



A nonstandard approach to Karamata uniform convergence theorem

Žarko Mijajlović^a, Danijela Branković^{b,*}

^aFaculty of Mathematics, University of Belgrade, Studentski trg 16, 11158 Belgrade, Serbia

^bSchool of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11120 Belgrade, Serbia

Abstract. A nonstandard proof of a generalization of Karamata uniform convergence theorem for slowly varying functions is presented. Properties of a related operator \mathcal{L} and its connection with slowly varying functions are discussed.

1. Introduction

This work is inspired by the possibility of application of the theory of regularly varying functions in study of asymptotics of cosmological parameters. Our particular aim was to consider the asymptotics of the expansion scale factor $a(t)$ in the Λ CDM (Lambda cold dark matter) model, see [8]. It appeared that the mathematical behavior of the scale factor $a(t)$ in certain epochs of evolution of the Universe is very connected to the properties of regularly varying functions. Some works in this area are [11], [13], [14] and [18]. For that cause, we extracted and studied certain general properties of regularly varying functions appearing in our cosmological studies which might be of an interest by themselves. Obtained results on these properties are presented in this paper.

We shall occasionally use here the methods of nonstandard analysis, as we did in [9] and [10]. Somewhat extended explanation of notions from nonstandard analysis is given so that a non-specialists in this area can read the paper, too. Only basic notions from model theory of first order logic will be assumed, see [3]. For more details of this subject one may consult [4], [12] and [19]. Some more recent papers on applications of nonstandard analysis are [2], [5], [6], [15].

If \mathcal{R} stands for the field of real numbers with some added functions and relations, then \mathcal{R}^* denotes a nonstandard extension of \mathcal{R} . We remind that $\mathcal{R} < \mathcal{R}^*$, i.e. \mathcal{R} is elementary embedded in \mathcal{R}^* . In other words all first order properties expressed in the expanded language $L = \{+, \cdot, \leq\} \cup \{\underline{s} : s \in S\}$, where S is the set of constants from \mathcal{R} and added functions and relations, are preserved from \mathcal{R} to \mathcal{R}^* and vice versa. We call it the transfer principle. Here \underline{s} is the name of an entity s from S . The symbol \underline{s} is interpreted as s in \mathcal{R} , i.e. $\underline{s}^{\mathcal{R}} = s$, while $s^* = \underline{s}^{\mathcal{R}^*}$. Due to $\mathcal{R} < \mathcal{R}^*$ if φ is a sentence of L , then $\mathcal{R} \models \varphi$ if and only if $\mathcal{R}^* \models \varphi$, where \models is the satisfaction relation. If $\varphi = \varphi(\underline{s}_1, \underline{s}_2, \dots, \underline{s}_n)$, we shall often write $\mathcal{R}^* \models \varphi(s_1^*, s_2^*, \dots, s_n^*)$ instead of $\mathcal{R}^* \models \varphi$. For easier reading we shall omit in some cases the star in s^* if this does not lead to ambiguity, for example in $+^*, \leq^*$, etc.

2020 *Mathematics Subject Classification.* Primary 26A12; Secondary 26E35.

Keywords. slowly varying functions, asymptotics, nonstandard analysis.

Received: 11 June 2023; Revised: 25 June 2023; Accepted: 04 July 2023

Communicated by Dragan S. Djordjević

* Corresponding author: Danijela Branković

Email addresses: zarkom@matf.bg.ac.rs (Žarko Mijajlović), danijela@etf.bg.ac.rs (Danijela Branković)

