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Biharmonic curves along Riemannian maps

Gizem Köprülü Karakaş[a], Bayram Şahin[a,*]

[a]*Ege University, Faculty of Science, Department of Mathematics, 35100, İzmir, Türkiye*

Abstract. In this paper, the transformation of a bi-harmonic curve on the total manifold into a bi-harmonic curve on the base manifold along a Riemannian map between Riemannian manifolds is examined. In this direction, first, necessary and sufficient conditions are obtained for the Riemannian map between two Riemannian manifolds for the curve on the total manifold to be bi-harmonic curve on the base manifold. Afterwards, the case that the total manifold is a complex space form was taken into consideration and the bi-harmonic character of the curve on the base manifold was examined by considering appropriate conditions on the basic notions of the Riemannian map.

1. Introduction

Many notions in differential geometry can be viewed as a map. Curves and surfaces, which are really the most basic notions in differential geometry, are also maps after all. For this reason, examining the behavior of curve, surface or submanifold along a map between two given manifolds will be very useful for us to understand both the geometry of the manifolds and the character of the map.

In this direction, the second author and his co-authors investigated the geometry of manifolds and the character of the map itself by examining the behavior of various curves (elastic curve, circle, helix) under a given immersion, submersion or Riemannian map, [16], [17], [18], [19].

Theory of harmonic maps has been applied into various fields in differential geometry. Harmonic maps $F : (M, g) \to (N, g_N)$ between Riemannian manifolds are the critical points of the energy $E(F) = \frac{1}{2}\int_M |dF|^2 \upsilon_g$, and they are therefore the solutions of the corresponding Euler-Lagrange equation. This equation is giving by the vanishing of the tension field $\tau(F) = trace\nabla dF$. On the other hand, Jiang [4] studied first and second variation formulas of the bienergy functional $E_2(F)$ whose critical points are called as biharmonic maps. There have been a rich literature on biharmonic maps like as harmonic maps. In [21], S. B. Wang studied the first variational formula of the tri-energy E_3. The critical points are called triharmonic maps. Notice that, every harmonic curve is a triharmonic curve. However, biharmonic curves are not necessary triharmonic curves and, vice versa, triharmonc curves do not need to be biharmonic, [9].

The authors of the present paper studied the behavior of biharmonic and triharmonic curves along a Riemann submersion between manifolds, [5], [6]. Using the behavior of the curve, they obtained results about the geometry of manifolds and the character of Riemann submersions.

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* Corresponding author: Bayram Şahin
Email addresses: koprulu1gizem@gmail.com (Gizem Köprülü Karakaş), bayram.sahin@ege.edu.tr (Bayram Şahin)

In this paper, we study biharmonic curves along Riemannian maps between Riemannian manifolds and we study curves along Riemannian maps from complex space form onto Riemannian manifolds. We considered the curve as horizontal curve. If the curve is considered as a general curve, it seems quite complicated the control the resulting equation in this case. In 2, we present the basic information needed for this paper. In 3, we investigate necessary and sufficient conditions for the curves along Riemannian maps from Riemannian manifolds to be biharmonic. Then, we investigate necessary and sufficient conditions for the Frenet curves along Riemannian maps from Riemannian manifolds to be biharmonic. In 4, we investigate necessary and sufficient conditions for the curves along Riemannian maps from complex space forms to be biharmonic. Then, we investigate necessary and sufficient conditions for the Frenet curves along Riemannian maps from complex space forms to be biharmonic.

2. Preliminaries

In this section, we recall some basic notions and results which will be needed throughout the paper [1], [2], [3], [8], [10], [11], [12], [14], [15], [20], [21], [22], [23].

Let $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a smooth map between Riemannian manifolds such that $0 < rankF \leqslant min\{m, n\}$, where $dimM = m$ and $dimN = n$. Then, the kernel space of F_* by $\mathcal{V}_p = kerF_{*p}$ at $p \in M$ and consider the orthogonal complementary space $\mathcal{H}_p = (kerF_{*p})^\perp$ to $kerF_{*p}$. Then T_pM of M at p has the following decomposition

$$T_pM = kerF_{*p} \oplus (kerF_{*p})^\perp = \mathcal{V}_p \oplus \mathcal{H}_p$$

Since $rankF \leqslant min\{m, n\}$, always we have $(rangeF_{*p})^\perp$. In this way, tangent space $T_{F(p)}N$ of N at $F(p) \in N$ has the following decomposition

$$T_{F(p)}N = rangeF_{*p} \oplus (rangeF_{*p})^\perp.$$

Now, a smooth map $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F^h_{*p_1}$ satisfies the equation

$$g_M(X, Y) = g_N(F_*X, F_*Y) \tag{1}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$. So that isometric immersions and Riemannian submersions are particular Riemannian map , respectively with $kerF_* = \{0\}$ and $(rangeF_*)^\perp = \{0\}$.
Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map between two Riemannian manifolds of dimensions m and n respectively. The second fundamental form of a map is defined by

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla}{}^F{}_X F_*Y - F_*(\overset{M}{\nabla}_X Y) \tag{2}$$

for any vector fields X, Y on M, where $\overset{M}{\nabla}$ is the Levi-Civita cennection of M and $\overset{N}{\nabla}{}^F$ is the pull-back of the connection $\overset{N}{\nabla}$ of N to the induced vector bundle $F^{-1}(TN)$. It is well known that ∇F_* is symmetric. It is known from [13] that, second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((kerF_*)^\perp)$, of a Riemannian map has no components in $rangeF_*$. Then, we have

$$(\nabla F_*)(X, Y) \in \Gamma((rangeF_*)^\perp), \forall X, Y \in \Gamma((kerF_*)^\perp).$$

Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N^n, g_N). Then we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F \tag{3}$$

$$\mathcal{T}_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{H}F \tag{4}$$

where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M. We can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew -symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions.
On the other hand, from (3) and (4) we have

$$\overset{M}{\nabla}_X Y = \mathcal{H}\overset{M}{\nabla}_X Y + \mathcal{A}_X Y \tag{5}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$, where $\overset{\wedge}{\nabla}_V W = \mathcal{V}\nabla_V W$.
We denote by $\overset{N}{\nabla}$ both the Levi-Civita connection of (N^n, g_N) and its pull-back along F. We denote by $(rangeF_*)^{\perp}$ the subbundle of $F^{-1}(TN)$ with fiber $(F_*(T_{p1}M))^{\perp}$-orthogonal complement of $F_*(T_{p1}M)$ for g_N over p_1. For any vector field X on M and any section V of $(rangeF_*)^{\perp}$, we define $\nabla^{F\perp}_X V$, which is the orthogonal projection of $\overset{N}{\nabla}_X V$ on $(rangeF_*)^{\perp}$. Then we have

$$\overset{N}{\nabla}_X V = -S_V F_* X + \nabla^{F\perp}_X V, \tag{6}$$

where $S_V F_* X$ is the tangential component of $\overset{N}{\nabla}_X V$. It is easy to see that, $S_V F_* X$ is bilinear in V and $F_* X$ and $S_V F_* X$ at p depends only on V_p and $F_{*p} X_p$.Then, we obtain

$$g_N(S_V F_* X, F_* Y) = g_N(V, (\nabla F_*)(X, Y)) \tag{7}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((rangeF_*)^{\perp})$. Since (∇F_*) is symmetric, it follows that S_V is a symmetric linear transformation of $rangeF_*$.
Let F be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then, F is an umbilical Riemannian map if and only if

$$(\nabla F_*)(X, Y) = g_M(X, Y)H_2,$$

for $X, Y \in \Gamma((kerF_*))^{\perp}$ and H_2 is vector field on $(rangeF_*)^{\perp}$.
By using (2) and (6), we have

$$\begin{aligned} R^N(F_*X, F_*Y)F_*Z &= F_*(R^M(X, Y)Z) - S_{(\nabla F_*)(Y,Z)}F_*X + S_{(\nabla F_*)(X,Z)}F_*Y \\ &+(\nabla_X(\nabla F_*))(Y, Z) - (\nabla_Y(\nabla F_*))(X, Z) \end{aligned} \tag{8}$$

for $X, Y, Z \in \Gamma((kerF_*)^{\perp})$, where R^M and R^N denote the curvature tensors of $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$ which are metric connections on M and N, respectively. Moreover $(\nabla_X(\nabla F_*))(Y, Z)$ is defined by

$$(\nabla_X(\nabla F_*))(Y, Z) = \nabla^{F\perp}_X(\nabla F_*)(Y, Z) - (\nabla F_*)(\overset{M}{\nabla}_X Y, Z) - (\nabla F_*)(Y, \overset{M}{\nabla}_X Z). \tag{9}$$

It is known that, F is a harmonic map if and only if the tension field $\tau(F) = trace(\nabla F_*) = 0$, which is called the harmonic equation or the Euler-Lagrange equation.
A map $F : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is a biharmonic map if the bitension field of F

$$\tau_2(F) = -\Delta_F \tau(F) + traceR(\tau(F), F_*)F_* \tag{10}$$

vanishes. The operator Δ_F is the rough Laplacian acting on $\Gamma(F^*TM)$ defined by

$$\Delta_F := -\sum_{i=1}^{n}(\nabla^F_{e_i}\nabla^F_{e_i} - \nabla^F_{\nabla^M_{e_i}e_i}),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N.

3. Biharmonic Curves along Riemannian Maps from Riemannian Manifolds

In this section, we study biharmonic curves along Riemannian maps from Riemannian manifolds. Then, we will investigate necessary and sufficient conditions for the curves along Riemannian maps from Riemannian manifolds to be biharmonic. We first note the following remarks. Let $\alpha : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g_M). If there exists orthonormal vector fields $E_1, E_2, \cdots, E_r$ along α such that

$$
\begin{aligned}
E_1 &= \alpha' = T, \\
\nabla_T E_1 &= \kappa_1 E_2, \\
\nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\
&\cdots \\
\nabla_T E_r &= -\kappa_{r-1} E_{r-1}.
\end{aligned} \tag{11}
$$

then α is called a Frenet curve of osculating order r, where $\kappa_1, \cdots, \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$.
A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if κ_1 is a nonzero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a helix of order r if $\kappa_1, \cdots\cdots \kappa_{r-1}$ are nonzero positive constants; a helix of order 3 is shortly called a helix [11], [23].

We recall the biharmonic equation for curves. Let $\alpha : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then the bitension field is given by [11]

$$
\tau_2(\alpha) = \nabla_T^3 T - R(T, \nabla_T T)T \tag{12}
$$

where $T = \alpha'$.
Let (M, g_M) be a Riemannian manifold and $\alpha : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then, using Frenet equations, the bitension field of α becomes [4]

$$
\begin{aligned}
&\tau_2(\alpha) = -3\kappa_1\kappa_1' E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 \\
&+\kappa_1\kappa_2\kappa_3 E_4.
\end{aligned} \tag{13}
$$

We first have the following result.

