Filomat 38:1 (2024), 271–293
https://doi.org/10.2298/FIL2401271L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On infinitesimal Hom-H-pseudobialgebras

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Abstract. The Hom-version of Lie H-pseudobialgebra was introduced in [21]. In this paper, we investigate infinitesimal Hom-H-pseudobialgebra, which is the Hom-associative analog of Hom-Lie H-pseudobialgebra. We first provide some examples of this new structure and present the construction theorems. We also consider the subclass of coboundary infinitesimal Hom-H-pseudobialgebras and the related Hom-associative pseudo-Yang-Baxter equation. Furthermore, the connection between infinitesimal Hom-H-pseudobialgebras and Hom-Lie H-pseudobialgebras is depicted. Finally, we show that, under suitable conditions, solutions of the Hom-associative pseudo-Yang-Baxter equation give rise to solutions of the Hom-classical pseudo-Yang-Baxter equation.

1. Introduction

The notion of conformal algebra introduced by Kac([12]) is widely used in the field of mathematical physics by providing an axiomatic description of the operator product expansion (OPE) of chiral fields in conformal field theory, and it came to be useful for the investigation of vertex algebras. Specifically, a Lie conformal algebra L is defined as a $\mathbb{C}[\partial]$-module (∂ is an indeterminate), endowed with a $\mathbb{C}$-linear map

$$L \otimes L \longrightarrow \mathbb{C}[\lambda] \otimes L, \quad a \otimes b \mapsto [a_\lambda b],$$

which satisfies the axioms similar to those of Lie algebra (see [5, 12]). In [3], Bakalov, D'Andrea and Kac studied "multi-dimensional" Lie conformal algebra by replacing the above polynomial algebra $\mathbb{C}[\partial]$ with any cocommutative Hopf algebra H. In fact, this multivariable generalization of conformal algebra is actually an algebra in pseudotensor category $\mathcal{M}^*(H)$, so it is called an H-pseudoalgebra, or simply a pseudoalgebra. When $H = k$, a pseudoalgebra is actually a usual algebra. A Lie algebra in this pseudotensor category is called a Lie H-pseudoalgebra, which is closely related to the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonlinear evolution equation ([6, 8, 9]). For other algebraic

2020 *Mathematics Subject Classification*. 16T10, 16T25, 17D30.
Keywords. infinitesimal Hom-H-pseudobialgebra, Hom-Lie H-pseudobialgebra, Hom-Yang-Baxter equation.
Received: 16 December 2022; Revised: 07 June 2023; Accepted: 10 July 2023
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No. 12301023), Doctoral Fund of Henan Institute of Technology (No. KQ2101), Project funded by China Postdoctoral Science Foundation (No. 2022M711076) and Postdoctoral research grant in Henan Province (No. 202103099).
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structures, such as associative algebra, symmetric algebra, Leibniz algebra, etc., in this pseudotensor category can be found in [3, 23, 24]. In particular, the fact that the annihilation algebra of associative H-pseudoalgebra Cend$H$ is the Drinfeld double of H makes us believe that there should be a deep connection between H-pseudoalgebra theory and quantum groups.

The study of Hom-algebras can be traced back to Hartwig, Larsson and Silvestrov's work in [10], where they introduced the structure of Hom-Lie algebras in the context of the deformations of Witt and Virasoro algebras. The main feature of Hom-type algebras is that the identities defining the structures are twisted by homomorphisms. Corresponding to Hom-Lie algebras are Hom-associative algebras [15], which give rise to Hom-Lie algebras via the commutator bracket. Other Hom-type algebraic structures were studied in [14, 16–18, 25, 26].

Infinitesimal bialgebras were introduced by Joni and Rota [11] in the context of the calculus of divided differences. Later, Aguiar established the basic theory of infinitesimal bialgebras in [1, 2]. Recently, this structures have been studied extensively, not only because of their important application to combinatorials [2, 7], but also because they are closely related to the theory of Lie bialgebras. In detail, the cobracket Δ in a Lie bialgebra is a 1-cocycle in Chevalley-Eilenberg cohomology, which is a 1-cocycle in Hochschild cohomology in an infinitesimal bialgebra. So any infinitesimal bialgebra can be seen as an associative analog of Lie bialgebra. Furthermore, Aguiar found the necessary and sufficient conditions so that an infinitesimal bialgebra gives rise to a Lie bialgebra. In [27], Yau studied the Hom-type generalization of infinitesimal bialgebras, called infinitesimal Hom-bialgebras and extended the main results in [1, 2] to Hom-case.

As a generalization of Lie bialgebra, Lie H-pseudobialgebra was introduced in [4]. There Boyallian and Liberati gave the classical Yang-Baxter equation and Drinfeld double in H-pseudoalgebras. In [21], Sun and Li generalized Lie H-pseudobialgebra and related classical Yang-Baxter equation to Hom-case by twisting the identities using morphisms of H-modules. Similar to the relationship between Lie bialgebra and infinitesimal bialgebra mentioned above, it is natural to consider the Hom-associative analog of Hom-Lie H-pseudobialgebra and related Hom-associative Yang-Baxter equation in a Hom-associative H-pseudoalgebra. This is our motivation to define infinitesimal Hom-H-pseudobialgebra.

The paper is organized as follows. In Section 2, we first define infinitesimal Hom-H-pseudobialgebra, which is the Hom-associative analog of Hom-Lie H-pseudobialgebra. Then we construct infinitesimal Hom-H-pseudobialgebras from infinitesimal Hom-bialgebras, infinitesimal H-pseudobialgebras and the given infinitesimal Hom-H-pseudobialgebras. Moreover, we obtain a class of infinitesimal Hom-H-pseudobialgebras. Section 3 is dedicated to the subclass of coboundary infinitesimal Hom-H-pseudobialgebras. We give the construction theorems and the characterizations of coboundary infinitesimal Hom-H-pseudobialgebras. In addition, we obtain related Hom-associative pseudo-Yang-Baxter equation (pseudo-AHYBE) in a Hom-associative H-pseudoalgebra. In Section 4, we construct a class of Hom-Lie H-pseudobialgebras from infinitesimal Hom-H-pseudobialgebras. We also study how coboundary infinitesimal Hom-H-pseudobialgebras give rise to coboundary Hom-Lie H-pseudobialgebras. In the last section, we show that under suitable conditions, solutions of pseudo-AHYBE give rise to solutions of the pseudo-CHYBE in related Hom-Lie H-pseudoalgebra.

Throughout this paper, k is an algebraically closed field. All vector spaces, linear maps, and tensor products are over k, H is a cocommutative Hopf algebra and $X = H^*$ is the dual of H. As usual, we adopt Sweedler's notations in [22]. For a coalgebra C, we write its comultiplication as $\Delta(c) = c_1 \otimes c_2$, for all $c \in C$. For any vector space V, we have the flipping maps $\sigma, \tau : f \otimes g \mapsto g \otimes f$ for all $f, g \in V$. Now we recall some useful definitions which will be used later.

2. Preliminaries

Definition 2.1 ([19]) A Hom-associative H-pseudoalgebra is a triple $(A, \mu = *, \alpha)$, where A is a left H-module, $\alpha \in \mathrm{Hom}_H(A, A)$ and $\mu \in \mathrm{Hom}_{H\otimes H}(A \otimes A, (H \otimes H) \otimes_H A)$, called the pseudoproduct, denote $\mu(a \otimes b) = a * b$, satisfying

$$(id \otimes_H \alpha) \circ \mu = \mu \circ (\alpha \otimes \alpha), \qquad \mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha),$$

i.e.,

$$(id \otimes_H \alpha)(a * b) = \alpha(a) * \alpha(b), \qquad \alpha(a) * (b * c) = (a * b) * \alpha(c) \in H^{\otimes 3} \otimes_H A$$

for all $a, b, c \in A$.

A morphism $f : (A, \mu_A, \alpha_A) \longrightarrow (B, \mu_B, \alpha_B)$ of Hom-associative H-pseudoalgebras is an H-linear map such that $\alpha_B \circ f = f \circ \alpha_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

For an arbitrary Hopf algebra H, we recall that the map $\mathcal{F} : H \otimes H \longrightarrow H \otimes H$, called the Fourier transform, defined by the formula

$$\mathcal{F}(f \otimes g) = (f \otimes 1)(S \otimes id)\Delta(g) = fS(g_1) \otimes g_2,$$

and $\mathcal{F}$ is a vector space isomorphism with an inverse given by

$$\mathcal{F}^{-1}(f \otimes g) = (f \otimes 1)\Delta(g) = fg_1 \otimes g_2.$$

Now we introduce another product $[a, b] \in H \otimes A$ defined as the Fourier transform of $a * b$:

$$[a, b] = \sum_i \mathcal{F}(f_i \otimes g_i)(1 \otimes e_i) = \sum_i f_i S(g_{i1}) \otimes g_{i2} e_i,$$

if $a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i = \sum_i f_i S(g_{i1}) \otimes 1 \otimes_H g_{i2} e_i$. In other words,

$$[a, b] = \sum_i h_i \otimes c_i, \ \text{ if } a * b = \sum_i (h_i \otimes 1) \otimes_H c_i.$$

Then for $x \in X = H^*$, the x-product in A is given by

$$a_x b = (< S(x), \cdot > \otimes id)[a, b] = \sum_i < S(x), h_i > c_i.$$

Using the properties of Fourier transform, we have the following equivalent definition of Hom-associative H-pseudoalgebra.

Definition 2.2 ([19]) A Hom-associative H-conformal algebra is a triple $(A, [\cdot, \cdot], \alpha)$, where A is a left H-module, $[\cdot, \cdot] : A \otimes A \longrightarrow H \otimes A$ is defined as above and $\alpha \in \mathrm{Hom}_H(A, A)$, satisfying the following properties ($\forall a, b, c \in A$ and $h \in H$):

(H-sesqui-linearity)

$$[ha, b] = (h \otimes 1)[a, b], \quad [a, hb] = (1 \otimes h_2)[a, b](S(h_1) \otimes 1).$$

(Hom-associativity)

$$[\alpha(a), [b, c]] = (\mathcal{F}^{-1} \otimes id)[[a, b], \alpha(c)]$$

in $H \otimes H \otimes A$, where $[a, [b, c]] = (\sigma \otimes id)(id \otimes [a, \cdot])[b, c]$, $[[a, b], c] = (id \otimes [\cdot, c])[a, b]$.

We can also reformulate Definition 2.2 in terms of the x-products.

Definition 2.3 ([19]) A Hom-associative H-conformal algebra is a triple $(A, \cdot_x \cdot, \alpha)$, where A is a left H-module, $\cdot_x \cdot : A \otimes A \longrightarrow A, a \otimes b \mapsto a_x b$ is defined as above and $\alpha \in \mathrm{Hom}_H(A, A)$, satisfying the following properties ($\forall a, b, c \in A, h \in H$ and $x \in X$):

(Locality)

for any basis $\{x_i\}$ of X, $a_{x_i} b \neq 0$ for only a finite number of i.

(H-sesqui-linearity)

$$ha_x b = a_{xh} b, \ a_x hb = h_2(a_{S(h_1)x} b).$$

(Hom-associativity)

$$\alpha(a)_x (b_y c) = (a_{x_2} b)_{y x_1} \alpha(c).$$

Definition 2.4 ([19]) Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra. A left Hom-A-module is a triple (M, ρ, β), where M is a left H-module, $\rho \in \mathrm{Hom}_{H\otimes H}(A \otimes M, (H \otimes H) \otimes_H M)$ (we denote $\rho(a \otimes m) = a * m$), $\beta \in \mathrm{Hom}_H(M, M)$, satisfying

$$(id \otimes_H \beta)(a * m) = \alpha(a) * \beta(m), \quad \alpha(a) * (b * m) = (a * b) * \beta(m), \ \forall\, a, b \in A, m \in M.$$

Similarly, we can define a right Hom-A-module by satisfying $(id \otimes_H \beta)(m * a) = \beta(m) * \alpha(a)$ and $\beta(m) * (a * b) = (m * a) * \alpha(b)$. Let $(A, *, \alpha_A)$ and $(B, *, \alpha_B)$ be two Hom-associative H-pseudoalgebras, then (M, ρ, β) is called a Hom-A-B-bimodule if (M, ρ, β) is both a left Hom-A-module and a right Hom-B-module, satisfying $(a*m)*\alpha_B(b) = \alpha_A(a)*(m*b)$ for all $a \in A, b \in B$ and $m \in M$. When $B = A$, we call (M, ρ, β) a Hom-A-bimodule.

Proposition 2.5 Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra. Then for each $n \geq 2$, $(A^{\otimes n}, \alpha^{\otimes n})$ are Hom-A-bimodules with the following structures:

$$a * (b_1 \otimes b_2 \otimes \cdots \otimes b_n) = \sum_i (f_i \otimes g_i) \otimes_H (e_i \otimes \alpha(b_2) \otimes \cdots \otimes \alpha(b_n)),$$
$$(b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n) * a = \sum_j (k_j \otimes l_j) \otimes_H (\alpha(b_1) \otimes \cdots \otimes \alpha(b_{n-1}) \otimes t_j),$$

where $\alpha(a) * b_1 = \sum_i (f_i \otimes g_i) \otimes_H e_i$ and $b_n * \alpha(a) = \sum_j (k_j \otimes l_j) \otimes_H t_j$, for all $a, b_1, b_2 \cdots, b_n \in A$.

Proof. It is a straightforward computation and we omit the details. □

Definition 2.6 ([19]) A Hom-Lie H-pseudoalgebra is a triple $(L, \mu = [*], \alpha)$, where L is a left H-module, $\alpha \in \mathrm{Hom}_H(L, L)$ and $\mu \in \mathrm{Hom}_{H\otimes H}(L \otimes L, H^{\otimes 2} \otimes_H L)$, called the pseudobracket, denote $\mu(a \otimes b) = [a * b]$, satisfying

(Multiplicativity)
$$(id \otimes_H \alpha)[a * b] = [\alpha(a) * \alpha(b)].$$

(Skew-commutativity)
$$[a * b] = -(\sigma \otimes_H id)[b * a].$$

(Hom-Jacobi identity)
$$[[a * b] * \alpha(c)] = [\alpha(a) * [b * c]] - ((\sigma \otimes id) \otimes_H id)[\alpha(b) * [a * c]],$$

for all $a, b, c \in L$.

In particular, for the one-dimensional Hopf algebra $H = k$, a Hom-Lie H-pseudoalgebra is just an ordinary Hom-Lie algebra over the field k. If $\alpha = id$, then a Hom-Lie H-pseudoalgebra is an ordinary Lie H-pseudoalgebra([3]).

Definition 2.7 ([21]) A Hom-Lie H-coalgebra is a triple (L, δ, α), where L is a left H-module, $\delta : L \longrightarrow L \otimes L$ is an H-linear map and $\alpha \in \mathrm{Hom}_H(L, L)$, satisfying

$$\sigma \circ \delta = -\delta,$$
$$\delta \circ \alpha = (\alpha \otimes \alpha) \circ \delta,$$
$$(\alpha \otimes \delta)\delta - \sigma_{12}(\alpha \otimes \delta)\delta = (\delta \otimes \alpha)\delta,$$

where $\sigma_{12}(a \otimes b \otimes c) = b \otimes a \otimes c$ for all $a, b, c \in L$.