We use the same symbols for the structures and their domains, e.g. \mathcal{N} denotes the set of natural numbers (non-negative integers) and the structure of natural numbers, too. Infinitesimal is an element $\varepsilon \in \mathcal{R}^*$ that is infinitely close to 0, i.e. for all $n \in \mathcal{N}^+$, $|\varepsilon| < 1/n$. The symbol $\mu(0)$ stands for the set of all infinitesimals and is called the monad of zero. If $a - b$ is an infinitesimal, then we write $a \approx b$. An element $H \in \mathcal{N}^* \setminus \mathcal{N}$ is a positive infinite integer if for all $n \in \mathcal{N}$, $H > n$, while $a \in \mathcal{R}^*$ is finite if for all infinite positive integers H , $|a| < H$. Elements of \mathcal{R}^* that are not finite are called infinite. Let $\mathcal{R}_{\text{fin}}^*$ denote all finite elements of \mathcal{R}^* . Then for $b \in \mathcal{R}_{\text{fin}}^*$ there is $a \in \mathcal{R}$ and an infinitesimal ε such that $b = a + \varepsilon$. Then $\text{st}(b) = a$, where st is the standard part function $\text{st}: \mathcal{R}_{\text{fin}}^* \rightarrow \mathcal{R}$. We remind that st is a homomorphism from the field $\mathcal{R}_{\text{fin}}^*$ to \mathcal{R} and for continuous functions as well. It is convenient to extend st to infinite elements of \mathcal{R}^* , taking for positive infinite $H \in \mathcal{R}^*$, $\text{st}(H) = +\infty$ and $\text{st}(-H) = -\infty$. Monad of $b \in \mathcal{R}$ is $\mu(b) = \mu(0) + b$. It is convenient to use the quasi-order \leq on \mathcal{R}^* defined by $x \leq y$, $x, y \in \mathcal{R}^*$ if and only if $x \leq^* y$ or $x \approx y$. If $\varphi(x)$ is a predicate formula which defines a set $X \subseteq \mathcal{R}$, then $\varphi^*(x)$, obtained by starring entities over \mathcal{R} appearing in φ , defines an internal set $X^* \subseteq \mathcal{R}^*$ associated to X . On internal subsets can be defined a finite-additive measure which can be extended by use of Caratheodory extension theorem to σ -additive measure on \mathcal{R}^* . This measure is called Loeb measure which is naturally related to Lebesgue measure. For other notation and terminology see [3], [19], [12].

The symbol \bigwedge denotes the universal quantifier, while \bigvee stands for the existential quantifier. For example, for $F: \mathcal{R} \times I \rightarrow \mathcal{R}$, $I \subseteq \mathcal{R}$, we have

$$\lim_{x \rightarrow +\infty} F(x, u) = 0 \quad \text{for all } u \in I \quad \text{if and only if}$$

$$\mathcal{R} \models \bigwedge_{u \in I} \bigwedge_{\varepsilon > 0} \bigvee_{x_0} \bigwedge_{x > x_0} |F(x, u)| < \varepsilon. \quad (1)$$

$$\lim_{x \rightarrow +\infty} F(x, u) = 0 \quad \text{uniformly for } u \in I \quad \text{if and only if}$$

$$\mathcal{R} \models \bigwedge_{\varepsilon > 0} \bigvee_{x_0} \bigwedge_{x > x_0} \bigwedge_{u \in I} |F(x, u)| < \varepsilon. \quad (2)$$

A real function F is said to be regularly varying at positive infinity if it is real-valued, positive and measurable on $[a, +\infty)$, for some $a > 0$, and if for each $\lambda > 0$

$$\lim_{x \rightarrow +\infty} \frac{F(\lambda x)}{F(x)} = \lambda^\rho \quad (3)$$

for some ρ , $-\infty < \rho < +\infty$.

Number ρ is called the index of regular variation. If $\rho = 0$, F is called slowly varying, or SV function. Notions of slowly varying functions and regular variations were introduced by Jovan Karamata [7]. Books [1] and [16] give detailed exposition of the theory of regular variation and slowly varying functions. The following theorem which refers to slowly varying functions is fundamental in this theory.

Theorem 1.1. (The Uniform Convergence Theorem, J. Karamata). *If F is a slowly varying function, then for every fixed $[a, b]$, $0 < a < b < +\infty$, the relation (3) holds uniformly with respect to $[a, b]$.*

We shall prove a generalization of the linear variant of this theorem using nonstandard methods. The linear form is obtained by transformation $f(x) = \ln F(e^x)$.

2. Regular variation in nonstandard analysis

First we prove a lemma on uniform convergence relative to a set I .