Theorem 3.1. *Let $F : (M(c_1), g_M) \to (N(c_2), g_N)$ be a Riemannian map from a real space form $(M(c_1), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(c_1), g_M)$ be a biharmonic horizontal curve. Then $F \circ \alpha : \gamma : I \to (N(c_2), g_N)$ is a biharmonic curve if and only if*

$$
\begin{aligned}
&-(\nabla F_*)(E_{1h}, {}^*F_* S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h})) - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h} \\
&+2\kappa_1'(\nabla F_*)(E_{1h}, E_{2h}) + (c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h}) \\
&-\kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}} E_{2v}) - \kappa_1 S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} = 0,
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&-F_* \overset{M^*}{\nabla}_{E_{1h}} F_* S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h} + \nabla^{F\perp}_{E_{1h}} \nabla^{F\perp}_{E_{1h}} (\nabla F_*)(E_{1h}, E_{1h}) \\
&-2\kappa_1' F_* \mathcal{A}_{E_{1h}} E_{2v} - \kappa_1\kappa_2 F_* \mathcal{A}_{E_{1h}} E_{3v} - \kappa_1 F_* \mathcal{H} \overset{M}{\nabla}_{E_{1h}} \mathcal{A}_{E_{1h}} E_{2v} \\
&+\kappa_1 \nabla^{F\perp}_{E_{1h}} (\nabla F_*)(E_{1h}, E_{2h}) - \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\
&+\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}) = 0.
\end{aligned} \tag{15}
$$

Proof. Let $F : (M(c_1), g_M) \to (N(c_2), g_N)$ be a Riemannian map from a real space form $(M(c_1), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(c_1), g_M)$ be a biharmonic horizontal curve. Then, we have the following equation,

$$
\alpha' = T = E_{1h}, \qquad \gamma' = F_* T = \tilde{T}, \tag{16}
$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Using (2) and (11) we get,

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = (\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1 F_* E_{2h}. \tag{17}$$

and

$$\begin{aligned}\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} &= -S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} + \nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1' F_* E_{2h}\\ &+\kappa_1((\nabla F_*)(E_{1h}, E_{2h}) + F_*\overset{M}{\nabla}_{E_{1h}}E_{2h}).\end{aligned} \tag{18}$$

From (5) and Frenet formulas, we have,

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h} = -\kappa_1 E_{1h} + \kappa_2 E_{3h} - \mathcal{A}_{E_{1h}}E_{2v}. \tag{19}$$

Using (19) in (18), we derive,

$$\begin{aligned}\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} &= -S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} + \nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1' F_* E_{2h}\\ &+\kappa_1(\nabla F_*)(E_{1h}, E_{2h}) - \kappa_1^2 F_*E_{1h} + \kappa_1\kappa_2 F_*E_{3h} - \kappa_1 F_*\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{20}$$

Taking the covariant dervivative of (20) , we get,

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= -\overset{N}{\nabla}_{F_*E_{1h}}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} + \overset{N}{\nabla}_{F_*E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h})\\ &+\overset{N}{\nabla}_{F_*E_{1h}}\kappa_1' F_*E_{2h} + \overset{N}{\nabla}_{F_*E_{1h}}\kappa_1(\nabla F_*)(E_{1h}, E_{2h}) - \overset{N}{\nabla}_{F_*E_{1h}}\kappa_1^2 F_*E_{1h}\\ &+\overset{N}{\nabla}_{F_*E_{1h}}\kappa_1\kappa_2 F_*E_{3h} - \overset{N}{\nabla}_{F_*E_{1h}}\kappa_1 F_*\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{21}$$

Since $S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} \in \Gamma(F_*(kerF_*)^\perp)$, we can write $F_*X = S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}$ for $X \in \Gamma((kerF_*)^\perp)$ where $X =^* F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}$.
Then using (2), we have,

$$\begin{aligned}\overset{N}{\nabla}_{F_*E_{1h}}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} &= (\nabla F_*)(E_{1h},{}^* F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}))\\ &+F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}.\end{aligned} \tag{22}$$

Then we have equation (23),

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}}E_{3v}. \tag{23}$$

Due (17), (19) and Frenet formulas, using (23), we arrive at

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= -3\kappa_1\kappa_1' F_*E_{1h} + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h} + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')F_*E_{3h}\\ &+\kappa_1\kappa_2\kappa_3 F_*E_{4h} - (\nabla F_*)(E_{1h},{}^* F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h})\\ &-F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}\\ &+\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + 2\kappa_1'(\nabla F_*)(E_{1h}, E_{2h}) - 2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v}\\ &-\kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1 S_{(\nabla F_*)(E_{1h},E_{2h})}F_*E_{1h}\\ &+\kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{2h}) - \kappa_1^2(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h})\\ &-\kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}}E_{2v}).\end{aligned} \tag{24}$$

It is easy to see that,

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, (\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1 F_*E_{2h})F_*E_{1h}, \tag{25}$$

Taking the vertical and horizontal parts of E_2, we find,

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}. \tag{26}$$

Hence, we obtain

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}). \tag{27}$$

Since F is Riemannian map, we have

$$\begin{aligned} &R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h} = F_*(R^M(E_{1h}, E_2)E_{1h}) - F_*(R^M(E_{1h}, E_{2v})E_{1h}) \\ &-S_{(\nabla F_*)(E_{2h}, E_{1h})}F_*E_{1h} + S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{2h} + (\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\ &-(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}) \end{aligned} \tag{28}$$