Definition 2.8 ([21]) A Hom-Lie H-pseudobialgebra is a quadruple $(L, [*], \delta, \alpha)$ such that $(L, [*], \alpha)$ is a Hom-Lie H-pseudoalgebra, (L, δ, α) is a Hom-Lie H-coalgebra, and they satisfy the compatible condition

$$\delta([a * b]) = [a * \delta(b)] - (\sigma \otimes_H id)[b * \delta(a)], \ \forall\, a, b \in L, \tag{2. 1}$$

where

$$[a * \delta(b)] = \sum_i (f_i S(g_{i1}) \otimes 1) \otimes_H (g_{i2} e_i \otimes \alpha(b_2)) + \sum_j (k_j S(l_{j1}) \otimes 1) \otimes_H (\alpha(b_1) \otimes l_{j2} t_j),$$

if

$$[\alpha(a) * b_1] = \sum_i (f_i \otimes g_i) \otimes_H e_i = \sum_i (f_i S(g_{i1}) \otimes 1) \otimes_H g_{i2} e_i$$

and

$$[\alpha(a) * b_2] = \sum_j (k_j \otimes l_j) \otimes_H t_j = \sum_j (k_j S(l_{j1}) \otimes 1) \otimes_H l_{j2} t_j.$$

A morphism $f : (L, [*], \delta, \alpha) \longrightarrow (L', [*]', \delta', \alpha')$ of Hom-Lie H-pseudobialgebras is an H-linear map such that $\alpha' \circ f = f \circ \alpha$, $f \circ [*] = [*]' \circ (f \otimes f)$ and $\delta' \circ f = (f \otimes f) \circ \delta$.

Remark 2.9 The compatible condition (2.1) is indeed the condition that δ is a 1-cocycle of the Hom-Lie H-pseudoalgebra $(L, [*], \alpha)$ with coefficient in $L \otimes L$ in the reduced complex ([19]).

Definition 2.10 ([21]) A coboundary Hom-Lie H-pseudobialgebra $(L, [*], \delta, , \alpha, r)$ consists of a Hom-Lie H-pseudobialgebra $(L, [*], \delta, \alpha)$ and an element $r = \sum_i u_i \otimes v_i \in L \otimes L$ such that $\alpha^{\otimes 2}(r) = r$ and

$$\delta(a) = \sum_i \mu(\{a, u_i\} \otimes \alpha(v_i) + \sigma_{12}(\alpha(u_i) \otimes \{a, v_i\})), \tag{2. 2}$$

where $\{a, b\}$ is the Fourier transform of $[a * b]$, and $\mu(h \otimes a \otimes b) = \Delta(h)(a \otimes b)$ for all $h \in H$ and $a, b \in L$. More precisely, suppose $[a * b] = \sum h^{a,b} \otimes 1 \otimes_H e_{a,b}$, then

$$\delta(a) = \sum h^{a,u_i}(e_{a,u_i} \otimes \alpha(v_i)) + \sum h^{a,v_i}(\alpha(u_i) \otimes e_{a,v_i}).$$

3. Infinitesimal Hom-H-pseudobialgebras

In this section, we define infinitesimal Hom-H-pseudobialgebras, which generalize both infinitesimal H-pseudobialgebras ([13]) and infinitesimal Hom-bialgebras ([27]). In addition, we construct an infinitesimal Hom-H-pseudobialgebra from the infinitesimal Hom-bialgebra, the infinitesimal H-pseudobialgebra and a given infinitesimal Hom-H-pseudobialgebra, respectively.

Definition 3.1 A Hom-coassociative H-coalgebra is a triple (A, Δ, α), where $\Delta : A \longrightarrow A \otimes A$ (we denote $\Delta(a) = a_1 \otimes a_2$) is an H-linear map and $\alpha \in \mathrm{Hom}_H(A, A)$, satisfying

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta, \ \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta.$$

More precisely, we have

$$\alpha(a_1) \otimes a_{21} \otimes a_{22} = a_{11} \otimes a_{12} \otimes \alpha(a_2), \quad \Delta(\alpha(a)) = \alpha(a_1) \otimes \alpha(a_2), \ \forall a \in A.$$

Remark 3.2 In fact, it is the definition of Hom-coassociative coalgebra compatible with an H-module structure on A.

Let V and W be two H-modules. An H-pseudolinear map from V to W is a k-linear map $\phi : V \longrightarrow (H \otimes H) \otimes_H W$ such that

$$\phi(hv) = ((1 \otimes h) \otimes_H 1)\phi(v), \quad \forall\, h \in H, v \in V.$$

We denote the vector space of all such ϕ by $\mathrm{Chom}(V, W)$. There is a left action of H on $\mathrm{Chom}(V, W)$ defined by

$$(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\phi(v).$$

In the special case $V = W$, we write $\text{Cend}(V) = \text{Chom}(V, V)$. Throughout the paper, unless otherwise specified, we always set $V^* = \text{Chom}(V, k)$.

Let L be a finite free H-module with a basis $\{a_i\}_{i=1}^n$. The dual basis to $\{a_i\}_{i=1}^n$ in $L^* = \text{Chom}(L, k)$ is defined as the set $\{a^j\}_{j=1}^n$, where each $a^j \in L^*$ is given by

$$a^i * a_j = (1 \otimes 1) \otimes_H \delta_{ij}.$$

It is easy to check that $\{a^j\}_{j=1}^n$ is a linearly independent set such that H-generates L^*. Under this assumption, we have

Theorem 3.3 (1) Let $(A = \bigoplus_{i=1}^N Ha_i, *, \alpha)$ be a finite free Hom-associative H-pseudoalgebra with the pseudoproduct

$$a_i * a_j = \sum_{k=1}^N (f_k^{ij} \otimes g_k^{ij}) \otimes_H a_k.$$

Let $A^* = \text{Chom}(A, k) = \bigoplus_{i=1}^N Ha^i$ be the dual of A, where $\{a^i\}$ is the dual basis corresponding to $\{a_i\}$. Let $\beta : A^* \longrightarrow A^*$ be the H-linear map dual to α. Define $\Delta : A^* \longrightarrow A^* \otimes A^*$ by

$$\Delta(a^k) = \sum_{i,j} S(f_k^{ij})a^i \otimes S(g_k^{ij})a^j$$

and extend it H-linearly, i.e., $\Delta(ha^k) = h\Delta(a^k)$. Then (A^*, Δ, β) is a Hom-coassociative H-coalgebra.

(2) Conversely, let (A, Δ, β) be a finite Hom-coassociative H-coalgebra. Then the left H-module $(A^* = \text{Chom}(A, k), \cdot_x\cdot, \alpha)$ is a Hom-associative H-conformal algebra with the x-product defined by

$$(f_x g)_y(a) = f_{x_2}(a_1) g_{yS(x_1)}(a_2),$$

for all $f, g \in A^*, a \in A$ and $x, y \in X$.

Proof. It is quite similar to the proof of Theorem 4.5 in [4]. □

Definition 3.4 An infinitesimal Hom-H-pseudobialgebra is a quadruple $(A, *, \Delta, \alpha)$, where $(A, *, \alpha)$ is a Hom-associative H-pseudoalgebra and (A, Δ, α) is a Hom-coassociative H-coalgebra, satisfying the compatible condition

$$\Delta(a * b) = a * \Delta(b) + \Delta(a) * b, \quad \forall\, a, b \in A, \tag{3. 1}$$

where

$$a * \Delta(b) = \sum_i f_i \otimes g_i \otimes_H (e_i \otimes \alpha(b_2)),$$
$$\Delta(a) * b = \sum_j m_j \otimes n_j \otimes_H (\alpha(a_1) \otimes t_j),$$

if $\alpha(a) * b_1 = \sum_i f_i \otimes g_i \otimes_H e_i$ and $a_2 * \alpha(b) = \sum_j m_j \otimes n_j \otimes_H t_j$.

Let $(A, \mu_A, \Delta_A, \alpha_A)$ and $(B, \mu_B, \Delta_B, \alpha_B)$ be two infinitesimal Hom-H-pseudoalgebras. An H-linear map $f : A \longrightarrow B$ is called the morphism of infinitesimal Hom-H-pseudobialgebras if f satisfies $f \circ \mu_A = \mu_B \circ (f \otimes f), \Delta_B \circ f = (f \otimes f) \circ \Delta_A$ and $\alpha_B \circ f = f \circ \alpha_A$.

Remark 3.5 (1) In particular, for the one dimensional Hopf algebra $H = k$, an infinitesimal Hom-H-pseudobialgebra is just an ordinary infinitesimal Hom-bialgebra ([27]). If $\alpha = id$, then an infinitesimal Hom-H-pseudobialgebra is actually an infinitesimal H-pseudobialgebra ([13]).

(2) The compatible condition (3.1) is indeed the condition that Δ is a 1-cocycle of the Hom-associative H-pseudoalgebra $(A, *, \alpha)$ with coefficients in $A \otimes A$ in the reduced complex ([19, 20]). So infinitesimal

Hom-H-pseudobialgebra is a Hom-associative analog of Hom-Lie H-pseudobialgebra.

Now we construct an infinitesimal Hom-H-pseudobialgebra from a given infinitesimal Hom-H-pseudobialgebra.

Proposition 3.6 Let H' be a Hopf subalgebra of H and $(A, *, \Delta, \alpha)$ be an infinitesimal Hom-H'-pseudobialgebra. Then $(Cur(A) = H \otimes_{H'} A, \tilde{*}, \delta, \beta = id \otimes_{H'} \alpha)$ is an infinitesimal Hom-H-pseudobialgebra with the following structures ($\forall\, f, g \in H, a, b \in A$):

$$(f \otimes_{H'} a) \tilde{*} (g \otimes_{H'} b) = \sum_i (f f_i \otimes g g_i) \otimes_H (1 \otimes_{H'} e_i),$$
$$\delta(f \otimes_{H'} a) = (f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2),$$

where $a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$.

Proof. By Proposition 2.16 in [19], $(Cur(A) = H \otimes_{H'} A, \tilde{*}, \beta)$ is a Hom-associative H-pseudoalgebra. For all $a \in A$ and $f \in H$, we have

$$\delta\beta(f \otimes_{H'} a) = \delta(f \otimes_{H'} \alpha(a)) = (f_1 \otimes_{H'} \alpha(a_1)) \otimes (f_2 \otimes_{H'} \alpha(a_2)) = (\beta \otimes \beta)\delta(f \otimes_{H'} a).$$

Since (A, Δ, α) is a Hom-coassociative H'-coalgebra, we get $(\alpha \otimes \Delta)\Delta(a) = (\Delta \otimes \alpha)\Delta(a)$, i.e., $\alpha(a_1) \otimes a_{21} \otimes a_{22} = a_{11} \otimes a_{12} \otimes \alpha(a_2)$. Using the above equation, we have

$$\begin{aligned}
(\beta \otimes \delta)\delta(f \otimes_{H'} a) &= (\beta \otimes \delta)((f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2)) \\
&= (f_1 \otimes_{H'} \alpha(a_1)) \otimes (f_2 \otimes_{H'} a_{21}) \otimes (f_3 \otimes_{H'} a_{22}) \\
&= (f_1 \otimes_{H'} a_{11}) \otimes (f_2 \otimes_{H'} a_{12}) \otimes (f_3 \otimes_{H'} \alpha(a_2)) \\
&= \delta(f_1 \otimes_{H'} a_1) \otimes \beta(f_2 \otimes_{H'} a_2) \\
&= (\delta \otimes \beta)\delta(f \otimes_{H'} a).
\end{aligned}$$

It follows that $(Cur(A), \delta, \beta)$ is a Hom-coassociative H-coalgebra. Finally, we verify the compatible condition (3.1). Suppose

$$a * b = \sum_i f_i \otimes g_i \otimes_{H'} e_i,\ \alpha(a) * b_1 = \sum_i p_i \otimes q_i \otimes_{H'} t_i,\ a_2 * \alpha(b) = \sum_i m_i \otimes n_i \otimes_{H'} r_i,$$

then we have

$$\Delta(a * b) = \Delta(a) * b + a * \Delta(b),$$

which is equivalent to

$$\sum_i f_i \otimes g_i \otimes_{H'} (e_{i1} \otimes e_{i2}) = \sum_i p_i \otimes q_i \otimes_{H'} (t_i \otimes \alpha(b_2)) + \sum_i m_i \otimes n_i \otimes_{H'} (\alpha(a_1) \otimes r_i).$$

Using the above equation, we get

$$\begin{aligned}
&(f \otimes_{H'} a) \tilde{*} \delta(g \otimes_{H'} b) + \delta(f \otimes_{H'} a) \tilde{*} (g \otimes_{H'} b) \\
=\ & \beta(f \otimes_{H'} a) \tilde{*} (g_1 \otimes_{H'} b_1) \otimes \beta(g_2 \otimes_{H'} b_2) + \beta(f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2) \tilde{*} \beta(g \otimes_{H'} b) \\
=\ & \sum_i (f p_i \otimes g_1 q_i) \otimes_H ((1 \otimes_{H'} t_i) \otimes (g_2 \otimes_{H'} \alpha(b_2))) + \sum_i (f_2 m_i \otimes g n_i) \otimes_H ((f_1 \otimes_{H'} \alpha(a_1)) \otimes (1 \otimes_{H'} r_i)) \\
=\ & \sum_i (f p_i \otimes g q_i) \otimes_H ((1 \otimes_{H'} t_i) \otimes (1 \otimes_{H'} \alpha(b_2))) + \sum_i (f m_i \otimes g n_i) \otimes_H ((1 \otimes_{H'} \alpha(a_1)) \otimes (1 \otimes_{H'} r_i)) \\
=\ & \sum_i (f f_i \otimes g g_i) \otimes_H ((1 \otimes_{H'} e_{i1}) \otimes (1 \otimes_{H'} e_{i2})) \\
=\ & \delta((f \otimes_{H'} a) \tilde{*} (g \otimes_{H'} b)).
\end{aligned}$$

This completes the proof. □

Remark 3.7 More generally, let $\phi : H' \longrightarrow H$ be a homomorphism of Hopf algebras, and $(A, *, \Delta, \alpha)$ an infinitesimal Hom-H'-pseudobialgebra. Then $(H \otimes_{H'} A, \tilde{*}, \delta, \beta = id \otimes_{H'} \alpha)$ is an infinitesimal Hom-H-pseudobialgebra with the strucutres:

$$(f \otimes_{H'} a)\tilde{*}(g \otimes_{H'} b) = \sum_i (f\phi(f_i) \otimes g\phi(g_i)) \otimes_H (1 \otimes_{H'} e_i),$$
$$\delta(f \otimes_{H'} a) = (f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2),$$

where $a * b = \sum_i f_i \otimes g_i \otimes_H e_i$ for all $a, b \in A$.

Corollary 3.8 Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra. Then $(Cur(A) = H \otimes A, \tilde{*}, \delta, id \otimes \alpha)$ is an infinitesimal Hom-H-pseudobialgebra with the following structures:

$$(f \otimes a)\tilde{*}(g \otimes b) = \sum_i (f \otimes g) \otimes_H (1 \otimes ab),$$
$$\delta(f \otimes a) = (f_1 \otimes a_1) \otimes (f_2 \otimes a_2),$$

for all $f, g \in H$ and $a, b \in A$.