Lemma 2.1. *Let $F: \mathcal{R} \times I \rightarrow \mathcal{R}$, $I \subseteq \mathcal{R}$. Then $\lim_{x \rightarrow +\infty} F(x, u) = 0$ uniformly with respect to $u \in I$ if and only if for all positive infinite $x \in \mathcal{R}^*$, $u \in I^*$, $F^*(x, u) \approx 0$.*

Proof (\Rightarrow) Suppose $\lim_{x \rightarrow +\infty} F(x, u) = 0$ uniformly for $u \in I$, i.e. (2) holds. Let $\varepsilon \in \mathcal{R}^+$ be arbitrary and $x_0 \in \mathcal{R}$ so that $\mathcal{R} \models \bigwedge_{x > x_0} \bigwedge_{u \in I} |F(x, u)| < \varepsilon$. By transfer principle, $\mathcal{R}^* \models \bigwedge_{x > x_0} \bigwedge_{u \in I^*} |F^*(x, u)| < \varepsilon$. Hence, for all positive infinite $x \in \mathcal{R}^*$, $\mathcal{R}^* \models \bigwedge_{u \in I^*} |F^*(x, u)| < \varepsilon$, and so for all $u \in I^*$, $|F^*(x, u)| < \varepsilon$. As $\varepsilon \in \mathcal{R}^+$ was chosen arbitrarily, it follows $F^*(x, u) \approx 0$.

(\Leftarrow) Suppose for all positive infinite $x \in \mathcal{R}^*$, $u \in I^*$, $F^*(x, u) \approx 0$. Then for arbitrary $\varepsilon \in \mathcal{R}^+$, all positive infinite x and all $u \in I^*$, $|F^*(x, u)| < \varepsilon$. Choose any positive infinite x_0 . As $x > x_0$ is also positive infinite it follows $\mathcal{R}^* \models \bigwedge_{x > x_0} \bigwedge_{u \in I^*} |F^*(x, u)| < \varepsilon$. Hence for all $\varepsilon \in \mathcal{R}^+$, $\mathcal{R}^* \models \bigvee_{x_0} \bigwedge_{x > x_0} \bigwedge_{u \in I^*} |F^*(x, u)| < \varepsilon$, i.e. $\mathcal{R} \models \bigvee_{x_0} \bigwedge_{x > x_0} \bigwedge_{u \in I} |F(x, u)| < \varepsilon$. It follows $\mathcal{R} \models \bigwedge_{\varepsilon > 0} \bigvee_{x_0} \bigwedge_{x > x_0} \bigwedge_{u \in I} |F(x, u)| < \varepsilon$, so $\lim_{x \rightarrow +\infty} F(x, u) = 0$ uniformly in respect to $u \in I$. \square

Corollary 2.2. $\lim_{x \rightarrow +\infty} F(x, u) = 0$ is not uniformly convergent relative to $u \in I$ if and only if there is a positive infinite $x^*, u^* \in I^*$ and $a \in \mathcal{R}^+$ such that $|F^*(x^*, u^*)| > a$.

In contrast to this corollary, note that for the ordinary convergence, $\lim_{x \rightarrow +\infty} F(x, u) = 0$ if and only if for all positive infinite $x \in \mathcal{R}^*$, $u \in I$, $F^*(x, u) \approx 0$.

Now we prove a generalization of Karamata uniform convergence theorem for slowly varying functions. In the next proof we assume the notion and properties of Loeb measure, a natural extension of Lebesgue measure into the nonstandard universe. If $A \subseteq \mathcal{R}$ is a measurable, $\lambda(A)$ denotes Lebesgue measure of A , while $\ell(A)$ denotes measure of a Loeb measurable set $A \subseteq \mathcal{R}^*$ in the nonstandard universe. The following well-known Fisher's lifting theorem gives a connection between Lebesgue and Leobe measure and basically states that $\text{st}^{-1}: \mathcal{R} \rightarrow \mathcal{R}_{\text{fin}}^*$ preserves measure.

Theorem 2.3. If A is a Lebesgue measurable subset of a finite closed interval of \mathcal{R} , then

$$\lambda(A) = \ell(\text{st}^{-1}(A)). \tag{4}$$

Now we proceed to the proof of the main theorem in Sec. 2, a generalization of Karamata uniform convergence theorem.

Theorem 2.4. Assume $H: S \times \mathcal{R} \rightarrow \mathcal{R}^+$ is a measurable, where $S = [d, +\infty]_{\mathcal{R}}$ for some $d \in \mathcal{R}$, $I = [0, 1]_{\mathcal{R}}$ and suppose H satisfy the following inequality on its domain:

$$H(x, u) \leq H(x + u, v) + H(x, u + v). \tag{5}$$

Further, let $m: S \rightarrow \mathcal{R}^+$ be a measurable and nondecreasing function and $G(x, u) = H(x, u)m(x)$. Then, if $\lim_{x \rightarrow +\infty} G(x, u) = 0$ for all $u \in \mathcal{R}$, then this convergence is uniform relative to $u \in I$.