On the other hand, since M is a space form, we obtain,

$$\begin{aligned} &R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} \\ &= R^N(F_*E_{1h}, (\nabla F_*)(E_{1h}, E_{1h}))F_*E_{1h} + R^N(F_*E_{1h}, \kappa_1 F_*E_{2h})F_*E_{1h} \end{aligned} \tag{29}$$

$$\begin{aligned} &= -c_2(\nabla F_*)(E_{1h}, E_{1h}) - c_1\kappa_1 F_*E_{2h} - \kappa_1 S_{(\nabla F_*)(E_{2h}, E_{1h})}F_*E_{1h} \\ &+\kappa_1 S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{2h} + \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\ &-\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}) \end{aligned} \tag{30}$$

Putting (24) and (30) in (12), we have,

$$\begin{aligned} &\tau_2(\gamma) = -3\kappa_1\kappa_1' F_*E_{1h} + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + c_1\kappa_1)F_*E_{2h} + (2\kappa_1'\kappa_2 \\ &+\kappa_1\kappa_2')F_*E_{3h} + \kappa_1\kappa_2\kappa_3 F_*E_{4h} - (\nabla F_*)(E_{1h}, {}^*F_*S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h}) \\ &-F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h} - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h} \\ &+\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + 2\kappa_1'(\nabla F_*)(E_{1h}, E_{2h}) - 2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} \\ &-\kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + \kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{2h}) \\ &+(c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h}) \\ &-\kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}}E_{2v}) + \kappa_1 S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{2h} \\ &+\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) - \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}). \end{aligned} \tag{31}$$

Since $\tau_2(\alpha) = 0$, we can write $F_*\tau_2(\alpha) = 0$. Then, using this equation in $\tau_2(\gamma)$, we get,

$$\begin{aligned} &\tau_2(\gamma) = -(\nabla F_*)(E_{1h}, {}^*F_*S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h}) - F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h} \\ &-S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{1h} + \nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + 2\kappa_1'(\nabla F_*)(E_{1h}, E_{2h}) \\ &-2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + \kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{2h}) \\ &+(c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h}) - \kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}}E_{2v}) \\ &+\kappa_1 S_{(\nabla F_*)(E_{1h}, E_{1h})}F_*E_{2h} + \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\ &-\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}). \end{aligned} \tag{32}$$

Then taking the $F_*((kerF_*)^\perp) = rangeF_*$ and $(rangeF_*)^\perp$ parts, we have (14) and (15). Thus $F \circ \alpha : \gamma : I \to (N, g_N)$ is a biharmonic curve if and only if (14) and (15) are satisfied. □

Theorem 3.2. *Let $F : (M(c_1), g_M) \to (N(c_2), g_N)$ be an umbilical Riemannian map from a real space form $(M(c_1), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(c_1), g_M)$ be a biharmonic horizontal curve and horizontal vector field $\mathcal{A}$ be a parallel. Then $F \circ \alpha : \gamma : I \to (N(c_2), g_N)$ is a biharmonic curve if and only if*

$$-\|H_2\|^2 - S_{\nabla^{F\perp}_{E_{1h}} H_2} F_* E_{1h} + (c_2 - \kappa_1) H_2 - \kappa_1 S_{H_2} F_* E_{2h} = 0,$$

$$-F_* \overset{M^*}{\nabla}_{E_{1h}} F_* S_{H_2} F_* E_{1h} + \nabla^{F\perp}_{E_{1h}} \nabla^{F\perp}_{E_{1h}} H_2 = 0.$$

Proof. The assertion follows from Theorem 3.1. □

In particular cases, we have the following results.

Theorem 3.3. *Let $F : (M(c_1), g_M) \to (N(c_2), g_N)$ be a Riemannian map from a real space form $(M(c_1), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(c_1), g_M)$ be a biharmonic horizontal curve and $\kappa_1 = constant \neq 0$ and horizontal vector field $\mathcal{A}$ be a parallel. Then $F \circ \alpha : \gamma : I \to (N(c_2), g_N)$ is a biharmonic curve if and only if*

$$\begin{aligned} &-(\nabla F_*)(E_{1h},{}^* F_*(S_{(\nabla F_*(E_{1h},E_{1h})} F_* E_{1h})) - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h} \\ &-\kappa_1 S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} + (c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) \\ &+\kappa_1 \kappa_2 (\nabla F_*)(E_{1h}, E_{3h}) = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &-F_* \mathcal{H} \overset{M^*}{\nabla}_{E_{1h}} F_*(S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h}) + \nabla^{F\perp}_{E_{1h}} \nabla^{F\perp}_{E_{1h}} (\nabla F_*)(E_{1h}, E_{1h}) \\ &+\kappa_1 \nabla^{F\perp}_{E_{1h}} (\nabla F_*)(E_{1h}, E_{2h}) - \kappa_1 (\nabla_{E_{1h}} (\nabla F_*))(E_{2h}, E_{1h}) \\ &+\kappa_1 (\nabla_{E_{2h}} (\nabla F_*))(E_{1h}, E_{1h}) = 0. \end{aligned} \tag{34}$$

Proof. The assertion follows from Theorem 3.1. □

Theorem 3.4. *Let $F : (M(c_1), g_M) \to (N, g_N)$ be a Riemannian map from a real space form $(M(c_1), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c_1), g_M)$ be a horizontal Frenet curve. Then Frenet curve $F \circ \alpha : \gamma : I \to (N, g_N)$ is a biharmonic curve if and only if*

$$\begin{aligned} &-3\tilde{\kappa_1}\tilde{\kappa_1}' F_* E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c_1 \tilde{\kappa_1}) F_* E_{2h} \\ &+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3} F_* E_{4h} - \tilde{\kappa_1} S_{(\nabla F_*)(E_{2h},E_{1h})} F_* E_{1h} + \tilde{\kappa_1} S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} = 0, \end{aligned} \tag{35}$$

$$\tilde{\kappa_1} (\nabla_{E_{1h}} (\nabla F_*))(E_{2h}, E_{1h}) - \tilde{\kappa_1} (\nabla_{E_{2h}} (\nabla F_*))(E_{1h}, E_{1h}) = 0. \tag{36}$$

where $\tilde{\kappa_1}, \cdots, \tilde{\kappa_{r-1}}$ are positive functions of γ on I.