Proof. It can be obtained directly by taking $H' = k$ in Proposition 3.6. □

We now show that how to construct an infinitesimal Hom-H-pseudobialgebra from an infinitesimal H-pseudobialgebra.

Theorem 3.9 Let $(A, \mu = *, \Delta)$ be an infinitesimal H-pseudobialgebra and $\alpha : A \longrightarrow A$ be a morphism of infinitesimal H-pseudobialgebras. Then $A_\alpha = (A, \mu_\alpha = *_\alpha, \Delta_\alpha, \alpha)$ is an infinitesimal Hom-H-pseudobialgebra, where $\mu_\alpha = (id \otimes_H \alpha)\mu$ and $\Delta_\alpha = \Delta \circ \alpha$.

Proof. By Theorem 2.9 in [19], (A, μ_α, α) is a Hom-associative H-pseudoalgebra. For all $h \in H, a \in A$, we have

$$\Delta_\alpha(ha) = \Delta \circ \alpha(ha) = \Delta(h\alpha(a)) = h\Delta_\alpha(a),$$

$$\Delta_\alpha \circ \alpha(a) = \alpha^2(a_1) \otimes \alpha^2(a_2) = (\alpha \otimes \alpha)(\alpha(a_1) \otimes \alpha(a_2)) = (\alpha \otimes \alpha)\Delta_\alpha(a)$$

and

$$\begin{aligned}(\alpha \otimes \Delta_\alpha)\Delta_\alpha(a) &= (\alpha \otimes \Delta_\alpha)(\alpha(a_1) \otimes \alpha(a_2)) \\ &= \alpha^2(a_1) \otimes \alpha^2(a_{21}) \otimes \alpha^2(a_{22}) \\ &= \alpha^2(a_{11}) \otimes \alpha^2(a_{12}) \otimes \alpha^2(a_2) \\ &= \alpha^2(a_1)_1 \otimes \alpha^2(a_1)_2 \otimes \alpha^2(a_2) \\ &= (\Delta_\alpha \otimes \alpha)\Delta_\alpha(a).\end{aligned}$$

Hence $(A, \Delta_\alpha, \alpha)$ is a Hom-coassociative H-coalgebra. It remains to check the compatible condition (3.1) in A_α. Suppose

$$a * b = \sum_i p_i \otimes q_i \otimes_H w_i, \quad a * b_1 = \sum_i f_i \otimes g_i \otimes_H e_i, \quad a_2 * b = \sum_i m_i \otimes n_i \otimes_H t_i.$$

Since $(A, *, \Delta)$ is an infinitesimal H-pseudobialgebra, we have $\Delta(a * b) = a * \Delta(b) + \Delta(a) * b$, which is equivalent to

$$\sum_i p_i \otimes q_i \otimes_H (w_{i1} \otimes w_{i2}) = \sum_i f_i \otimes g_i \otimes_H (e_i \otimes b_2) + \sum_i m_i \otimes n_i \otimes_H (a_1 \otimes t_i).$$

According to the above equation and the fact that $a *_\alpha b = (id \otimes_H \alpha)(a * b) = \alpha(a) * \alpha(b)$, we have

$$\begin{aligned}
& a *_\alpha \Delta_\alpha(b) + \Delta_\alpha(a) *_\alpha b \\
= \; & \alpha(a) *_\alpha \alpha(b_1) \otimes \alpha^2(b_2) + \alpha^2(a_1) \otimes \alpha(a_2) *_\alpha \alpha(b) \\
= \; & (id \otimes_H \alpha^2)(a * b_1) \otimes \alpha^2(b_2) + \alpha^2(a_1) \otimes (id \otimes_H \alpha^2)(a_2 * b) \\
= \; & \sum_i f_i \otimes g_i \otimes_H (\alpha^2(e_i) \otimes \alpha^2(b_2)) + \sum_i m_i \otimes n_i \otimes_H (\alpha^2(a_1) \otimes \alpha^2(t_i)) \\
= \; & \sum_i p_i \otimes q_i \otimes_H (\alpha^2(w_{i1}) \otimes \alpha^2(w_{i2})) \\
= \; & \Delta_\alpha(\sum_i p_i \otimes q_i \otimes_H \alpha(w_i)) \\
= \; & \Delta_\alpha(a *_\alpha b),
\end{aligned}$$

which completes the proof. □

More generally, we have

Theorem 3.10 Let $(A, \mu = *, \delta, \alpha)$ be an infinitesimal Hom-H-pseudobialgebra. Then so is $A^n = (A, \mu_n = *_n, \Delta^n, \alpha^{2^n})$ for each $n \geq 0$, where $\mu_n = (id \otimes_H \alpha^{2^n-1}) \circ \mu$ and $\Delta^n = \Delta \circ \alpha^{2^n-1}$.

Proof. First note that $A^0 = A, A^1 = (A, \mu_1 = (id \otimes_H \alpha) \circ \mu, \Delta^1 = \Delta \circ \alpha, \alpha^2)$, and $A^{n+1} = (A^n)^1$. Therefore, by an induction argument, it suffices to prove the case $n = 1$, i.e., that A^1 is an infinitesimal Hom-H-pseudobialgebra. One can check directly that (A, μ_1, α^2) is a Hom-associative H-pseudoalgebra and that (A, Δ^1, α^2) is a Hom-coassociative H-coalgebra. It remains to establish the compatible condition (3.1) in A^1. We denote $a * b = h^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$ for all $a, b \in A$, where we avoid the sum that is implicity understood. Then (3.1) can be rewritten as

$$\begin{aligned}
h^{a,b} \otimes g^{a,b} \otimes_H (e_{a,b})_1 \otimes (e_{a,b})_2 \;=\; & h^{a,\alpha(b_1)} \otimes g^{a,\alpha(b_1)} \otimes_H e_{a,\alpha(b_1)} \otimes \alpha(b_2) \\
& + h^{a_2,\alpha(b)} \otimes g^{a_2,\alpha(b)} \otimes_H \alpha(a_1) \otimes e_{a_2,\alpha(b)}.
\end{aligned}$$

Using the above equation, we have

$$\begin{aligned}
& a *_1 \Delta^1(b) + \Delta^1(a) *_1 b \\
= \; & \alpha^2(a) *_1 \alpha(b_1) \otimes \alpha^3(b_2) + (\alpha(a_1) \otimes \alpha(a_2)) *_1 b \\
= \; & h^{\alpha^2(a),\alpha(b_1)} \otimes g^{\alpha^2(a),\alpha(b_1)} \otimes_H \alpha(e_{\alpha^2(a),\alpha(b_1)}) \otimes \alpha^3(b_2) \\
& + h^{\alpha(a_2),\alpha^2(b)} \otimes g^{\alpha(a_2),\alpha^2(b)} \otimes \alpha^3(a_1) \otimes \alpha(e_{\alpha(a_2),\alpha^2(b)}) \\
= \; & h^{\alpha(a),b_1} \otimes g^{\alpha(a),b_1} \otimes_H \alpha^2(e_{\alpha(a),b_1}) \otimes \alpha^3(b_2) \\
& + h^{a_2,\alpha(b)} \otimes g^{a_2,\alpha(b)} \otimes \alpha^3(\alpha(a_1)) \otimes \alpha^2(e_{a_2,\alpha(b)}) \\
= \; & h^{\alpha(a),b_1} \otimes g^{\alpha(a),b_1} \otimes_H \alpha^2(e_{\alpha(a),b_1} \otimes \alpha(b_2)) \\
& + h^{a_2,\alpha(b)} \otimes g^{a_2,\alpha(b)} \otimes \alpha^2(\alpha(a_1) \otimes e_{a_2,\alpha(b)}) \\
= \; & h^{a,b} \otimes g^{a,b} \otimes_H \alpha^2((e_{a,b})_1 \otimes (e_{a,b})_2) \\
= \; & \Delta^1(a *_1 b),
\end{aligned}$$

as desired. □

Example 3.11 Let H be a commutative Hopf algebra and $G(H)$ the set of all grouplike elements of H. Suppose that $H\{e_1, e_2\}$ is a free associative H-pseudoalgebra with pseudoproduct given by $e_1 * e_1 = a \otimes_H e_2, e_1 * e_2 = e_2 * e_1 = e_2 * e_2 = 0, \forall\, a \in H \otimes H$. Define $\Delta : A \longrightarrow A \otimes A$ as follows:

$$\Delta(e_1) = e_1 \otimes e_1, \;\; \Delta(e_2) = e_2 \otimes e_2,$$

and extend it H-linearly, i.e., $\Delta(he_i) = h\Delta(e_i)$ for $i = 1, 2$. Then $(A, *, \Delta)$ is an infinitesimal H-pseudobialgebra. We define $\alpha : A \longrightarrow A$ by

$$\alpha(e_1) = ge_1, \alpha(e_2) = ge_2 \text{ with } g \in G(H).$$

Then α is an endomorphism of infinitesimal H-pseudobialgebra. By Theorem 3.9, $(H\{e_1, e_2\}, \mu_\alpha, \Delta_\alpha, \alpha)$ is an infinitesimal Hom-H-pseudobialgebra with

$$\begin{aligned}
&\mu_\alpha(e_1 \otimes e_1) = a \otimes_H ge_2, \quad \mu_\alpha(e_1 \otimes e_2) = \mu_\alpha(e_2 \otimes e_1) = \mu_\alpha(e_2 \otimes e_2) = 0,\\
&\Delta_\alpha(e_1) = \Delta(ge_1) = ge_1 \otimes ge_1, \quad \Delta_\alpha(e_2) = \Delta(ge_2) = ge_2 \otimes ge_2.
\end{aligned}$$

Moreover, we can obtain a class of infinitesimal Hom-H-pseudobialgebras by Theorem 3.10.

4. Coboundary infinitesimal Hom-H-pseudobialgebras

Suppose that (A, α) is a Hom-associative H-pseudoalgebra and M is a left Hom-A-module. By Lemma 2.3 in [3], for all $a \in A$ and $m \in M$, $a * m$ can be written uniquely in the form $\sum_i (h_i \otimes 1) \otimes_H c_i$, where $\{h_i\}$ is a fixed k-basis of H. In the following, we always write

$$a * m = h^{a,m} \otimes 1 \otimes_H c_{a,m},$$

for convenience. Similarly, for a right Hom-A-module N, we set

$$n * a = 1 \otimes l^{n,a} \otimes_H e_{n,a},$$

for all $a \in A$ and $n \in N$.

Proposition 4.1 Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra and $r = \sum_i u_i \otimes v_i \in A \otimes A$ α-invariant (i.e., $(\alpha^{\otimes 2})r = r$). Define the map $\Delta_r : A \longrightarrow A \otimes A$ by

$$\Delta_r(a) = h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a}), \tag{4. 1}$$

where $a * u_i = h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i}$, $v_i * a = 1 \otimes g^{v_i,a} \otimes_H m_{v_i,a}$. Then Δ_r satisfies the compatible condition (3.1).

Proof. For all $a, b \in A$, suppose $a * b = h^{a,b} \otimes 1 \otimes_H c_{a,b}$ for convenience. By Proposition 2.5, $(A \otimes A, \alpha \otimes \alpha)$ is a Hom-A-bimodule, so we have

$$(a * b) * (\alpha(u_i) \otimes \alpha(v_i)) = \alpha(a) * (b * (u_i \otimes v_i)), \quad ((u_i \otimes v_i) * a) * \alpha(b) = (\alpha(u_i) \otimes \alpha(v_i)) * (a * b),$$

which are equivalent to

$$\begin{aligned}
& h^{\alpha(a),\alpha(b)} h_1^{c_{\alpha(a),\alpha(b)},\alpha(u_i)} \otimes h_2^{c_{\alpha(a),\alpha(b)},\alpha(u_i)} \otimes 1 \otimes_H (c_{c_{\alpha(a),\alpha(b)},\alpha(u_i)} \otimes \alpha^2(v_i))\\
= \;& h^{\alpha^2(a),c_{\alpha(b),u_i}} \otimes h^{\alpha(b),u_i} \otimes 1 \otimes_H (c_{\alpha^2(a),c_{\alpha(b),u_i}} \otimes \alpha^2(v_i))
\end{aligned}$$

and

$$\begin{aligned}
& 1 \otimes g^{v_i,\alpha(a)} \otimes g^{m_{v_i,\alpha(a)},\alpha^2(b)} \otimes_H (\alpha^2(u_i) \otimes m_{m_{v_i,\alpha(a)},\alpha^2(b)})\\
= \;& 1 \otimes h^{a,b} g_1^{\alpha(v_i),\alpha(c_{a,b})} \otimes g_2^{\alpha(v_i),\alpha(c_{a,b})} \otimes_H (\alpha^2(u_i) \otimes m_{\alpha(v_i),\alpha(c_{a,b})}),
\end{aligned}$$

respectively. Combining with the above two equations, we have

$$\begin{aligned}
& a * \Delta_r(b) + \Delta_r(a) * b \\
= \ & a * (h^{b,u_i}(c_{b,u_i} \otimes \alpha(v_i))) - a * (g^{v_i,b}(\alpha(u_i) \otimes m_{v_i,b})) + h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) * b \\
& -(g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a})) * b \\
= \ & h^{\alpha(a),c_{b,u_i}} \otimes h^{b,u_i} \otimes_H (c_{\alpha(a),c_{b,u_i}} \otimes \alpha^2(v_i)) - h^{\alpha(a),\alpha(u_i)} \otimes g^{v_i,b} \otimes_H (c_{\alpha(a),\alpha(u_i)} \otimes \alpha(m_{v_i,b})) \\
& +h^{a,u_i} \otimes g^{\alpha(v_i),\alpha(b)} \otimes_H (\alpha(c_{a,u_i}) \otimes m_{\alpha(v_i),\alpha(b)}) - g^{v_i,a} \otimes g^{m_{v_i,a},\alpha(b)} \otimes_H (\alpha^2(u_i) \otimes m_{m_{v_i,a},\alpha(b)}) \\
= \ & h^{\alpha(a),c_{b,u_i}} \otimes h^{b,u_i} \otimes_H (c_{\alpha(a),c_{b,u_i}} \otimes \alpha^2(v_i)) - g^{v_i,a} \otimes g^{m_{v_i,a},\alpha(b)} \otimes_H (\alpha^2(u_i) \otimes m_{m_{v_i,a},\alpha(b)}) \\
= \ & h^{a,b} h_1^{c_{a,b},\alpha(u_i)} \otimes h_2^{c_{a,b},\alpha(u_i)} \otimes_H (c_{c_{a,b},\alpha(u_i)} \otimes \alpha^2(v_i)) \\
& -h^{a,b} g_1^{\alpha(v_i),c_{a,b}} \otimes g_2^{\alpha(v_i),c_{a,b}} \otimes_H (\alpha^2(u_i) \otimes m_{\alpha(v_i),c_{a,b}}) \\
= \ & h^{a,b} h_1^{c_{a,b},u_i} \otimes h_2^{c_{a,b},u_i} \otimes_H (c_{c_{a,b},u_i} \otimes \alpha(v_i)) - h^{a,b} g_1^{v_i,c_{a,b}} \otimes g_2^{v_i,c_{a,b}} \otimes_H (\alpha(u_i) \otimes m_{v_i,c_{a,b}}) \\
= \ & h^{a,b} \otimes 1 \otimes_H h^{c_{a,b},u_i}(c_{c_{a,b},u_i} \otimes \alpha(v_i)) - h^{a,b} \otimes 1 \otimes_H g^{v_i,c_{a,b}}(\alpha(u_i) \otimes m_{v_i,c_{a,b}}) \\
= \ & h^{a,b} \otimes 1 \otimes_H \Delta_r(c_{a,b}) \\
= \ & \Delta_r(a * b).
\end{aligned}$$

This completes the proof. □

With the definition of Δ_r in the above proposition, we have

Definition 4.2 A coboundary infinitesimal Hom-H-pseudobialgebra is a quintuple $(A, *, \Delta_r, \alpha, r)$, where $(A, *, \Delta_r, \alpha)$ is an infinitesimal Hom-H-pseudobialgebra and $r = \sum_i u_i \otimes v_i \in A \otimes A$ is α-invariant.

