-\infty, 0]$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda, \quad \forall t \in I$ *and suppose that* (4.11) *holds. Then the Neumann Problem* (4.3) *has a nonnegative solution* u *for all* $\lambda < \lambda_0^N$.

Theorem 4.9. *Suppose that* (H_1) *and* (H_2) *holds. Moreover, assume that*

$$(\omega_D^1)'[\lambda](0) \neq 0. \tag{4.12}$$

Then the Mixed Problem 1 (4.4) *has a least one solution* $u \in X$.

Corollary 4.10. *Assume that* $f : I \times [0,\infty) \to (-\infty, 0]$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda,\ \forall t \in I$ *and suppose that* (4.12) *holds. Then for all* $\lambda < \lambda_0^{M_1}$ *the Mixed Problem 1* (4.4) *has a nonnegative solution* u.

Theorem 4.11. *Suppose that* (H_1) *and* (H_2) *holds. Moreover, assume that*

$$(\omega_N^1)'[\lambda](0) \neq 0. \tag{4.13}$$

Then the Mixed Problem 2 (4.5) *has a least one solution* $u \in X$.

Corollary 4.12. *Assume that* $f : I \times [0,\infty) \to (-\infty, 0]$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda,\ \forall t \in I$ *and suppose that* (4.13) *holds. Then for all* $\lambda < \lambda_0^{M_2}$ *the Mixed Problem 2* (4.5) *has a nonnegative solution* u.

In the case that the Green's functions G_P and G_N are nonnegative on $I \times I$ we arrive at the following results.

Corollary 4.13. *Assume that* $f : I \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda,\ \forall t \in I$ *and suppose that* (4.10) *holds. Then the Periodic Problem* (4.2) *has a nonnegative solution* u *for all* λ *such that* $\lambda_0^P < \lambda \leq \lambda_0^A$ *where* λ_0^A *denotes the smallest eigenvalue of the antiperiodic equation*

$$u''(t) + (a(t) + \lambda)\, u(t) = 0,\ \ a.e.,\ t \in I,\ \ u(0) = -u(T),\ \ u'(0) = -u'(T).$$

Corollary 4.14. *Assume that* $f : I \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\lim_{u\to\infty} \frac{f(t,u)}{u} = -\lambda,\ \forall t \in I$ *and suppose that* (4.11) *holds. Then the Neumann Problem* (4.3) *has a nonnegative solution* u *for all* λ *such that* $\lambda_0^P < \lambda \leq \min\{\lambda_0^{M_1}, \lambda_0^{M_2}\}$.

Remark 4.15. *It should be noted that conditions* (4.10), (4.11), (4.12), *and* (4.13) *can be replaced by their equivalents using the characterizations of Theorems 3.7, 3.15, 3.30 and 3.33, respectively. For example, the condition* (4.10) *is equivalent to* $\omega_F^1[\lambda](0) + (\omega_F^2)'[\lambda](0) \neq 2$ *since equalities* 1) *and* 2) *of Theorem 3.7 characterize the spectrum of the Periodic problem.*

Remark 4.16. *We must note that conditions* $\lambda < \lambda_0^P$, $\lambda < \lambda_0^N$, $\lambda < \lambda_0^{M_1}$, $\lambda < \lambda_0^{M_2}$, $\lambda_0^P < \lambda \leq \lambda_0^A$ *and* $\lambda_0^P < \lambda \leq \min\{\lambda_0^{M_1}, \lambda_0^{M_2}\}$ *imply that* $G_P[\lambda] < 0$, $G_N[[\lambda] < 0$, $G_{M_1}[\lambda] < 0$, $G_{M_2}[\lambda] < 0$, $G_P[\lambda] \geq 0$ *and* $G_N[\lambda] > 0$ *on* $(0,T) \times (0,T)$ *as can be seen in [4, Lemma 37], [4, Theorem 18], [4, Corollary 14], [4, Corollary 13], [4, Lemma 37] and [4, Theorem 18], respectively.*

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