Proof Assume $\lim_{x \rightarrow +\infty} G(x, u) = 0$ for all $u \in \mathcal{R}$, but this convergence is not uniform relative to $u \in I$. Then by Corollary 2.2 there are positive infinite $x_0^* \in \mathcal{R}^*$, $u_0^* \in I^*$ and $a \in \mathcal{R}^+$ such that $G^*(x_0^*, u_0^*) > a$. Let $[0, 2]^* = [0, 2]_{\mathcal{R}^*}$ and define

$$\begin{aligned} U &= \{u \in [0, 2]^* \mid G^*(x_0^*, u) < a/3\} \\ V &= \{v \in [0, 2]^* \mid G^*(x_0^* + u_0^*, v) < a/3\}. \end{aligned} \tag{6}$$

As sets U and V are internal and G is λ -measurable, U and V are ℓ -measurable. For positive infinite x and $u \in [0, 2]$ we have $G^*(x, u) \approx 0$, hence $[0, 2] \subseteq U \cap V \subseteq [0, 2]^*$. Then Fisher's theorem implies

$$\ell(U) = 2, \quad \ell(V) = 2. \tag{7}$$

Let $V_0 = V + u_0^*$. As measure ℓ is invariant under translation, we have $\ell(V_0) = \ell(V)$, i.e.

$$\ell(V_0) = 2. \tag{8}$$

Further, $U, V_0 \subseteq [0, 3]^*$, hence $U \cap V_0 \neq \emptyset$, as otherwise

$$3 = \ell([0, 3]) \geq \ell(U \cup V_0) = 4, \text{ a contradiction.}$$

So, let $b^* \in U \cap V_0$. Hence, $b^* \in U$ and for some $v^* \in V$, $b^* = v^* + u_0^*$ and so $v^* = b^* - u_0^*$. Then

As $v^* \in V$, we have $G^*(x_0^* + u_0^*, v^*) < a/3$,

As $b^* \in U$, we have $G^*(x_0^*, b^*) < a/3$.

Hence,

$$H^*(x_0^* + u_0^*, v^*)m(x_0^* + u_0^*) < a/3 \tag{9}$$

and

$$H^*(x_0^*, b^*)m(x_0^*) < a/3. \tag{10}$$

By the inequality (5) we have

$$\begin{aligned} H^*(x_0^*, u_0^*) &\leq H^*(x_0^* + u_0^*, v^*) + H^*(x_0^*, u_0^* + v^*) \\ &= H^*(x_0^* + u_0^*, v^*) + H^*(x_0^*, b^*) \end{aligned} \tag{11}$$

As $m(x)$ is nondecreasing we have $m(x_0^*) \leq m(x_0^* + u_0^*)$, so

$$H^*(x_0^*, u_0^*)m(x_0^*) \leq H^*(x_0^* + u_0^*, v^*)m(x_0^* + u_0^*) + H^*(x_0^*, b^*)m(x_0^*), \tag{12}$$

i.e.

$$a < G^*(x_0^*, u_0^*) \leq G^*(x_0^* + u_0^*, v^*) + G^*(x_0^*, b^*) < 2a/3, \tag{13}$$

a contradiction. □

The uniform convergence is preserved under translation and homothety, hence we have:

Corollary 2.5. *The previous theorem is still true if the interval $I = [0, 1]_{\mathcal{R}}$ is replaced by any finite interval $I' = [a, b]_{\mathcal{R}}$, $0 < a < b$.*

If we take $H(x, u) = |f(x + u) - f(x)|$ and $m(x) = 1$, where $f(x) = \ln(F(e^x))$ and $F(x)$ is a slowly varying function, we immediately obtain Karamata theorem 1.1.

3. Operator \mathcal{L}

We introduce operator \mathcal{L} , which have an important role in the analysis of regular variation. The operator \mathcal{L} may be defined on the set of Lebesgue integrable functions, but due to the nature of physical parameters that are studied in this paper, our attention will be turned only towards to its restriction to at least twice differentiable real functions, i.e to the space $C^2(\mathcal{R})$.