Remark 4.3 A coboundary infinitesimal H-pseudobialgebra is a coboundary infinitesimal Hom-H-pseudobialgebra with $\alpha = id$.

Theorem 4.4 Let $(A, *, \Delta, r)$ be a coboundary infinitesimal H-pseudobialgebra and $\alpha : A \longrightarrow A$ a morphism of associative H-pseudoalgebras such that $\alpha^{\otimes 2}(r) = r$. Then $A_\alpha = (A, *_\alpha, \Delta_\alpha, \alpha, r)$ is a coboundary infinitesimal Hom-H-pseudobialgebra, where $a *_\alpha b = (id \otimes_H \alpha)(a * b)$ and $\Delta_\alpha = \Delta \circ \alpha$.

Proof. Let $r = \sum_i u_i \otimes v_i$. Using $\alpha^{\otimes 2}(r) = r$ and $(id \otimes_H \alpha)(a * b) = \alpha(a) * \alpha(b)$, we obtain

$$\begin{aligned}
\Delta(\alpha(a)) &= h^{\alpha(a),u_i}(c_{\alpha(a),u_i} \otimes v_i) - g^{v_i,\alpha(a)}(u_i \otimes m_{v_i,\alpha(a)}) \\
&= h^{\alpha(a),\alpha(u_i)}(c_{\alpha(a),\alpha(u_i)} \otimes \alpha(v_i)) - g^{\alpha(v_i),\alpha(a)}(\alpha(u_i) \otimes m_{\alpha(v_i),\alpha(a)}) \\
&= h^{a,u_i}(\alpha(c_{a,u_i}) \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes \alpha(m_{v_i,a})) \\
&= \alpha^{\otimes 2}(\Delta(a)).
\end{aligned}$$

Hence α is a morphism of infinitesimal H-pseudobialgebras. By Theorem 3.9, A_α is an infinitesimal Hom-H-pseudobialgebra. In what follows, we prove that Δ_α satisfies condition (4.1). Note that $a *_\alpha u_i = \alpha(a) * \alpha(u_i) = h^{\alpha(a),\alpha(u_i)} \otimes 1 \otimes_H c_{\alpha(a),\alpha(u_i)}$ and $v_i *_\alpha a = 1 \otimes g^{\alpha(v_i),\alpha(a)} \otimes_H m_{\alpha(v_i),\alpha(a)}$, then we have

$$\Delta_\alpha(a) = \Delta(\alpha(a)) = h^{\alpha(a),\alpha(u_i)}(c_{\alpha(a),\alpha(u_i)} \otimes \alpha(v_i)) - g^{\alpha(v_i),\alpha(a)}(\alpha(u_i) \otimes m_{\alpha(v_i),\alpha(a)}).$$

This completes the proof. □

More generally, we have

Theorem 4.5 Let $(A, \mu = *, \Delta_r, \alpha, r)$ be a coboundary infinitesimal Hom-H-pseudobialgebra. Then so is $A^n = (A, \mu_n = *_n, \Delta_r^n, \alpha^{2^n}, r)$ for each $n \geq 0$, where $\mu_n = \alpha^{2^n - 1} \circ \mu, \Delta_r^n = \Delta_r \circ \alpha^{2^n - 1}$.

Proof. By Theorem 3.10, we already known that A^n is an infinitesimal Hom-H-pseudobialgebra. Moreover, since r is fixed by $\alpha \otimes \alpha$, i.e., $(\alpha \otimes \alpha)r = r$, it is also fixed by $(\alpha^{2^n})^{\otimes 2}$. Thus it remains to show that Δ_r^n has the form of (4.1). Let $r = \sum_i u_i \otimes v_i$, $a * u_i = h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i}$ and $v_i * a = 1 \otimes g^{v_i,a} \otimes_H m_{v_i,a}$, we have

$$\begin{aligned}
\Delta_r^n(a) &= \Delta_r(\alpha^{2^n-1}(a)) \\
&= h^{\alpha^{2^n-1}(a),u_i}(c_{\alpha^{2^n-1}(a),u_i} \otimes \alpha(v_i)) - g^{v_i,\alpha^{2^n-1}(a)}(\alpha(u_i) \otimes m_{v_i,\alpha^{2^n-1}(a)}) \\
&= h^{\alpha^{2^n-1}(a),\alpha^{2^n-1}(u_i)}(c_{\alpha^{2^n-1}(a),\alpha^{2^n-1}(u_i)} \otimes \alpha^{2^n}(v_i)) \\
&\quad -g^{\alpha^{2^n-1}(v_i),\alpha^{2^n-1}(a)}(\alpha^{2^n}(u_i \otimes m_{\alpha^{2^n-1}(v_i),\alpha^{2^n-1}(a)}) \\
&= h^{a,u_i}(\alpha^{2^n-1}(c_{a,u_i}) \otimes \alpha^{2^n}(v_i)) - g^{v_i,a}(\alpha^{2^n}(u_i) \otimes \alpha^{2^n-1}(m_{v_i,a})),
\end{aligned}$$

as desired. □

In Proposition 2.5, we are interest in the case of $n = 3$, that is, $(A^{\otimes 3}, \alpha^{\otimes 3})$ is a Hom-A-bimodule with the following structures ($\forall\ a, x, y, z \in A$):

$$a * (x \otimes y \otimes z) = (h^{\alpha(a),x} \otimes 1) \otimes_H (c_{\alpha(a),x} \otimes \alpha(y) \otimes \alpha(z)),$$
$$(x \otimes y \otimes z) * a = (1 \otimes l^{z,\alpha(a)}) \otimes_H (\alpha(x) \otimes \alpha(y) \otimes e_{z,\alpha(a)}),$$

where $\alpha(a) * x = h^{\alpha(a),x} \otimes 1 \otimes_H c_{\alpha(a),x}$ and $z * \alpha(a) = 1 \otimes l^{z,\alpha(a)} \otimes_H e_{z,\alpha(a)}$. Here we denote

$$a \triangleright (x \otimes y \otimes z) = h^{\alpha(a),x} \otimes c_{\alpha(a),x} \otimes \alpha(y) \otimes \alpha(z),$$
$$(x \otimes y \otimes z) \triangleleft a = l^{z,\alpha(a)} \otimes \alpha(x) \otimes \alpha(y) \otimes e_{z,\alpha(a)}.$$

Under this assumption, we have the main result of this section.

Theorem 4.6 Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra. Suppose that $r = \sum_i u_i \otimes v_i \in A \otimes A$ satisfies $\alpha^{\otimes 2}(r) = r$. Then $(A, *, \Delta_r, \alpha)$ is a coboundary infinitesimal Hom-H-pseudobialgebra if and only if

$$\mu_3(a \triangleright A_\alpha(r) - A_\alpha(r) \triangleleft a) = 0, \tag{4. 2}$$

where $\mu_3(h \otimes a \otimes b \otimes c) = ((\Delta \otimes id)\Delta(h))(a \otimes b \otimes c)$ and

$$\begin{aligned}
A_\alpha(r) &= \mu^4_{-1}([u_i, u_j] \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu_2^{3,4}(\alpha(u_i) \otimes [v_i, u_j] \otimes \alpha(v_j)) \\
&\quad +\mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_j, v_i]),
\end{aligned}$$

where μ^l_{-k} means that the element of H that appears in its argument in the k-th place acts via the antipode on the element of A located in the l-th entry, $\mu_k^{r,s}$ means that the element of H located in the k-th place in its argument acts on the elements of $A \otimes A$ formed by the elements in the r-th and s-th places. For example, $\mu^4_{-1}(h \otimes a \otimes b \otimes c) = a \otimes b \otimes S(h)c$, $\mu_3^{1,4}(a \otimes b \otimes h \otimes c) = h_1 a \otimes b \otimes h_2 c$ for all $h \in H$ and $a, b, c \in A$.

Proof. Suppose that $a * u_i = h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i}$, $v_i * a = 1 \otimes g^{v_i,a} \otimes_H m_{v_i,a}$. Then we have

$$\begin{aligned}
\Delta_r(\alpha(a)) &= h^{\alpha(a),u_i}(c_{\alpha(a),u_i} \otimes \alpha(v_i)) - g^{v_i,\alpha(a)}(\alpha(u_i) \otimes m_{v_i,\alpha(a)}) \\
&= h^{\alpha(a),\alpha(u_i)}(c_{\alpha(a),\alpha(u_i)} \otimes \alpha^2(v_i)) - g^{\alpha(v_i),\alpha(a)}(\alpha^2(u_i) \otimes m_{\alpha(v_i),\alpha(a)}) \\
&= h^{a,u_i}(\alpha(c_{a,u_i}) \otimes \alpha^2(v_i)) - g^{v_i,a}(\alpha^2(u_i) \otimes \alpha(m_{v_i,a})) \\
&= (\alpha \otimes \alpha)(h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a})) \\
&= (\alpha \otimes \alpha)\Delta_r(a).
\end{aligned}$$

By Proposition 4.1, Δ_r satisfies the compatible condition (3.1). So we only need to prove that Δ_r is Hom-coassociative if and only if $\mu_3(a \triangleright A_\alpha(r) - A_\alpha(r) \triangleleft a) = 0$, i.e.,

$$(\Delta_r \otimes \alpha)\Delta_r(a) - (\alpha \otimes \Delta_r)\Delta_r(a) - \mu_3(a \triangleright A_\alpha(r) - A_\alpha(r) \triangleleft a) = 0 \tag{4. 3}$$

holds. In fact, we have

$$
\begin{aligned}
& (\Delta_r \otimes \alpha)\Delta_r(a) - (\alpha \otimes \Delta_r)\Delta_r(a) \\
= \ & (\Delta_r \otimes \alpha)(h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a})) \\
& -(\alpha \otimes \Delta_r)(h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a})) \\
= \ & h_1^{a,u_i}\Delta_r(c_{a,u_i}) \otimes h_2^{a,u_i}\alpha^2(v_i) - g_1^{v_i,a}\Delta_r(\alpha(u_i)) \otimes g_2^{v_i,a}\alpha(m_{v_i,a}) \\
& -h_1^{a,u_i}\alpha(c_{a,u_i}) \otimes h_2^{a,u_i}\Delta_r(\alpha(v_i)) + g_1^{v_i,a}\alpha^2(u_i) \otimes g_2^{v_i,a}\Delta_r(m_{v_i,a}) \\
= \ & h^{a,u_i}(h^{c_{a,u_i},u_j}(c_{c_{a,u_i},u_j} \otimes \alpha(v_j)) \otimes \alpha^2(v_i)) && (4.\ 4) \\
& -h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \otimes m_{v_j,c_{a,u_i}}) \otimes \alpha^2(v_i)) && (4.\ 5) \\
& -g^{v_i,a}(h^{\alpha(u_i),u_j}(c_{\alpha(u_i),u_j} \otimes \alpha(v_j)) \otimes \alpha(m_{v_i,a})) && (4.\ 6) \\
& +g^{v_i,a}(g^{v_j,\alpha(u_i)}(\alpha(u_j) \otimes m_{v_j,\alpha(u_i)}) \otimes \alpha(m_{v_i,a})) && (4.\ 7) \\
& -h^{a,u_i}(\alpha(c_{a,u_i}) \otimes h^{\alpha(v_i),u_j}(c_{\alpha(v_i),u_j} \otimes \alpha(v_j))) && (4.\ 8) \\
& +h^{a,u_i}(\alpha(c_{a,u_i}) \otimes g^{v_j,\alpha(v_i)}(\alpha(u_j) \otimes m_{v_j,\alpha(v_i)})) && (4.\ 9) \\
& +g^{v_i,a}(\alpha^2(u_i) \otimes h^{m_{v_i,a},u_j}(c_{m_{v_i,a},u_j} \otimes \alpha(v_j))) && (4.\ 10) \\
& -g^{v_i,a}(\alpha^2(u_i) \otimes g^{v_j,m_{v_i,a}}(\alpha(u_j) \otimes m_{v_j,m_{v_i,a}})) && (4.\ 11)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mu_3(a \triangleright A_\alpha(r) - A_\alpha(r) \triangleleft a) \\
= \ & \mu_3(a \triangleright (\mu_{-1}^4([u_i, u_j] \otimes \alpha(v_j) \otimes \alpha(v_i)))) && (4.\ 12) \\
& -\mu_3(a \triangleright (\mu_2^{3,4}(\alpha(u_i) \otimes [v_i, u_j] \otimes \alpha(v_j)))) && (4.\ 13) \\
& +\mu_3(a \triangleright (\mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_j, v_i]))) && (4.\ 14) \\
& -\mu_3((\mu_{-1}^4([u_i, u_j] \otimes \alpha(v_j) \otimes \alpha(v_i))) \triangleleft a) && (4.\ 15) \\
& +\mu_3((\mu_2^{3,4}(\alpha(u_i) \otimes [v_i, u_j] \otimes \alpha(v_j))) \triangleleft a) && (4.\ 16) \\
& -\mu_3((\mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_j, v_i])) \triangleleft a). && (4.\ 17)
\end{aligned}
$$