Proof. Let $F : (M(c_1), g_M) \to (N, g_N)$ be a Riemannian map from a real space form $(M(c_1), g_M)$ to a Riemannian manifold (N, g_N). Since $\alpha : I \to (M(c_1), g_M)$ is a horizontal Frenet curve, we have,

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_* T = \tilde{T},$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Then we have Frenet formulas of γ as follows

$$\begin{aligned} \overset{N}{\nabla}_{\tilde{T}} \tilde{T} &= \tilde{\kappa_1} F_* E_{2h} \\ \overset{N}{\nabla}_{\tilde{T}} F_* E_{2h} &= -\tilde{\kappa_1} F_* E_{1h} + \tilde{\kappa_2} F_* E_{3h} \\ &\cdots \\ \overset{N}{\nabla}_{\tilde{T}} F_* E_{rh} &= -\tilde{\kappa}_{r-1} F_* E_{(r-1)h}. \end{aligned} \tag{37}$$

We calculate $\overset{N}{\nabla}_{\tilde{T}}\tilde{T}$ as follows.

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} \quad = \quad \overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h} = \tilde{\kappa_1}F_*E_{2h}. \tag{38}$$

Then, using Frenet formulas of γ, we get

$$\begin{aligned}
&\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \tilde{\kappa_1}'F_*E_{2h} + \tilde{\kappa_1}\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{2h}\\
&\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = -\tilde{\kappa_1}^2F_*E_{1h} + \tilde{\kappa_1}'F_*E_{2h} + \tilde{\kappa_1}\tilde{\kappa_2}F_*E_{3h}.
\end{aligned} \tag{39}$$

We calculate, $\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}$ as follows.

$$\begin{aligned}
&\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2)F_*E_{2h} + (2\tilde{\kappa_1}'\tilde{\kappa_2}\\
&+\tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h}.
\end{aligned} \tag{40}$$

Then, using the Frenet formulas, we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \tilde{\kappa_1}F_*E_{2h})F_*E_{1h} = \tilde{\kappa_1}R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h}. \tag{41}$$

Now, taking the vertical and horizontal parts of E_2, we find,

$$\begin{aligned}
&R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_2)E_{1h}\\
&= R^M(E_{1h}, \kappa_1 E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}.
\end{aligned} \tag{42}$$

Then, we applied to F_* both sides in (42), we obtain,

$$\begin{aligned}
&F_*(R^M(T, \overset{M}{\nabla}_T T)T)\\
&= F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}).
\end{aligned} \tag{43}$$

Since F is a Riemannian map, we have

$$\begin{aligned}
&\kappa_1 R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h} = \kappa_1 F_*(R^M(E_{1h}, E_2)E_{1h})\\
&-\kappa_1 F_*(R^M(E_{1h}, E_{2v})E_{1h}) - \kappa_1 S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + \kappa_1 S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h}\\
&+\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) - \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}).
\end{aligned} \tag{44}$$

Using, Riemannian curvature tensor of M, we get

$$\begin{aligned}
&\kappa_1 R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h} = -c_1\kappa_1 F_*E_{2h} - \kappa_1 S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h}\\
&+\kappa_1 S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} + \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h})\\
&-\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h})
\end{aligned} \tag{45}$$

Then, using (35) into (41), we have,

$$\begin{aligned}
&R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = -c_1\tilde{\kappa_1}F_*E_{2h} - \tilde{\kappa_1}S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h}\\
&+\tilde{\kappa_1}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} + \tilde{\kappa_1}(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h})\\
&-\tilde{\kappa_1}(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}).
\end{aligned} \tag{46}$$

Thus putting (40) and (46) in (12), we have,

$$\begin{aligned}
&\tau_2(\gamma) = -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c_1\tilde{\kappa_1})F_*E_{2h}\\
&+(2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h} - \tilde{\kappa_1}S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h}\\
&+\tilde{\kappa_1}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} + \tilde{\kappa_1}(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h})\\
&-\tilde{\kappa_1}(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}).
\end{aligned} \tag{47}$$

Then taking the $F_*((kerF_*)^{\perp}) = rangeF_*$ and $(rangeF_*)^{\perp}$ parts, we have (33) and (34). Thus $F \circ \alpha : \gamma : I \to (N, g_N)$ is a biharmonic curve if and only if (33) and (34) are satisfied. □

Corollary 3.5. *Let $F : (M(c_1), g_M) \to (N, g_N)$ be an umbilical Riemannian map from a real space form $(M(c_1), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha : I \to (M(c_1), g_M)$ be a horizontal Frenet curve. Then Frenet curve $F \circ \alpha : \gamma : I \to (N, g_N)$ is a biharmonic curve such that $\tilde{\kappa_1} = constant \neq 0$ if and only if*

$$\tilde{\kappa_1}^2 + \tilde{\kappa_2}^2 = \| H_2 \|^2 + c_1,$$
$$\tilde{\kappa_2} = constant,$$
$$\tilde{\kappa_2}\tilde{\kappa_3} = 0.$$