Definition 3.1. $\mathcal{L}(h)(x) = \frac{1}{\ln(x)} \int_{x_0}^x \frac{h(t)}{t} dt, \quad x_0 > 1, h \in C^2(\mathcal{R}).$

As we are interested in asymptotics at positive infinity, the exact value of x_0 is not of some importance. We may even assume, with a proper adaptation of the argument function h , that $x_0 = 1$. Namely, if the function $h(x)$ in the above definition of \mathcal{L} is bounded in some neighborhood of 1, what is in this paper almost of the only interest, then by the l'Hopital's rule

$$\lim_{x \rightarrow 1} \mathcal{L}(h)(x) = h(1), \tag{14}$$

so we can take in the above definition $x_0 = 1$. From now on we assume $x_0 = 1$, if it is not otherwise specified. Obviously, \mathcal{L} is a linear operator over the space $C^2(\mathcal{R})$. This operator has many interesting properties and some of them reflect more or less well-known theorems on regularly varying functions.

It is convenient to denote by \mathcal{R}_α the class of regularly varying functions of index α . Hence $\mathcal{R}_0 = SV$ is the class of all slowly varying functions. By \mathcal{Z}_0 we shall denote the class of zero functions at $+\infty$, i.e. $\varepsilon \in \mathcal{Z}_0$ if and only if $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$.

Theorem 3.2. Let \mathcal{L}' denote the restriction of \mathcal{L} to the appropriate domain. Then

1. $\mathcal{L}' : \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$.
2. $\mathcal{L}' : \mathcal{R}_0 \rightarrow \mathcal{R}_0$.
3. $\mathcal{L}' : \mathcal{R}_\alpha \rightarrow \mathcal{R}_\alpha$, $\alpha \in \mathbb{R}$.
4. $\mathcal{L}' : B(\mathcal{R}) \rightarrow B(\mathcal{R})$, where $B(\mathcal{R})$ is the set of real bounded functions.

Proof. The statement 1. may be obtained by use of l'Hopital's rule. Statement 2. immediately follows from the statements 1.5.9a and 1.5.9b in [1]. Statement 3. follows from theorems 1.5.10 and 1.5.11 in [1]. Finally, if h is bounded, then there are a and b such that $a \leq h(x) \leq b$, $x > 1$. Hence $a \leq \mathcal{L}(h)(x) \leq b$, for $x > 1$. \square

The following proposition gives us one interesting property of the linear operator \mathcal{L} .

Proposition 3.3. Linear operator \mathcal{L} is invertible.

Proof. Let $f(x) = \mathcal{L}(g)(x)$, where $g \in C^2(\mathcal{R})$. By definition of the operator \mathcal{L} , it follows

$$f(x) \ln(x) = \int_1^x \frac{g(t)}{t} dt. \tag{15}$$

Differentiating the equation (15) over variable x , we infer

$$\frac{f(x)}{x} + f'(x) \ln(x) = \frac{g(x)}{x}. \tag{16}$$

From the equation (16) we obtain

$$g(x) = f(x) + x f'(x) \ln(x), \tag{17}$$

what proves the proposition. \square

Limit $\lim_{x \rightarrow +\infty} \mathcal{L}(h)(x)$ acts as a proper generalization of $\lim_{x \rightarrow +\infty} h(x)$. Namely, by easy application of l'Hopital's rule we have

Proposition 3.4. Let c be a real number and suppose $\lim_{x \rightarrow +\infty} h(x) = c$. Then $\lim_{x \rightarrow +\infty} \mathcal{L}(h)(x) = c$.

The example $h(x) = \sin(x)$ shows that $\lim_{x \rightarrow +\infty} \mathcal{L}(h)(x)$ is a proper extension of ordinary limit. Namely, $h(x)$ diverges at positive infinity, while $\lim_{x \rightarrow +\infty} \mathcal{L}(h)(x)$ converges.

If a slowly varying function $L(x)$ is given by integral representation (see [7])

$$L(x) = g(x) e^{\int_{\varepsilon_0}^x \frac{\varepsilon(t)}{t} dt}, \quad \text{where } \varepsilon \in \mathcal{Z}_0 \text{ and } g(x) \rightarrow g_0 \text{ as } x \rightarrow +\infty,$$

then

$$\frac{\ln(L(x))}{\ln(x)} = \frac{\ln(g(x))}{\ln(x)} + \mathcal{L}(\varepsilon)(x),$$

wherefrom by previous two propositions

$$\frac{\ln(L(x))}{\ln(x)} \rightarrow 0 \text{ as } x \rightarrow +\infty. \tag{18}$$

This statement can be found already in [16]. Now we give an application to asymptotics of certain integrals.