Using the property of Fourier transform, we obtain

$$
\begin{aligned}
[[a, u_i], \alpha(u_j)] &= (\mathcal{F} \otimes id)[\alpha(a), [u_i, u_j]] \\
&= (\mathcal{F} \otimes id)(h^{\alpha(a),c_{u_i,u_j}} \otimes h^{u_i,u_j} \otimes_H c_{\alpha(a),c_{u_i,u_j}}) \\
&= h^{\alpha(a),c_{u_i,u_j}} S(h_1^{u_i,u_j}) \otimes h_2^{u_i,u_j} \otimes_H c_{\alpha(a),c_{u_i,u_j}}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
& h^{a,u_i}(h^{c_{a,u_i},u_j} \cdot (c_{c_{a,u_i},u_j} \otimes \alpha(v_j)) \otimes \alpha^2(v_i)) \\
= \ & \mu_3(\mu_2^{3,4}([[a, u_i], u_j] \otimes \alpha(v_j) \otimes \alpha^2(v_i))) \\
= \ & \mu_3(\mu_2^{3,4}(h^{\alpha(a),c_{u_i,u_j}} S(h_1^{u_i,u_j}) \otimes h_2^{u_i,u_j} \otimes c_{\alpha(a),c_{u_i,u_j}} \otimes \alpha^2(v_j) \otimes \alpha^2(v_i))) \\
= \ & \mu_3(h^{\alpha(a),c_{u_i,u_j}} S(h_1^{u_i,u_j}) \otimes h_2^{u_i,u_j}(c_{\alpha(a),c_{u_i,u_j}} \otimes \alpha^2(v_j)) \otimes \alpha^2(v_i)) \\
= \ & h^{\alpha(a),c_{u_i,u_j}} S(h_1^{u_i,u_j})(h_2^{u_i,u_j}(c_{\alpha(a),c_{u_i,u_j}} \otimes \alpha^2(v_j)) \otimes \alpha^2(v_i)) \\
= \ & h^{\alpha(a),c_{u_i,u_j}}(c_{\alpha(a),c_{u_i,u_j}} \otimes \alpha^2(v_j) \otimes S(h^{u_i,u_j})\alpha^2(v_i)) \\
= \ & \mu_3(a \triangleleft (\mu_{-1}^4([u_i, u_j] \otimes \alpha(v_j) \otimes \alpha(v_i)))).
\end{aligned}
$$

Hence (4.4) − (4.12) = 0. Since

$$
\begin{aligned}
& g^{v_i,a}(h^{\alpha(u_i),u_j}(c_{\alpha(u_i),u_j}\otimes\alpha(v_j))\otimes\alpha(m_{v_i,a}))\\
= \;& g^{v_i,a}h_1^{\alpha(u_i),u_j}(c_{\alpha(u_i),u_j}\otimes\alpha(v_j)\otimes S(h_2^{\alpha(u_i),u_j})\alpha(m_{v_i,a}))\\
= \;& g^{v_i,a}h_1^{\alpha(u_i),\alpha(u_j)}(c_{\alpha(u_i),\alpha(u_j)}\otimes\alpha^2(v_j)\otimes S(h_2^{\alpha(u_i),\alpha(u_j)})\alpha(m_{v_i,a}))\\
= \;& g^{v_i,a}h_1^{u_i,u_j}(\alpha(c_{u_i,u_j})\otimes\alpha^2(v_j)\otimes S(h_2^{u_i,u_j})\alpha(m_{v_i,a}))\\
= \;& g^{\alpha(v_i),\alpha(a)}h_1^{u_i,u_j}(\alpha(c_{u_i,u_j})\otimes\alpha^2(v_j)\otimes S(h_2^{u_i,u_j})m_{\alpha(v_i),\alpha(a)})\\
= \;& \mu_3(g^{\alpha(v_i),\alpha(a)}h_1^{u_i,u_j}\otimes\alpha(c_{u_i,u_j})\otimes\alpha^2(v_j)\otimes S(h_2^{u_i,u_j})m_{\alpha(v_i),\alpha(a)})\\
= \;& \mu_3(\mu_{-1}^4([u_i,u_j]\otimes\alpha(v_j)\otimes\alpha(v_i))\triangleleft a).
\end{aligned}
$$

It follows that (4.6) − (4.15) = 0. Similarly, we have

$$
\begin{aligned}
& g^{v_i,a}(g^{v_j,\alpha(u_i)}(\alpha(u_j)\otimes m_{v_j,\alpha(u_i)})\otimes\alpha(m_{v_i,a}))\\
= \;& g^{v_i,a}g_1^{v_j,\alpha(u_i)}(\alpha(u_j)\otimes m_{v_j,\alpha(u_i)}\otimes S(g_2^{v_j,\alpha(u_i)})\alpha(m_{v_i,a}))\\
= \;& g^{\alpha(v_i),\alpha(a)}g_1^{v_j,\alpha(u_i)}(\alpha(u_j)\otimes m_{v_j,\alpha(u_i)}\otimes S(g_2^{v_j,\alpha(u_i)})m_{\alpha(v_i),\alpha(a)})\\
= \;& g^{\alpha(v_i),\alpha(a)}g_1^{\alpha(v_j),\alpha(u_i)}(\alpha^2(u_j)\otimes m_{\alpha(v_j),\alpha(u_i)}\otimes S(g_2^{\alpha(v_j),\alpha(u_i)})m_{\alpha(v_i),\alpha(a)})\\
= \;& g^{\alpha(v_i),\alpha(a)}g_1^{v_j,u_i}(\alpha^2(u_j)\otimes\alpha(m_{v_j,u_i})\otimes S(g_2^{v_j,u_i})m_{\alpha(v_i),\alpha(a)})\\
= \;& \mu_3((\alpha(u_j)\otimes m_{v_j,u_i}\otimes S(g^{v_j,u_i})\alpha(v_i))\triangleleft a)\\
= \;& \mu_3(\mu_2^{3,4}(\alpha(u_j)\otimes[v_j,u_i]\otimes\alpha(v_i))\triangleleft a)
\end{aligned}
$$

and

$$
\begin{aligned}
& h^{a,u_i}(\alpha(c_{a,u_i})\otimes g^{v_j,\alpha(v_i)}(\alpha(u_j)\otimes m_{v_j,\alpha(v_i)}))\\
= \;& h^{a,u_i}g_1^{v_j,\alpha(v_i)}(S(g_2^{v_j,\alpha(v_i)})\alpha(c_{a,u_i})\otimes\alpha(u_j)\otimes m_{v_j,\alpha(v_i)})\\
= \;& h^{\alpha(a),\alpha(u_i)}g_1^{v_j,\alpha(v_i)}(S(g_2^{v_j,\alpha(v_i)})c_{\alpha(a),\alpha(u_i)}\otimes\alpha(u_j)\otimes m_{v_j,\alpha(v_i)})\\
= \;& \mu_3(h^{\alpha(a),\alpha(u_i)}g_1^{v_j,\alpha(v_i)}\otimes S(g_2^{v_j,\alpha(v_i)})c_{\alpha(a),\alpha(u_i)}\otimes\alpha(u_j)\otimes m_{v_j,\alpha(v_i)})\\
= \;& \mu_3(h^{\alpha(a),\alpha(u_i)}g_1^{\alpha(v_j),\alpha(v_i)}\otimes S(g_2^{\alpha(v_j),\alpha(v_i)})c_{\alpha(a),\alpha(u_i)}\otimes\alpha^2(u_j)\otimes m_{\alpha(v_j),\alpha(v_i)})\\
= \;& \mu_3(h^{\alpha(a),\alpha(u_i)}g_1^{v_j,v_i}\otimes S(g_2^{v_j,v_i})c_{\alpha(a),\alpha(u_i)}\otimes\alpha^2(u_j)\otimes\alpha(m_{v_j,v_i}))\\
= \;& \mu_3(a\triangleright(S(g^{v_j,v_i})\alpha(u_i)\otimes\alpha(u_j)\otimes m_{v_j,v_i}))\\
= \;& \mu_3(a\triangleright\mu_3^{1,4}(\alpha(u_i)\otimes\alpha(u_j)\otimes[v_j,v_i])).
\end{aligned}
$$

Hence (4.7) − (4.16) = 0 and (4.9) − (4.14) = 0. Clearly,

$$
\begin{aligned}
& h^{a,u_i}(\alpha(c_{a,u_i})\otimes h^{\alpha(v_i),u_j}(c_{\alpha(v_i),u_j}\otimes\alpha(v_j)))\\
= \;& h^{\alpha(a),\alpha(u_i)}(c_{\alpha(a),\alpha(u_i)}\otimes h^{\alpha(v_i),\alpha(u_j)}(c_{\alpha(v_i),\alpha(u_j)}\otimes\alpha^2(v_j)))\\
= \;& h^{\alpha(a),\alpha(u_i)}(c_{\alpha(a),\alpha(u_i)}\otimes h^{v_i,u_j}(\alpha(c_{v_i,u_j})\otimes\alpha^2(v_j)))\\
= \;& \mu_3(a\triangleright(\alpha(u_i)\otimes h^{v_i,u_j}(c_{v_i,u_j}\otimes\alpha(v_j))))\\
= \;& \mu_3(a\triangleright\mu_2^{3,4}(\alpha(u_i)\otimes[v_i,u_j]\otimes\alpha(v_j))).
\end{aligned}
$$

So we have (4.8) − (4.13) = 0. Using the Hom-associativity of A, we have $(v_i * a) * \alpha(u_j) = \alpha(v_i) * (a * u_j)$, which is equivalent to

$$
\begin{aligned}
& 1\otimes g^{v_i,a}\otimes S(h_1^{m_{v_i,a},\alpha(u_j)})\otimes_H h_2^{m_{v_i,a},\alpha(u_j)}c_{m_{v_i,a},\alpha(u_j)}\\
= \;& 1\otimes h^{a,u_j}g_1^{\alpha(v_i),c_{a,u_j}}\otimes g_2^{\alpha(v_i),c_{a,u_j}}\otimes_H m_{\alpha(v_i),c_{a,u_j}}.
\end{aligned}
$$

Applying $\varepsilon \otimes id \otimes S \otimes id$ to the above equation, we obtain

$$g^{v_i,a} \otimes h_1^{m_{v_i,a},\alpha(u_j)} \otimes_H h_2^{m_{v_i,a},\alpha(u_j)} c_{m_{v_i,a},\alpha(u_j)} = h^{a,u_j} g_1^{\alpha(v_i),c_{a,u_j}} \otimes S(g_2^{\alpha(v_i),c_{a,u_j}}) \otimes_H m_{\alpha(v_i),c_{a,u_j}}.$$

Thus we get that (4.10) + (4.5) is

$$\begin{aligned}
& g^{v_i,a}(\alpha^2(u_i) \otimes h^{m_{v_i,a},u_j}(c_{m_{v_i,a},u_j} \otimes \alpha(v_j))) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \otimes m_{v_j,c_{a,u_i}}) \otimes \alpha^2(v_i)) \\
= \; & g^{v_i,a}(\alpha^2(u_i) \otimes h_2^{m_{v_i,a},u_j} c_{m_{v_i,a},u_j} \otimes h_1^{m_{v_i,a},u_j} \alpha(v_j)) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \otimes m_{v_j,c_{a,u_i}}) \\
& \otimes \alpha^2(v_i)) \\
= \; & g^{v_i,a}(\alpha^2(u_i) \otimes h_2^{m_{v_i,a},\alpha(u_j)} c_{m_{v_i,a},\alpha(u_j)} \otimes h_1^{m_{v_i,a},\alpha(u_j)} \alpha^2(v_j)) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \\
& \otimes m_{v_j,c_{a,u_i}}) \otimes \alpha^2(v_i)) \\
= \; & h^{a,u_j} g_1^{\alpha(v_i),c_{a,u_j}}(\alpha^2(u_i) \otimes m_{\alpha(v_i),c_{a,u_j}} \otimes S(g_2^{\alpha(v_i),c_{a,u_j}})\alpha^2(v_j)) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \\
& \otimes m_{v_j,c_{a,u_i}}) \otimes \alpha^2(v_i)) \\
= \; & h^{a,u_j}(g^{\alpha(v_i),c_{a,u_j}}(\alpha^2(u_i) \otimes m_{\alpha(v_i),c_{a,u_j}}) \otimes \alpha^2(v_j)) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \otimes m_{v_j,c_{a,u_i}}) \\
& \otimes \alpha^2(v_i)) \\
= \; & h^{a,u_j}(g^{v_i,c_{a,u_j}}(\alpha(u_i) \otimes m_{v_i,c_{a,u_j}}) \otimes \alpha^2(v_j)) - h^{a,u_i}(g^{v_j,c_{a,u_i}}(\alpha(u_j) \otimes m_{v_j,c_{a,u_i}}) \otimes \alpha^2(v_i)) \\
= \; & 0.
\end{aligned}$$

Using the Hom-associativity of A, we have $\alpha(v_j) * (v_i * a) = (v_j * v_i) * \alpha(a)$, which is equivalent to

$$\begin{aligned}
& 1 \otimes g_1^{\alpha(v_j),m_{v_i,a}} \otimes g^{v_i,a} g_2^{\alpha(v_j),m_{v_i,a}} \otimes_H m_{\alpha(v_j),m_{v_i,a}} \\
= \; & 1 \otimes g^{v_j,v_i} \otimes g^{m_{v_j,v_i},\alpha(a)} \otimes_H m_{m_{v_j,v_i},\alpha(a)}.
\end{aligned}$$

Then we get

$$\begin{aligned}
& g^{v_i,a}(\alpha^2(u_i) \otimes g^{v_j,m_{v_i,a}}(\alpha(u_j) \otimes m_{v_j,m_{v_i,a}})) \\
= \; & g^{v_i,a} g_2^{v_j,m_{v_i,a}}(S(g_1^{v_j,m_{v_i,a}})\alpha^2(u_i) \otimes \alpha(u_j) \otimes m_{v_j,m_{v_i,a}}) \\
= \; & g^{v_i,a} g_2^{\alpha(v_j),m_{v_i,a}}(S(g_1^{\alpha(v_j),m_{v_i,a}})\alpha^2(u_i) \otimes \alpha^2(u_j) \otimes m_{\alpha(v_j),m_{v_i,a}}) \\
= \; & g^{m_{v_j,v_i},\alpha(a)}(S(g^{v_j,v_i})\alpha^2(u_i) \otimes \alpha^2(u_j) \otimes m_{m_{v_j,v_i},\alpha(a)}) \\
= \; & \mu_3((S(g^{v_j,v_i})\alpha(u_i) \otimes \alpha(u_j) \otimes m_{v_j,v_i}) \triangleleft a) \\
= \; & \mu_3(\mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_j, v_i]) \triangleleft a).
\end{aligned}$$

So (4.11) − (4.17) = 0. Finally, it is easy to check that we have canceled all the terms of the left-hand side of equation (4.3). This completes the proof. □

As a by-product of Theorem 4.6, we have

Definition 4.7 Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra and $r = \sum_i u_i \otimes v_i \in A \otimes A$. The equation

$$\begin{aligned}
A_\alpha(r) \; = \; & \mu_{-1}^4([u_i, u_j] \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu_2^{3,4}(\alpha(u_i) \otimes [v_i, u_j] \otimes \alpha(v_j)) \\
& + \mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_j, v_i])
\end{aligned} \tag{4. 18}$$

is called *the Hom-associative pseudo-Yang-Baxter equation* (*pseudo*-HAYBE) and r is called a solution of the *pseudo*-HAYBE if $A_\alpha(r) = 0$.

Remark 4.8 The *pseudo*-HAYBE is just associative pseudo-Yang-Baxter equation (*pseudo*-AYBE) in an associative H-pseudoalgebra when $\alpha = id$.

Theorem 4.9 Let $(A, \mu = *)$ be an associative H-pseudoalgebra, $r = \sum_i u_i \otimes v_i \in A \otimes A$ a solution of *pseudo*-AYBE, and $\alpha : A \longrightarrow A$ a morphism of associative H-pseudoalgebra. Then $r^n = (\alpha^n \otimes \alpha^n)(r)$ is a solution of *pseudo*-HAYBE in the Hom-associative H-pseudoalgebra $A_\alpha = (A, \mu_\alpha = *_\alpha = (id_{H^{\otimes 2}} \otimes_H \alpha)\mu, \alpha)$ for each $n \geq 0$.