Proof. The assertion follows from Theorem 3.4. □

4. Biharmonic Curves along Riemannian Maps from Complex Space Forms

In this section, we study biharmonic curves along Riemannian maps from complex space forms. Then, we will investigate necessary and sufficient conditions for the curves along Riemannian maps from complex space forms to be biharmonic. We first recall the complex space form and related notions. An almost complex manifold (M, J) endowed with a Riemannian metric g_M satisfying

$$g_M(JX, JY) = g_M(X, Y) \tag{48}$$

Let $M^m(4c)$ be a complex space form of holomorphic sectional curvature $4c$ [22]. Let us denote by J the complex structure and by g_M the Riemannian metric on $M^m(4c)$. Then its curvature operator is given by

$$\begin{aligned} R^{M^m(4c)}(X,Y)Z = c\{&g_M(Y,Z)X - g_M(X,Z)Y + g_M(JY,Z)JX \\ &-g_M(JX,Z)JY + 2g_M(X,JY)JZ\} \end{aligned} \tag{49}$$

for $X, Y, Z \in \chi(M)$ [11] . Following S. Maeda and Y. Ohnita [7], we define the complex torsions of the curve α by $\tau_{ij} = g(E_i, JE_j)$, $1 \leq i < j \leq r$. A helix of order r is called a holomorphic helix of order r if all the complex torsions are constant.

Let (M, g_M) be a complex space form and $\alpha : I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then, using Frenet equations, the bitension field of α becomes [1], [12]

$$\begin{aligned} \tau_2(\alpha) = &-3\kappa_1\kappa_1^{'}E_1 + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})E_3 \\ &+\kappa_1\kappa_2\kappa_3E_4 - 3c\kappa_1\tau_{12}JE_1. \end{aligned} \tag{50}$$

For a Riemannian map from a complex space form to a real space form, we have the following result.

Theorem 4.1. *Let $F : (M(4c_1), g_M) \to (N(c_2), g_N)$ be a Riemannian map from a complex space form $(M(4c_1), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(4c_1), g_M)$ be a biharmonic horizontal curve. Then $F \circ \alpha : \gamma : I \to (N(c_2), g_N)$ is a biharmonic curve if and only if*

$$\begin{aligned} &-(\nabla F_*)(E_{1h},{}^* F_* S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h}) - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h} \\ &+2\kappa_1^{'}(\nabla F_*)(E_{1h}, E_{2h}) - \kappa_1 S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} \\ &+(c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h}) \\ &-\kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}} E_{2v}) = 0, \end{aligned} \tag{51}$$

$$\begin{aligned} &3c_1\kappa_1\tau_{12mix}F_*JE_{1h} - F_*\overset{M^*}{\nabla}_{E_{1h}} F_* S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{1h} \\ &+\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} \\ &-\kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} + \kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{2h}) \\ &-\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) + \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}) = 0. \end{aligned} \tag{52}$$

where $\tau_{12mix} = g_M(E_{1h}, JE_{2v})$.

Proof. Let $F : (M(4c), g_M) \to (N, g_N)$ be a Riemannian map from a complex space form $(M(4c), g_M)$ to a real space form $(N(c_2), g_N)$. Let $\alpha : I \to (M(4c), g_M)$ be a biharmonic horizontal curve. Then, we have the following equation.

$$\tau_2(\alpha) = -3\kappa_1\kappa_1^{'}E_1 + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})E_3 \\ +\kappa_1\kappa_2\kappa_3E_4 - 3c\kappa_1\tau_{12}JE_1. \tag{53}$$

Since α is horizontal curve, we have

$$\alpha^{'} = T = E_{1h}, \qquad \gamma^{'} = F_*T = \tilde{T}, \tag{54}$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma^{'} = \tilde{T}$ is the unit tangent vector field along the curve. Then we have the equation (24). On the other hand, using (12), we obtain,

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, (\nabla F_*)(E_{1h}, E_{1h}))F_*E_{1h} + R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h}. \tag{55}$$

and

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(T, \kappa_1E_2)T, \tag{56}$$

respectively. Now, taking the vertical and horizontal parts of E_2 in (56), we find,

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1E_{2h})E_{1h}. \tag{57}$$

Since F is a Riemannian map, we derive,

$$R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h} = F_*(R^M(E_{1h}, E_2)E_{1h}) - F_*(R^M(E_{1h}, E_{2v})E_{1h}) \\ -S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} + (\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\ -(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}) \tag{58}$$

Using (49), we get

$$R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h} = -c_1\kappa_1F_*E_{2h} - 3c_1\kappa_1\tau_{12\mathcal{H}}F_*JE_{1h} \\ -\kappa_1S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + \kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} \\ +\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) - \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}). \tag{59}$$

Thus putting (60) in (56), we have,

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = -c_2F_*(\nabla F_*)(E_{1h}, E_{1h}) - c_1\kappa_1F_*E_{2h} - 3c_1\kappa_1\tau_{12\mathcal{H}}F_*JE_{1h} \\ -\kappa_1S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + \kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} \\ +\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) - \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h})$$

Then, putting this equation and (24) in (12), we have,

$$\tau_2(\gamma) = -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c_1\kappa_1)F_*E_{2h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} \\ +\kappa_1\kappa_2\kappa_3F_*E_{4h} - 3c_1\tau_{12\mathcal{H}}F_*JE_{1h} - (\nabla F_*)(E_{1h},{}^*F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}) \\ -F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} \\ +\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{1h}) + 2\kappa_1^{'}(\nabla F_*)(E_{1h}, E_{2h}) - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} \\ -\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} \\ +\kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h}, E_{2h}) + (c_2 - \kappa_1^2)(\nabla F_*)(E_{1h}, E_{1h}) \\ +\kappa_1\kappa_2(\nabla F_*)(E_{1h}, E_{3h}) - \kappa_1(\nabla F_*)(E_{1h}, \mathcal{A}_{E_{1h}}E_{2v}) - \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h}, E_{1h}) \\ +\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h}, E_{1h}). \tag{60}$$