Proof. By Theorem 2.9 in [19], A_α is a Hom-associative H-pseudoalgebra. Note that $a *_\alpha b = (id \otimes_H \alpha)(a \otimes b) = \alpha(a) \otimes \alpha(b)$ for all $a, b \in A$, and we write $[a, b]_\alpha$ as the Fourier transform of $a *_\alpha b$, then $[a, b]_\alpha = [\alpha(a), \alpha(b)]$. Moreover, for integer $n \geq 0$, $[a, b]_{\alpha^n} = [\alpha^n(a), \alpha^n(b)] = (id \otimes \alpha^n)[a, b]$. Using the above equation, we have

$$\begin{aligned}
& A_\alpha(r^n) = \mu^4_{-1}([\alpha^n(u_i), \alpha^n(u_j)]_\alpha \otimes \alpha^{n+1}(v_j) \otimes \alpha^{n+1}(v_i)) \\
& -\mu^{3,4}_2(\alpha^{n+1}(u_i) \otimes [\alpha^n(v_i), \alpha^n(u_j)]_\alpha \otimes \alpha^{n+1}(v_j)) \\
& +\mu^{1,4}_3(\alpha^{n+1}(u_i) \otimes \alpha^{n+1}(u_j) \otimes [\alpha^n(v_j), \alpha^n(v_i)]_\alpha) \\
= \; & \mu^4_{-1}([u_i, u_j]_{\alpha^{n+1}} \otimes \alpha^{n+1}(v_j) \otimes \alpha^{n+1}(v_i)) \\
& -\mu^{3,4}_2(\alpha^{n+1}(u_i) \otimes [v_i, u_j]_{\alpha^{n+1}} \otimes \alpha^{n+1}(v_j)) \\
& +\mu^{1,4}_3(\alpha^{n+1}(u_i) \otimes \alpha^{n+1}(u_j) \otimes [v_j, v_i]_{\alpha^{n+1}}) \\
= \; & (\alpha^{n+1})^{\otimes 3}(\mu^4_{-1}([u_i, u_j] \otimes v_j \otimes v_i - \mu^{3,4}_2(u_i \otimes [v_i, u_j] \otimes v_j) \\
& +\mu^{1,4}_3(u_i \otimes u_j \otimes [v_j, v_i])) \\
= \; & (\alpha^{n+1})^{\otimes 3} A(r) = 0.
\end{aligned}$$

This completes the proof. □

5. From infinitesimal Hom-H-pseudobialgebras to Hom-Lie H-pseudobialgebras

In this section, we construct (coboundary) infinitesimal Hom-H-pseudobialgebras from (coboundary) Hom-Lie H-pseudobialgebras.

Let $(A, *, \alpha)$ be an infinitesimal Hom-H-pseudoalgebra. By Proposition 2.5, we have the following actions of A on $A \otimes A$:

$$\begin{aligned}
a * (b \otimes c) &= h^{\alpha(a),b} \otimes g^{\alpha(a),b} \otimes_H e_{\alpha(a),b} \otimes \alpha(c), \\
(b \otimes c) * a &= h^{c,\alpha(a)} \otimes g^{c,\alpha(a)} \otimes_H \alpha(b) \otimes e_{c,\alpha(a)},
\end{aligned}$$

if $a * b = h^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$. Here we write $[a, b \otimes c]_\bullet = a * (b \otimes c) - (\sigma \otimes_H id)((b \otimes c) * a)$. Under this assumption, we have

Definition 5.1 Let $(A, *, \Delta, \alpha)$ be an infinitesimal Hom-H-pseudobialgebra. Define the map $\mathcal{B} : A \otimes A \longrightarrow (H \otimes H) \otimes_H (A \otimes A)$, called the balanceator of A, by

$$\mathcal{B}(a, b) = [a, \Delta^{op}(b)]_\bullet + (\sigma \otimes_H \tau)[b, \Delta^{op}(a)]_\bullet,$$

for all $a, b \in A$. More precisely, we have

$$\begin{aligned}
\mathcal{B}(a, b) \;=\; & h^{\alpha(a),b_2} \otimes g^{\alpha(a),b_2} \otimes_H (e_{\alpha(a),b_2} \otimes \alpha(b_1)) - g^{b_1,\alpha(a)} \otimes h^{b_1,\alpha(a)} \otimes_H (\alpha(b_2) \otimes e_{b_1,\alpha(a)}) \\
& +g^{\alpha(b),a_2} \otimes h^{\alpha(b),a_2} \otimes_H (\alpha(a_1) \otimes e_{\alpha(b),a_2}) - h^{a_1,\alpha(b)} \otimes g^{a_1,\alpha(b)} \otimes_H (e_{a_1,\alpha(b)} \otimes \alpha(a_2)),
\end{aligned}$$

if $a * b = h^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$.

The balanceator is said to be symmetric if $\mathcal{B}(a, b) = (\sigma \otimes_H id)\mathcal{B}(b, a)$ for all $a, b \in A$.

Theorem 5.2 Let $(A,*,\Delta,\alpha)$ be an infinitesimal Hom-H-pseudobialgebra. Define $[a*b]=a*b-(\sigma\otimes_H id)(b*a)$ and $\delta=(id-\tau)\Delta$. Then we have

$$\delta([a*b])=[a*\delta(b)]-(\sigma\otimes_H id)[b*\delta(a)]+\mathcal{B}(a,b)-(\sigma\otimes_H id)\mathcal{B}(b,a),$$

for all $a,b\in A$.

Proof. For all $a,b\in A$, we have

$$\begin{aligned}
&[a*\delta(b)]+\mathcal{B}(a,b)\\
=\;&[\alpha(a)*b_1]\otimes\alpha(b_2)+\alpha(b_1)\otimes[\alpha(a)*b_2]-[\alpha(a)*b_2]\otimes\alpha(b_1)-\alpha(b_2)\otimes[\alpha(a)*b_1]\\
=\;&\alpha(a)*b_1\otimes\alpha(b_2)-(\sigma\otimes id)(b_1*\alpha(a)\otimes\alpha(b_2))+\alpha(b_1)\otimes(\alpha(a)*b_2)\\
&-(\sigma\otimes id)(\alpha(b_1)\otimes b_2*\alpha(a))-\alpha(a)*b_2\otimes\alpha(b_1)+(\sigma\otimes id)(b_2*\alpha(a)\otimes\alpha(b_1))\\
&-\alpha(b_2)\otimes\alpha(a)*b_1+(\sigma\otimes id)(\alpha(b_2)\otimes b_1*\alpha(a))+\mathcal{B}(a,b)\\
=\;&h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(e_{\alpha(a),b_1}\otimes\alpha(b_2))-g^{b_1,\alpha(a)}\otimes h^{b_1,\alpha(a)}\otimes_H(e_{b_1,\alpha(a)}\otimes\alpha(b_2))\\
&+h^{\alpha(a),b_2}\otimes g^{\alpha(a),b_2}\otimes_H(\alpha(b_1)\otimes e_{\alpha(a),b_2})-g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(\alpha(b_1)\otimes e_{b_2,\alpha(a)})\\
&-h^{\alpha(a),b_2}\otimes g^{\alpha(a),b_2}\otimes_H(e_{\alpha(a),b_2}\otimes\alpha(b_1))+g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(e_{b_2,\alpha(a)}\otimes\alpha(b_1))\\
&-h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(\alpha(b_2)\otimes e_{\alpha(a),b_1})+g^{b_1,\alpha(a)}\otimes h^{b_1,\alpha(a)}\otimes_H(\alpha(b_2)\otimes e_{b_1,\alpha(a)})\\
=\;&h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(e_{\alpha(a),b_1}\otimes\alpha(b_2))-g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(\alpha(b_1)\otimes e_{b_2,\alpha(a)})\\
&+g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(e_{b_2,\alpha(a)}\otimes\alpha(b_1))-h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(\alpha(b_2)\otimes e_{\alpha(a),b_1})\\
&+(id\otimes_H\tau)([a,\Delta^{op}(b)]_\bullet)-[a,\Delta^{op}(b)]_\bullet+\mathcal{B}(a,b).
\end{aligned}\tag{5. 1}$$

Interchanging the roles of a and b in the above equation, we get

$$\begin{aligned}
&[b*\delta(a)]+\mathcal{B}(b,a)\\
=\;&h^{\alpha(b),a_1}\otimes g^{\alpha(b),a_1}\otimes_H(e_{\alpha(b),a_1}\otimes\alpha(a_2))-g^{a_2,\alpha(b)}\otimes h^{a_2,\alpha(b)}\otimes_H(\alpha(a_1)\otimes e_{a_2,\alpha(b)})\\
&+g^{a_2,\alpha(b)}\otimes h^{a_2,\alpha(b)}\otimes_H(e_{a_2,\alpha(b)}\otimes\alpha(a_1))-h^{\alpha(b),a_1}\otimes g^{\alpha(b),a_1}\otimes_H(\alpha(a_2)\otimes e_{\alpha(b),a_1})\\
&+(id\otimes_H\tau)([b,\Delta^{op}(a)]_\bullet)-[b,\Delta^{op}(a)]_\bullet+\mathcal{B}(b,a).
\end{aligned}\tag{5. 2}$$

Using (5.1) and (5.2), we compute

$$\begin{aligned}
&[a*\delta(b)]-(\sigma\otimes_H id)[b*\delta(a)]+\mathcal{B}(a,b)-(\sigma\otimes_H id)\mathcal{B}(b,a)\\
=\;&h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(e_{\alpha(a),b_1}\otimes\alpha(b_2))-g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(\alpha(b_1)\otimes e_{b_2,\alpha(a)})\\
&+g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(e_{b_2,\alpha(a)}\otimes\alpha(b_1))-h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(\alpha(b_2)\otimes e_{\alpha(a),b_1})\\
&+(id\otimes_H\tau)([a,\Delta^{op}(b)]_\bullet)-[a,\Delta^{op}(b)]_\bullet+\mathcal{B}(a,b)\\
&-g^{\alpha(b),a_1}\otimes h^{\alpha(b),a_1}\otimes_H(e_{\alpha(b),a_1}\otimes\alpha(a_2))+h^{a_2,\alpha(b)}\otimes g^{a_2,\alpha(b)}\otimes_H(\alpha(a_1)\otimes e_{a_2,\alpha(b)})\\
&-h^{a_2,\alpha(b)}\otimes g^{a_2,\alpha(b)}\otimes_H(e_{a_2,\alpha(b)}\otimes\alpha(a_1))+g^{\alpha(b),a_1}\otimes h^{\alpha(b),a_1}\otimes_H(\alpha(a_2)\otimes e_{\alpha(b),a_1})\\
&-(\sigma\otimes_H\tau)([b,\Delta^{op}(a)]_\bullet)+(\sigma\otimes_H id)[b,\Delta^{op}(a)]_\bullet-(\sigma\otimes_H id)\mathcal{B}(b,a)\\
=\;&h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(e_{\alpha(a),b_1}\otimes\alpha(b_2))+h^{a_2,\alpha(b)}\otimes g^{a_2,\alpha(b)}\otimes_H(\alpha(a_1)\otimes e_{a_2,\alpha(b)})\\
&-h^{\alpha(a),b_1}\otimes g^{\alpha(a),b_1}\otimes_H(\alpha(b_2)\otimes e_{\alpha(a),b_1})-h^{a_2,\alpha(b)}\otimes g^{a_2,\alpha(b)}\otimes_H(e_{a_2,\alpha(b)}\otimes\alpha(a_1))\\
&-g^{\alpha(b),a_1}\otimes h^{\alpha(b),a_1}\otimes_H(e_{\alpha(b),a_1}\otimes\alpha(a_2))-g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(\alpha(b_1)\otimes e_{b_2,\alpha(a)})\\
&+g^{\alpha(b),a_1}\otimes h^{\alpha(b),a_1}\otimes_H(\alpha(a_2)\otimes e_{\alpha(b),a_1})+g^{b_2,\alpha(a)}\otimes h^{b_2,\alpha(a)}\otimes_H(e_{b_2,\alpha(a)}\otimes\alpha(b_1))\\
=\;&(id-\tau)\Delta(a*b)-(id-\tau)\Delta((\sigma\otimes id)(b*a))\\
=\;&(id-\tau)\Delta(a*b-(\sigma\otimes id)(b*a))\\
=\;&\delta([a*b]).
\end{aligned}$$

This completes the proof. □

The following result can be obtained directly by Theorem 5.2.

Corollary 5.3 Let $(A, *, \Delta, \alpha)$ be an infinitesimal Hom-H-pseudobialgebra. $[*]$ and δ are defined as in Theorem 5.2. Then $(A, [*], \delta, \alpha)$ is a Hom-Lie H-pseudobialgebra if and only if the balanceator $\mathcal{B}$ is symmetric.

Next we consider the infinitesimal Hom-H-pseudoalgebra whose balanceator is symmetric.

Theorem 5.4 Let $(A, \mu = *, \Delta)$ be an infinitesimal H-pseudobialgebra with the balanceator $\mathcal{B}$ and $\alpha : A \longrightarrow A$ an endomorphism of infinitesimal H-pseudoalgebras. Then the balanceator $\mathcal{B}_\alpha$ of the infinitesimal Hom-H-pseudoalgebra $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \Delta_\alpha = \Delta \circ \alpha, \alpha)$ is given by $\mathcal{B}_\alpha = (\alpha^2)^{\otimes 2} \circ \mathcal{B}$. In particular, if $\mathcal{B}$ is symmetric, then so is $\mathcal{B}_\alpha$. Conversely, if $\mathcal{B}_\alpha$ is symmetric and $\alpha^{\otimes 2}$ is injective, then $\mathcal{B}$ is symmetric.

Proof. For all $x, y \in A$, we have $\Delta_\alpha(y) = \alpha(y_1) \otimes \alpha(y_2)$ and

$$\begin{aligned}[x, \Delta_\alpha^{op}(y)]_\bullet &= \mu_\alpha(\alpha(x) \otimes \alpha(y)) \otimes \alpha^2(y_1) - (\sigma \otimes_H id)(\alpha^2(y_2) \otimes \mu_\alpha(\alpha(y_1) \otimes \alpha(a))) \\ &= \alpha^2(x) * \alpha^2(y_2) \otimes \alpha^2(y_1) - (\sigma \otimes_H id)(\alpha^2(y_2) \otimes \alpha^2(y_1) * \alpha^2(a)) \\ &= (id \otimes_H (\alpha^2)^{\otimes 2})[x, \Delta^{op}(y)].\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{B}_\alpha(x, y) &= [x, \Delta_\alpha^{op}(y)]_\bullet + (\sigma \otimes_H \sigma)[y, \Delta_\alpha^{op}(x)]_\bullet \\ &= (id \otimes_H \alpha^2 \otimes \alpha^2)([x, \Delta^{op}(y)] + (\sigma \otimes_H \sigma)[y, \Delta^{op}(x)]) \\ &= (id \otimes_H (\alpha^2)^{\otimes 2})\mathcal{B}(x, y),\end{aligned}$$

as desired. The last two assertions of the theorem follow immediately from the first one, which completes the proof. □

Example 5.5 Let $(A, *, \Delta)$ be an infinitesimal H-pseudobialgebra that is both commutative and cocommutative. Then its balanceator $\mathcal{B} = 0$ ([13]). If α is a morphism of A, then it follows from Theorem 5.4 that $\mathcal{B}_\alpha = 0$, which is trivially symmetric. In this case, applying Corollary 5.3 to infinitesimal Hom-H-pseudobialgebra A_α yields a Hom-Lie H-pseudobialgebra.

Theorem 5.6 Let $(A, \mu = *, \Delta, \alpha)$ be an infinitesimal Hom-H-pseudobialgebra with the balanceator $\mathcal{B}$. Then for each $n \geq 1$, the balanceator $\mathcal{B}^n$ of A^n is given by

$$\mathcal{B}^n = (id \otimes_H (\alpha^{2(2^n-1)})^{\otimes 2}) \circ \mathcal{B},$$

where $A^n = (A, \mu^{(n)} = \alpha^{2^n-1} \circ \mu, \Delta^{(n)} = \Delta \circ \alpha^{2^n-1}, \alpha^{2^n})$ is the infinitesimal Hom-H-pseudobialgebra in Theorem 3.10. In particular, if $\mathcal{B}$ is symmetric, then so is $\mathcal{B}^n$. Conversely, if $\mathcal{B}^n$ is symmetric for some n and $\alpha^{\otimes 2}$ is injective, then $\mathcal{B}$ is symmetric.

Proof. Consider the case $n = 1$. In $A^1 = (A, \mu^{(1)} = \alpha \circ \mu, \Delta^{(1)} = \Delta \circ \alpha, \alpha^2)$, we have

$$\begin{aligned}&[x, \Delta^{(1)op}(y)]_\bullet \\ =\ & \mu^{(1)}(x \otimes (\alpha(y_2) \otimes \alpha(y_1))) - (\sigma \otimes id)\mu^{(1)}((\alpha(y_2) \otimes \alpha(y_1)) \otimes x) \\ =\ & \mu^{(1)}(\alpha^2(x) \otimes \alpha(y_2)) \otimes \alpha^3(y_1) - (\sigma \otimes id)(\alpha^3(y_2) \otimes \mu^{(1)}(\alpha(y_1) \otimes \alpha^2(x))) \\ =\ & (id \otimes_H (\alpha^2)^{\otimes 2})(\mu(\alpha(x) \otimes y_2) \otimes \alpha(y_1) - (\sigma \otimes id)(\alpha(y_2) \otimes \mu(y_1 \otimes \alpha(x)))).\end{aligned}$$

Therefore, in A^1 we have

$$\begin{aligned}\mathcal{B}^1(x, y) &= [x, \Delta^{(1)op}(y)]_\bullet + (\sigma \otimes \tau)[y, \Delta^{(1)op}(x)]_\bullet \\ &= (id \otimes_H (\alpha^2)^{\otimes 2})\mathcal{B}(x, y),\end{aligned}$$

as desired. For the general case, we proceed inductively, noting that $A^n = (A^{n-1})^1$ and using the case just proved repeatedly, we have

$$\begin{aligned}\mathcal{B}^n &= id \otimes_H ((\alpha^{2^{n-1}})^2)^{\otimes 2} \circ \mathcal{B}^{n-1} \\ &= (id \otimes_H (\alpha^{2^n})^{\otimes 2}) \circ (id \otimes_H (\alpha^{2^n-2})^{\otimes 2}) \circ \mathcal{B} \\ &= (id \otimes_H (\alpha^{2(2^n-1)})^{\otimes 2}) \circ \mathcal{B}.\end{aligned}$$

The last two assertions follow immediately from the first one, which completes the proof. □

The following result can be obtained directly by Corollary 5.3 and Theorem 5.6.

Corollary 5.7 Let $(A, \mu = *, \Delta, \alpha)$ be an infinitesimal Hom-H-pseudobialgebra whose balanceator $\mathcal{B}$ is symmetric. Then $(A^n)_{HLie} = (A, \mu^{(n)} \circ (id - \tau), (id - \tau) \circ \Delta^{(n)}, \alpha^{2^n})$ is a Hom-Lie H-pseudobialgebra for each $n \geq 1$, where $\mu^{(n)} = \alpha^{2^n-1} \circ \mu$ and $\Delta^{(n)} = \Delta \circ \alpha^{2^n-1}$.

Now we restrict to the subclass of coboundary infinitesimal Hom-H-pseudobialgebras. In the following we give a sufficient condition under which a coboundary infinitesimal Hom-H-pseudobialgebra gives rise to a coboundary Hom-Lie H-pseudobialgebra.

A 2-tensor $r \in A \otimes A$ is said to be symmetric (resp, anti-symmetric) if $r = r^{op}$ (resp, $r = -r^{op}$), where $r^{op} = \sigma(r)$.

Proposition 5.8 Let $(A, \mu = *, \Delta_r, r, \alpha)$ be a coboundary infinitesimal Hom-H-pseudobialgebra and $r = \sum_i u_i \otimes v_i$ anti-symmetric. Then the balancertor $\mathcal{B}$ of A is 0.

Proof. Suppose that $a * u_i = h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i}$ and $v_i * a = 1 \otimes g^{v_i,a} \otimes_H m_{v_i,a}$, then we have

$$\begin{aligned}
& [a, \Delta_r^{op}(b)]_\bullet = a * \Delta_r^{op}(b) - (\sigma \otimes id)(\Delta_r^{op}(b) * a) \\
= \ & a * (h^{b,u_i}(\alpha(v_i) \otimes c_{b,u_i})) - a * (g^{v_i,b}(m_{v_i,b} \otimes \alpha(u_i))) - (\sigma \otimes id)((h^{b,u_i}(\alpha(v_i) \otimes c_{b,u_i})) * a) \\
& +(\sigma \otimes id)((g^{v_i,b}(m_{v_i,b} \otimes \alpha(u_i))) * a) \\
= \ & (1 \otimes h^{b,u_i} \otimes_H 1)(a * (\alpha(v_i) \otimes c_{b,u_i})) - (1 \otimes g^{v_i,b} \otimes_H 1)(a * (m_{v_i,b} \otimes \alpha(u_i))) \\
& -(\sigma \otimes id)((h^{b,u_i} \otimes 1 \otimes_H 1)((\alpha(v_i) \otimes c_{b,u_i}) * a)) \\
& +(\sigma \otimes id)((g^{v_i,b} \otimes 1 \otimes_H 1)((m_{v_i,b} \otimes \alpha(u_i)) * a)) \\
= \ & h^{\alpha(a),\alpha(v_i)} \otimes h^{b,u_i} \otimes_H (c_{\alpha(a),\alpha(v_i)} \otimes \alpha(c_{b,u_i})) - h^{\alpha(a),m_{v_i,b}} \otimes g^{v_i,b} \otimes_H (c_{\alpha(a),m_{v_i,b}} \otimes \alpha^2(u_i)) \\
& -g^{c_{b,u_i},\alpha(a)} \otimes h^{b,u_i} \otimes_H \alpha^2(v_i) \otimes m_{c_{b,u_i},\alpha(a)} + g^{\alpha(u_i),\alpha(a)} \otimes g^{v_i,b} \otimes_H \alpha(m_{v_i,b}) \otimes m_{\alpha(u_i),\alpha(a)}.
\end{aligned}$$

Interchanging the roles of a and b in the above equation, we obtain

$$\begin{aligned}
& [b, \Delta_r^{op}(a)]_\bullet \\
= \ & h^{\alpha(b),\alpha(v_i)} \otimes h^{a,u_i} \otimes_H (c_{\alpha(b),\alpha(v_i)} \otimes \alpha(c_{a,u_i})) - h^{\alpha(b),m_{v_i,a}} \otimes g^{v_i,a} \otimes_H (c_{\alpha(b),m_{v_i,a}} \otimes \alpha^2(u_i)) \\
& -g^{c_{a,u_i},\alpha(b)} \otimes h^{a,u_i} \otimes_H \alpha^2(v_i) \otimes m_{c_{a,u_i},\alpha(b)} + g^{\alpha(u_i),\alpha(b)} \otimes g^{v_i,a} \otimes_H \alpha(m_{v_i,a}) \otimes m_{\alpha(u_i),\alpha(b)}.
\end{aligned}$$

Using the above two equations, we compute

$$\begin{aligned}
& \mathcal{B}(a, b) = [a, \Delta_r^{op}(b)]_\bullet + (\sigma \otimes \tau)([b, \Delta_r^{op}(a)]_\bullet) \\
= \ & h^{\alpha(a),\alpha(v_i)} \otimes h^{b,u_i} \otimes_H c_{\alpha(a),\alpha(v_i)} \otimes \alpha(c_{b,u_i}) - h^{\alpha(a),m_{v_i,b}} \otimes g^{v_i,b} \otimes_H c_{\alpha(a),m_{v_i,b}} \otimes \alpha^2(u_i) \\
& -g^{c_{b,u_i},\alpha(a)} \otimes h^{b,u_i} \otimes_H \alpha^2(v_i) \otimes m_{c_{b,u_i},\alpha(a)} + g^{\alpha(u_i),\alpha(a)} \otimes g^{v_i,b} \otimes_H \alpha(m_{v_i,b}) \otimes m_{\alpha(u_i),\alpha(a)} \\
& +h^{a,u_i} \otimes h^{\alpha(b),\alpha(v_i)} \otimes_H \alpha(c_{a,u_i}) \otimes c_{\alpha(b),\alpha(v_i)} - g^{v_i,a} \otimes h^{\alpha(b),m_{v_i,a}} \otimes_H \alpha^2(u_i) \otimes c_{\alpha(b),m_{v_i,a}} \\
& -h^{a,u_i} \otimes g^{c_{a,u_i},\alpha(b)} \otimes_H m_{c_{a,u_i},\alpha(b)} \otimes \alpha^2(v_i) + g^{v_i,a} \otimes g^{\alpha(u_i),\alpha(b)} \otimes_H m_{\alpha(u_i),\alpha(b)} \otimes \alpha(m_{v_i,a}) \\
= \ & -h^{\alpha(a),\alpha(u_i)} \otimes h^{b,v_i} \otimes_H c_{\alpha(a),\alpha(u_i)} \otimes \alpha(c_{b,v_i}) + h^{\alpha(a),m_{u_i,b}} \otimes g^{u_i,b} \otimes_H c_{\alpha(a),m_{u_i,b}} \otimes \alpha^2(v_i) \\
& +g^{c_{b,v_i},\alpha(a)} \otimes h^{b,v_i} \otimes_H \alpha^2(u_i) \otimes m_{c_{b,v_i},\alpha(a)} - g^{\alpha(v_i),\alpha(a)} \otimes g^{u_i,b} \otimes_H \alpha(m_{u_i,b}) \otimes m_{\alpha(v_i),\alpha(a)} \\
& +h^{a,u_i} \otimes h^{\alpha(b),\alpha(v_i)} \otimes_H \alpha(c_{a,u_i}) \otimes c_{\alpha(b),\alpha(v_i)} - g^{v_i,a} \otimes h^{\alpha(b),m_{v_i,a}} \otimes_H \alpha^2(u_i) \otimes c_{\alpha(b),m_{v_i,a}} \\
& -h^{a,u_i} \otimes g^{c_{a,u_i},\alpha(b)} \otimes_H m_{c_{a,u_i},\alpha(b)} \otimes \alpha^2(v_i) + g^{v_i,a} \otimes g^{\alpha(u_i),\alpha(b)} \otimes_H m_{\alpha(u_i),\alpha(b)} \otimes \alpha(m_{v_i,a}) \\
= \ & h^{\alpha(a),m_{u_i,b}} \otimes g^{u_i,b} \otimes_H c_{\alpha(a),m_{u_i,b}} \otimes \alpha^2(v_i) - h^{a,u_i} \otimes g^{c_{a,u_i},\alpha(b)} \otimes_H m_{c_{a,u_i},\alpha(b)} \otimes \alpha^2(v_i) \\
& +g^{c_{b,v_i},\alpha(a)} \otimes h^{b,v_i} \otimes_H \alpha^2(u_i) \otimes m_{c_{b,v_i},\alpha(a)} - g^{v_i,a} \otimes h^{\alpha(b),m_{v_i,a}} \otimes_H \alpha^2(u_i) \\
& \otimes c_{\alpha(b),m_{v_i,a}}.
\end{aligned} \tag{5. 3}$$

Due to the Hom-associativity of A, we have $\alpha(a) * (u_i * b) = (a * u_i) * \alpha(b)$ and $(b * v_i) * \alpha(a) = \alpha(b) * (v_i * a)$, which are equivalent to

$$h^{\alpha(a),m_{u_i,b}} \otimes 1 \otimes g^{u_i,b} \otimes_H c_{\alpha(a),m_{u_i,b}} = h^{a,u_i} \otimes 1 \otimes g^{c_{a,u_i},\alpha(b)} \otimes_H m_{c_{a,u_i},\alpha(b)} \tag{5. 4}$$

and

$$h^{b,v_i} \otimes 1 \otimes g^{c_{b,v_i},\alpha(a)} \otimes_H m_{c_{b,v_i},\alpha(a)} = h^{\alpha(b),m_{v_i,a}} \otimes 1 \otimes g^{v_i,a} \otimes_H c_{\alpha(b),m_{v_i,a}}, \tag{5. 5}$$

respectively. Using (5.4) and (5.5), all the terms on the right-hand side of (5.3) are canceled. Thus $\mathcal{B} = 0$, as required. □

Theorem 5.9 Let $(A, \mu = *, \Delta_r, r, \alpha)$ be a coboundary infinitesimal Hom-H-pseudobialgebra with $(id \otimes_H \alpha)(a * b) = \alpha(a) * \alpha(b)$ and $r = \sum_i u_i \otimes v_i$ anti-symmetric. Then $A_{HLie} = (A, [*], \delta_r, r, \alpha)$ is a coboundary Hom-Lie H-pseudobialgebra, where $[a * b] = a * b - (\sigma \otimes id)(b * a)$, $\delta_r(a) = (id - \tau)\Delta_r(a)$ for all $a, b \in A$.