Since $\tau_2(\alpha)=0$, we can write $F_*\tau_2(\alpha)=0$. Then, using this equation in $\tau_2(\gamma)$, we have,

$$\begin{aligned}
\tau_2(\gamma) &= 3c_1\kappa_1\tau_{12mix}F_*JE_{1h} - (\nabla F_*)(E_{1h},{}^*F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}) \\
&-F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} \\
&+\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h}) + 2\kappa_1'(\nabla F_*)(E_{1h},E_{2h}) - 2\kappa_1'F_*\mathcal{A}_{E_{1h}}E_{2v} \\
&-\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} \\
&+\kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{2h}) + (c_2-\kappa_1^2)(\nabla F_*)(E_{1h},E_{1h}) + \kappa_1\kappa_2(\nabla F_*)(E_{1h},E_{3h}) \\
&-\kappa_1(\nabla F_*)(E_{1h},\mathcal{A}_{E_{1h}}E_{2v}) - \kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) \\
&+\kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}). \qquad (61)
\end{aligned}$$

Then taking the $F_*((kerF_*)^\perp)=rangeF_*$ and $(rangeF_*)^\perp$ parts, we have (51) and (52). Thus $F\circ\alpha:\gamma:I\to(N,g_N)$ is a biharmonic curve if and only if (51) and (52) are satisfied. □

Theorem 4.2. *Let $F:(M(4c_1),g_M)\to(N(c_2),g_N)$ be an umbilical Riemannian map from a complex space form $(M(4c_1),g_M)$ to a real space form $(N(c_2),g_N)$. Let $\alpha:I\to(M(4c_1),g_M)$ be a biharmonic horizontal curve and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha:\gamma:I\to(N(c_2),g_N)$ is a biharmonic curve if and only if*

$$-\|H_2\|^2 - S_{\nabla^{F\perp}_{E_{1h}}H_2}F_*E_{1h} - \kappa_1S_{H_2}F_*E_{2h} + (c_2-\kappa_1^2)H_2 = 0,$$

$$3c_1\kappa_1\tau_{12mix}F_*JE_{1h} - F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{H_2}F_*E_{1h} + \nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}H_2 = 0.$$

where $\tau_{12mix}=g_M(E_{1h},JE_{2v})$.

Proof. The assertions follows from Theorem 4.1. □

In particular, if κ_1 =constant$\neq 0$, then we have the following result.

Theorem 4.3. *Let $F:(M(4c_1),g_M)\to(N(c_2),g_N)$ be a Riemannian map from a complex space form $(M(4c_1),g_M)$ to a real space form $(N(c_2),g_N)$. Let $\alpha:I\to(M(4c_1),g_M)$ be a biharmonic horizontal curve and κ_1 =constant$\neq 0$ and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha:\gamma:I\to(N(c_2),g_N)$ is a biharmonic curve if and only if*

$$\begin{aligned}
&-(\nabla F_*)(E_{1h},{}^*F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h}) - S_{\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} \\
&-\kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h} + (c_2-\kappa_1^2)(\nabla F_*)(E_{1h},E_{1h}) \\
&+\kappa_1\kappa_2(\nabla F_*)(E_{1h},E_{3h}) = 0, \qquad (62)
\end{aligned}$$

$$\begin{aligned}
&3c_1\kappa_1\tau_{12mix}F_*JE_{1h} - F_*\overset{M^*}{\nabla}_{E_{1h}}F_*S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{1h} \\
&+\nabla^{F\perp}_{E_{1h}}\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{1h}) + \kappa_1\nabla^{F\perp}_{E_{1h}}(\nabla F_*)(E_{1h},E_{2h}) \\
&-\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) + \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}) = 0. \qquad (63)
\end{aligned}$$

Proof. Since κ_1 =constant$\neq 0$, we have $\kappa_1'=0$. The parallelity of $\mathcal{A}$ implies that $\mathcal{A}=0$. Then the assertion follows from Theorem 4.1. □

Theorem 4.4. *Let $F:(M(4c_1),g_M)\to(N,g_N)$ be a Riemannian map from a complex space form $(M(4c_1),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha:I\to(M(4c_1),g_M)$ be a horizontal Frenet curve. Then Frenet curve $F\circ\alpha:\gamma:I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\begin{aligned}
&-3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c_1\tilde{\kappa_1})F_*E_{2h} \\
&+(2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h} + 3c_1\tilde{\kappa_1}\tau_{12\mathcal{H}}F_*JE_{1h} \\
&-\tilde{\kappa_1}(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) + \tilde{\kappa_1}(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}) = 0, \qquad (64)
\end{aligned}$$

$$\tilde{\kappa_1} S_{(\nabla F_*)(E_{2h},E_{1h})} F_* E_{1h} - \tilde{\kappa_1} S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} = 0. \tag{65}$$

where $\tilde{\kappa_1}, \cdots, \tilde{\kappa_{r-1}}$ are positive functions of γ on I.