Proof. By Proposition 5.8, balanceator $\mathcal{B}$ of A is 0. Thus Corollary 5.3 implies that A_{HLie} is a Hom-Lie H-pseudobialgebra. It remains to show that δ_r satisfies condition (2.2). since

$$\begin{aligned} & [a * u_i] = a * u_i - (\sigma \otimes id)(u_i * a) \\ = \quad & h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i} - g^{u_i,a} \otimes 1 \otimes_H m_{u_i,a}, \end{aligned}$$

we have

$$\{a, u_i\} = h^{a,u_i} \otimes c_{a,u_i} - g^{u_i,a} \otimes m_{u_i,a}. \tag{5. 6}$$

Similarly, we get

$$\{a, v_i\} = h^{a,v_i} \otimes c_{a,v_i} - g^{v_i,a} \otimes m_{v_i,a}. \tag{5. 7}$$

According to (5.6)-(5.7) and the condition $r = -r^{op}$, we have

$$\begin{aligned} \delta_r(a) &= (id - \tau)\Delta_r(a) \\ &= (id - \tau)(h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a})) \\ &= h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a}) \\ &\quad - h^{a,u_i}(\alpha(v_i) \otimes c_{a,u_i}) + g^{v_i,a}(m_{v_i,a} \otimes \alpha(u_i)) \\ &= h^{a,u_i}(c_{a,u_i} \otimes \alpha(v_i)) - g^{u_i,a}(m_{u_i,a} \otimes \alpha(v_i)) + h^{a,v_i}(\alpha(u_i) \otimes c_{a,v_i}) \\ &\quad - g^{v_i,a}(\alpha(u_i) \otimes m_{v_i,a}) \\ &= \mu(\{a, u_i\} \otimes \alpha(v_i) + \sigma_{12}(\alpha(u_i) \otimes \{a, v_i\}), \end{aligned}$$

as desired. □

6. Pseudo-Yang-Baxter equations in Hom-*H*-pseudoalgebras

Recall that the Hom-classical pseudo-Yang-Baxter equation (*pseudo*-CHYBE) in a Hom-Lie H-pseudoalgebra $(L, [*], \alpha)$ for $r = \sum_i u_i \otimes v_i \in L^{\otimes 2}$ has the form ([21])

$$\begin{aligned} & [[r, r]]_\alpha = \mu^3_{-1}(\{u_j, u_i\} \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu^4_{-2}(\alpha(u_i) \otimes \{u_j, v_i\} \otimes \alpha(v_j)) \\ & -\mu^2_{-3}(\alpha(u_i) \otimes \alpha(u_j) \otimes \{v_j, v_i\}). \end{aligned}$$

Now we establish the relationship between solutions of the *pseudo*-AHYBE and solutions of the *pseudo*-CHYBE in related Hom-Lie H-pseudoalgebra.

Theorem 6.1 Let $(A, *, \alpha)$ be a Hom-associative H-pseudoalgebra and r a solution of the *pseudo*-HAYBE. Suppose that r is either symmetric or anti-symmetric, then r is a solution of the *pseudo*-CHYBE in the Hom-Lie H-pseudoalgebra $A_{HLie} = (A, [*], \alpha)$, where $[a * b] = a * b - (\sigma \otimes_H id)(b * a)$ for all $a, b \in A$.

Proof. For all $a, b \in A$, we write $a * b = h^{a,b} \otimes g^{a,b} \otimes_H e_{a,b}$. Then

$$[a * b] = a * b - (\sigma \otimes id)(b * a) = h^{a,b} \otimes g^{a,b} \otimes_H e_{a,b} - g^{b,a} \otimes h^{b,a} \otimes_H e_{b,a}.$$

Define the 3-tensor

$$\begin{aligned} A_\alpha(r)' &= \mu_1^{2,4}([u_j, u_i] \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu_2^{1,3}(\alpha(u_j) \otimes [u_i, v_j] \otimes \alpha(v_i)) \\ &\quad + \mu_{-3}^{1}(\alpha(u_i) \otimes \alpha(u_j) \otimes [v_i, v_j]), \end{aligned}$$

that is,

$$\begin{aligned} A_\alpha(r)' &= h_2^{u_j,u_i} e_{u_j,u_i} \otimes \alpha(v_j) \otimes h_1^{u_j,u_i} S(g^{u_j,u_i})\alpha(v_i) \\ &\quad - h_2^{u_i,v_j} S(g^{u_i,v_j})\alpha(u_j) \otimes h_1^{u_i,v_j} e_{u_i,v_j} \otimes \alpha(v_i) \\ &\quad + g_1^{v_i,v_j} S(h^{v_i,v_j})\alpha(u_i) \otimes \alpha(u_j) \otimes g_2^{v_i,v_j} e_{v_i,v_j}. \end{aligned}$$

We first check that $A_\alpha(r)' = \sigma_{13}(A_\alpha(r))$. Using the (anti-)symmetry of r, we have

$$\begin{aligned} & \sigma_{13}(A_\alpha(r)) \\ = \; & \sigma_{13}(\mu_{-1}^{4}(\{u_i, u_j\} \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu_2^{3,4}(\alpha(u_i) \otimes \{v_i, u_j\} \otimes \alpha(v_j)) \\ & + \mu_3^{1,4}(\alpha(u_i) \otimes \alpha(u_j) \otimes \{v_j, v_i\})) \\ = \; & \sigma_{13}(g_2^{u_i,u_j} e_{u_i,u_j} \otimes \alpha(v_j) \otimes g_1^{u_i,u_j} S(h^{u_i,u_j})\alpha(v_i) \\ & - \alpha(u_i) \otimes h_1^{v_i,u_j} e_{v_i,u_j} \otimes h_2^{v_i,u_j} S(g^{v_i,u_j})\alpha(v_j) + h_1^{v_j,v_i} S(g^{v_j,v_i})\alpha(u_i) \otimes \alpha(u_j) \otimes h_2^{v_j,v_i} e_{v_j,v_i}) \\ = \; & g_1^{u_i,u_j} S(h^{u_i,u_j})\alpha(v_i) \otimes \alpha(v_j) \otimes g_2^{u_i,u_j} e_{u_i,u_j} - h_2^{v_i,u_j} S(g^{v_i,u_j})\alpha(v_j) \otimes h_1^{v_i,u_j} e_{v_i,u_j} \otimes \alpha(u_i) \\ & + h_2^{v_j,v_i} e_{v_j,v_i} \otimes \alpha(u_j) \otimes h_1^{v_j,v_i} S(g^{v_j,v_i})\alpha(u_i) \\ = \; & g_1^{v_i,v_j} S(h^{v_i,v_j})\alpha(u_i) \otimes \alpha(u_j) \otimes g_2^{v_i,v_j} e_{v_i,v_j} - h_2^{u_i,v_j} S(g^{u_i,v_j})\alpha(u_j) \otimes h_1^{u_i,v_j} e_{u_i,v_j} \otimes \alpha(v_i) \\ & + h_2^{u_j,u_i} e_{u_j,u_i} \otimes \alpha(v_j) \otimes h_1^{u_j,u_i} S(g^{u_j,u_i})\alpha(v_i) \\ = \; & A_\alpha(r)'. \end{aligned}$$

Now, we compute

$$\begin{aligned} & [[r, r]]_\alpha \\ = \; & \mu_{-1}^{3}(\{u_j, u_i\} \otimes \alpha(v_j) \otimes \alpha(v_i)) - \mu_{-2}^{4}(\alpha(u_i) \otimes \{u_j, v_i\} \otimes \alpha(v_j)) \\ & - \mu_{-3}^{2}(\alpha(u_i) \otimes \alpha(u_j) \otimes \{v_j, v_i\}) \\ = \; & \mu_{-1}^{3}(h^{u_j,u_i} S(g_1^{u_j,u_i}) \otimes g_2^{u_j,u_i} e_{u_j,u_i} \otimes \alpha(v_j) \otimes \alpha(v_i) \\ & - g^{u_i,u_j} S(h_1^{u_i,u_j}) \otimes h_2^{u_i,u_j} e_{u_i,u_j} \otimes \alpha(v_j) \otimes \alpha(v_i)) \\ & + \mu_{-2}^{4}(\alpha(u_i) \otimes g^{v_i,u_j} S(h_1^{v_i,u_j}) \otimes h_2^{v_i,u_j} e_{v_i,u_j} \otimes \alpha(v_j) \\ & - \alpha(u_i) \otimes h^{u_j,v_i} S(g_1^{u_j,v_i}) \otimes g_2^{u_j,v_i} e_{u_j,v_i} \otimes \alpha(v_j)) \\ & + \mu_{-3}^{2}(\alpha(u_i) \otimes \alpha(u_j) \otimes g^{v_i,v_j} S(h_1^{v_i,v_j}) \otimes h_2^{v_i,v_j} e_{v_i,v_j} \\ & - \alpha(u_i) \otimes \alpha(u_j) \otimes h^{v_j,v_i} S(g_1^{v_j,v_i}) \otimes g_2^{v_j,v_i} e_{v_j,v_i}) \\ = \; & g_2^{u_j,u_i} e_{u_j,u_i} \otimes g_1^{u_j,u_i} S(h^{u_j,u_i})\alpha(v_j) \otimes \alpha(v_i) - h_2^{u_i,u_j} e_{u_i,u_j} \otimes h_1^{u_i,u_j} S(g^{u_i,u_j})\alpha(v_j) \otimes \alpha(v_i) \\ & + \alpha(u_i) \otimes h_2^{v_i,u_j} e_{v_i,u_j} \otimes h_1^{v_i,u_j} S(g^{v_i,u_j})\alpha(v_j) - \alpha(u_i) \otimes g_2^{u_j,v_i} e_{u_j,v_i} \otimes g_1^{u_j,v_i} S(h^{u_j,v_i})\alpha(v_j) \\ & + \alpha(u_i) \otimes h_1^{v_i,v_j} S(g^{v_i,v_j})\alpha(u_j) \otimes h_2^{v_i,v_j} e_{v_i,v_j} - \alpha(u_i) \otimes g_1^{v_j,v_i} S(h^{v_j,v_i})\alpha(u_j) \otimes g_2^{v_j,v_i} e_{v_j,v_i} \end{aligned}$$

$$\begin{aligned}
&= h_2^{u_j,u_i} e_{u_j,u_i} \otimes \alpha(v_j) \otimes h_1^{u_j,u_i} S(g^{u_j,u_i})\alpha(v_i) - g_2^{u_i,u_j} e_{u_i,u_j} \otimes \alpha(v_j) \otimes g_1^{u_i,u_j} S(h^{u_i,u_j})\alpha(v_i) \\
&\quad + \alpha(u_i) \otimes h_1^{v_i,u_j} e_{v_i,u_j} \otimes h_2^{v_i,u_j} S(g^{v_i,u_j})\alpha(v_j) - h_2^{u_i,v_j} S(g^{u_i,v_j})\alpha(u_j) \otimes h_1^{u_i,v_j} e_{u_i,v_j} \otimes \alpha(v_i) \\
&\quad + g_1^{v_i,v_j} S(h^{v_i,v_j})\alpha(u_i) \otimes \alpha(u_j) \otimes g_2^{v_i,v_j} e_{v_i,v_j} - h_1^{v_j,v_i} S(g^{v_j,v_i})\alpha(u_i) \otimes \alpha(u_j) \otimes h_2^{v_j,v_i} e_{v_j,v_i} \\
&= A_\alpha(r)' - A_\alpha(r) \\
&= (\sigma_{13} - id)(A_\alpha(r)) \\
&= 0,
\end{aligned}$$

finishing the proof. □

7. Further discussion

We study infinitesimal Hom-H-pseudobialgebras by twisting the identities using morphisms of H-modules. Further research on infinitesimal Hom-H-pseudobialgebras would be important for several reasons. Firstly, it would help to deepen our understanding of the structure and properties of these algebraic objects. Secondly, as is well known, the annihilation algebra of the associative H-pseudoalgebra Cend H is nothing else but the Drinfeld Double of H. This fact leads us to believe that there should be a deep connection between the theory of infinitesimal Hom-H-pseudobialgebras and quantum groups. Finally, we can obtain infinitesimal Hom-H-pseudobialgebra from a given infinitesimal Hom-bialgebra or infinitesimal H-pseudobialgebra by using Corollary 3.8 and Theorem 3.9, respectively. Therefore, we expect that the further research of infinitesimal Hom-H-pseudobialgebras can provide a new perspective for the study of infinitesimal Hom-bialgebras and infinitesimal H-pseudobialgebras. Overall, the study of infinitesimal Hom-H-pseudobialgebras is a promising area of research with many potential avenues for exploration.

References

[1] M. Aguiar, *Infinitesimal Hopf algebras*, Contemp. Math. **267** (2000), 1–30.
[2] M. Aguiar, *On the associative analog of Lie bialgebras*, J. Algebra **244** (2001), 492–532.
[3] B. Bakalov, A. D'Andrea, V. G. Kac, *Theory of finite pseudoalgebras*, Adv. Math. **162** (2001), 1–140.
[4] C. Boyallian, J. Liberati, *On pseudo-bialgebras*, J. Algebra **372** (2012), 1–34.
[5] A. D'Andrea, V. G. Kac, *Structure theory of finite conformal algebras*, Selecta Math. **4** (1998), 377–418.
[6] I. Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, Nonlinear Science: Theory and Applications. John Wiley and Sons, Ltd., Chichester, 1993, xii+176 pp.
[7] R. Ehrenborg, M. Readdy, *Coproducts and the cd-index*, J. Algebraic Combin. **8** (1998), 273–299.
[8] I. M. Gel'Fand, I. Y. Dorfman, *Hamiltonian operators and infinite-dimensional Lie algebra*, Funktsional. Anal. i Prilozhen **15** (1981), 23–40.
[9] I. M. Gel'Fand, I. Y. Dorfman, *Hamiltonian operators and the classical Yang-Baxter equation*, Funktsional. Anal. i Prilozhen **16** (1982), 1–9.
[10] J. T. Hartwig, D. Larsson, S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314–361.
[11] S. A. Joni, G. C. Rota, *Coalgebras and bialgebras in combinatorics*, Stud. Appl. Math. **61** (1979), 93–139.
[12] V. G. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. Providence, RI: American Mathematical Society, 1998, vi+201 pp.
[13] L. L. Liu, S. H. Wang, G. D. Shi, Z. T. Guo, *Infinitesimal H-pseudobialgebra and associative pseudo-Yang-Baxter equation*, Comm. Algebra **50** (2022), 1071–1091.
[14] T. S. Ma, H. Y. Yang, *Drinfeld double for infinitesimal BiHom-bialgebras*, Adv. Appl. Clifford Algebr. **30** (2020), 42, 22 pp.
[15] A. Makhlouf, S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51–64.
[16] A. Makhlouf, S. Silvestrov, *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, Generalized Lie theory in mathematics, physics and beyond, 189-206, Springer, Berlin, 2009.
[17] A. Makhlouf, S. Silvestrov, *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. **9** (2010), 553–589.
[18] L. N. Song, R. Tang, *Cohomologies, deformations and extensions of n-Hom-Lie algebras*, J. Geom. Phys. **141** (2019), 65–78.
[19] Q. X. Sun, *Generalization of H-pseudoalgebraic structures*, J. Math. Phys. **53** (2012), 012105, 18 pp.
[20] Q. X. Sun, K. D. A. Akinocho, *The cohomology structure of Hom-H-pseudoalgebras*, Bull. Braz. Math. Soc. **50** (2019), 211–231.
[21] Q. X. Sun, F. Li, *A generalization of Lie H-pseudobialgebras*, Theoret. and Math. Phys. **192** (2017), 939–957.
[22] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series W. A. Benjamin, Inc., New York, 1969, vii+336 pp.
[23] Z. X. Wu, *Graded left symmetric pseudo-algebras*, Comm. Algebra **43** (2015), 3869–3897.
[24] Z. X. Wu, *Leibniz H-pseudoalgebras*, J. Algebra **437** (2015), 1–33.

[25] D. Yau, *The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A **42** (2009), 165202, 12 pp.
[26] D. Yau, *The Hom-Yang-Baxter equation and Hom-Lie algebras*, J. Math. Phys. **52** (2011), 053502, 19 pp.
[27] D. Yau, *Infinitesimal Hom-bialgebras and Hom-Lie bialgebras*, arXiv:1001.5000v1. 2010.