Proof. Let $F : (M(4c_1), g_M) \to (N, g_N)$ be a Riemannian map from a complex space form $(M(4c_1), g_M)$ to a Riemannian manifold (N, g_N). Since $\alpha : I \to (M(4c_1), g_M)$ is a horizontal Frenet curve, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_* T = \tilde{T},$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Then we have Frenet formulas of γ as follows

$$\begin{aligned} \overset{N}{\nabla}_{\tilde{T}} \tilde{T} &= \tilde{\kappa_1} F_* E_{2h} \\ \overset{N}{\nabla}_{\tilde{T}} F_* E_{2h} &= -\tilde{\kappa_1} F_* E_{1h} + \tilde{\kappa_2} F_* E_{3h} \\ &\cdots \\ \overset{N}{\nabla}_{\tilde{T}} F_* E_{rh} &= -\tilde{\kappa}_{r-1} F_* E_{(r-1)h}. \end{aligned} \tag{66}$$

We calculate $\overset{N}{\nabla}_{\tilde{T}} \tilde{T}$ as follows.

$$\overset{N}{\nabla}_{\tilde{T}} \tilde{T} = \overset{N}{\nabla}_{F_* E_{1h}} F_* E_{1h} = \tilde{\kappa_1} F_* E_{2h}. \tag{67}$$

Then, using Frenet formulas of γ, we get,

$$\begin{aligned} \overset{N}{\nabla}{}^2_{\tilde{T}} \tilde{T} &= \tilde{\kappa_1}' F_* E_{2h} + \kappa_1 \overset{N}{\nabla}_{F_* E_{1h}} F_* E_{2h} \\ &= -\tilde{\kappa_1}^2 F_* E_{1h} + \tilde{\kappa_1}' F_* E_{2h} + \tilde{\kappa_1}\tilde{\kappa_2} F_* E_{3h}. \end{aligned} \tag{68}$$

We calculate, $\overset{N}{\nabla}{}^3_{\tilde{T}} \tilde{T}$ as follows.

$$\begin{aligned} &\overset{N}{\nabla}{}^3_{\tilde{T}} \tilde{T} = -3\tilde{\kappa_1}\tilde{\kappa_1}' F_* E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2) F_* E_{2h} + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}') F_* E_{3h} \\ &+ \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3} F_* E_{4h}. \end{aligned} \tag{69}$$

Then, using the Frenet formulas, we obtain,

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}} \tilde{T})\tilde{T} = R^N(F_* E_{1h}, \tilde{\kappa_1} F_* E_{2h}) F_* E_{1h} = \tilde{\kappa_1} R^N(F_* E_{1h}, F_* E_{2h}) F_* E_{1h}. \tag{70}$$

Now, taking the vertical and horizontal parts of E_2, we find,

$$\begin{aligned} &R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_2) E_{1h} \\ &= R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h}) E_{1h}. \end{aligned} \tag{71}$$

Then, we applied to F_* both sides in (71), we obtain,

$$\begin{aligned} &F_*(R^M(T, \overset{M}{\nabla}_T T)T) \\ &= F_*(R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h}) + F_*(R^M(E_{1h}, \kappa_1 E_{2h}) E_{1h}). \end{aligned} \tag{72}$$

Since F is a Riemannian map, we have

$$\begin{aligned} &R^N(F_* E_{1h}, F_* E_{2h}) F_* E_{1h} = F_*(R^M(E_{1h}, E_2) E_{1h}) - F_*(R^M(E_{1h}, E_{2v}) E_{1h}) \\ &- S_{(\nabla F_*)(E_{2h},E_{1h})} F_* E_{1h} + S_{(\nabla F_*)(E_{1h},E_{1h})} F_* E_{2h} + (\nabla_{E_{1h}} (\nabla F_*))(E_{2h}, E_{1h}) \\ &- (\nabla_{E_{2h}} (\nabla F_*))(E_{1h}, E_{1h}) \end{aligned} \tag{73}$$

Using, (49), we get

$$R^N(F_*E_{1h},\kappa_1F_*E_{2h})F_*E_{1h} = -c_1\kappa_1F_*E_{2h} - 3c_1\kappa_1\tau_{12\mathcal{H}}F_*JE_{1h}$$
$$-\kappa_1S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + \kappa_1S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h}$$
$$+\kappa_1(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) - \kappa_1(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}) \quad (74)$$

Then, using (74) into (70), we have,

$$R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = -c_1\tilde{\kappa_1}F_*E_{2h} - 3c_1\tilde{\kappa_1}\tau_{12\mathcal{H}}F_*JE_{1h}$$
$$-\tilde{\kappa_1}S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} + \tilde{\kappa_1}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h}$$
$$+\tilde{\kappa_1}(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) - \tilde{\kappa_1}(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}) \quad (75)$$

Thus putting (69) and (75) in (12), we have,

$$\tau_2(\gamma) = -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c_1\tilde{\kappa_1})F_*E_{2h}$$
$$+(2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h} + 3c_1\tilde{\kappa_1}\tau_{12\mathcal{H}}F_*JE_{1h}$$
$$+\tilde{\kappa_1}S_{(\nabla F_*)(E_{2h},E_{1h})}F_*E_{1h} - \tilde{\kappa_1}S_{(\nabla F_*)(E_{1h},E_{1h})}F_*E_{2h}$$
$$-\tilde{\kappa_1}(\nabla_{E_{1h}}(\nabla F_*))(E_{2h},E_{1h}) + \tilde{\kappa_1}(\nabla_{E_{2h}}(\nabla F_*))(E_{1h},E_{1h}). \quad (76)$$

Then taking the $F_*((kerF_*)^\perp) = rangeF_*$ and $(rangeF_*)^\perp$ parts, we have (64) and (65). Thus $F\circ\alpha : \gamma : I \to (N, g_N)$ is a biharmonic curve if and only if (64) and (65) are satisfied. □

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