Theorem 3.5. Let $h(x)$ be a positive and M -bounded Lebesgue integrable function, $M > 0$ and $\lambda > 1$. Assume

$$\lim_{x \rightarrow +\infty} (\mathcal{L}(h)(\lambda x) - \mathcal{L}(h)(x)) \ln(x) = 0 \tag{19}$$

uniformly with respect to λ . Then

$$\int_x^{\lambda x} \frac{h(t)}{t} dt \approx \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^x \frac{h(t)}{t} dt. \tag{20}$$

Proof Let

$$I = \frac{\ln(x)}{\ln(\lambda) + \ln(x)} \int_1^{\lambda x} \frac{h(t)}{t} dt - \int_1^x \frac{h(t)}{t} dt. \tag{21}$$

First observe that

$$\begin{aligned} \frac{\ln(x)}{\ln(\lambda) + \ln(x)} \int_1^{\lambda x} \frac{h(t)}{t} dt = \\ \int_1^x \frac{h(t)}{t} dt + \int_x^{\lambda x} \frac{h(t)}{t} dt - \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^{\lambda x} \frac{h(t)}{t} dt, \end{aligned} \tag{22}$$

so

$$I = \int_x^{\lambda x} \frac{h(t)}{t} dt - \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^{\lambda x} \frac{h(t)}{t} dt. \tag{23}$$

Further,

$$\int_1^{\lambda x} \frac{h(t)}{t} dt = \int_1^x \frac{h(t)}{t} dt + \int_x^{\lambda x} \frac{h(t)}{t} dt \tag{24}$$

and

$$\int_x^{\lambda x} \frac{h(t)}{t} dt \leq M \int_x^{\lambda x} \frac{dt}{t} = M \ln(\lambda). \tag{25}$$

Hence,

$$\frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_x^{\lambda x} \frac{h(t)}{t} dt \leq \frac{M \ln(\lambda)^2}{\ln(\lambda) + \ln(x)} \rightarrow 0 \quad \text{as } x \rightarrow +\infty. \tag{26}$$

Hence, by (23) and (26)

$$I = \int_x^{\lambda x} \frac{h(t)}{t} dt - \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^x \frac{h(t)}{t} dt + \varepsilon(x), \tag{27}$$

where $\varepsilon(x) \rightarrow 0$ as $x \rightarrow +\infty$. By (19) and the assumption that this convergence is uniform with respect to λ , there is $\xi(x)$ which does not depend on λ so that $I = \xi(x)$ and $\xi(x) \rightarrow 0$ as $x \rightarrow +\infty$. Hence

$$\int_x^{\lambda x} \frac{h(t)}{t} dt - \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^x \frac{h(t)}{t} dt + \varepsilon(x) = \xi(x), \tag{28}$$

so

$$\int_x^{\lambda x} \frac{h(t)}{t} dt \approx \frac{\ln(\lambda)}{\ln(\lambda) + \ln(x)} \int_1^x \frac{h(t)}{t} dt. \tag{29}$$

□

The following theorem gives us a better insight in the convergence of the limit appearing in the previous theorem.

Theorem 3.6. Let $f(x)$ be a measurable function defined on $[a, +\infty)$ for some real number a and $S \subseteq \mathcal{R}^+$ a measurable set of positive measure. If for all $\lambda \in S$

$$\lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0, \tag{30}$$

then (30) holds for all $\lambda \in \mathcal{R}^+$.

In the proof of the theorem we follow ideas presented in [16] and it is achieved by proving next lemmas.

Lemma 3.7. *Let $f(x)$ and S be as in the theorem 3.6 and suppose (30) for all $\lambda \in S$. Then there are $a, b \in \mathcal{R}^+$ such that $a < b$ and $[a, b] \subseteq S$.*

Proof of Lemma We show

$$\lambda, \mu \in S \text{ implies } \lambda\mu \in S. \tag{31}$$

Suppose $\lambda, \mu \in S$. In the following expression

$$(f(\lambda\mu x) - f(x)) \ln(x) = (f(\lambda\mu x) - f(\lambda x)) \ln(x) + (f(\lambda x) - f(x)) \ln(x) \tag{32}$$

we have $\lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0$. Further,

$$\lim_{x \rightarrow +\infty} (f(\lambda\mu x) - f(\lambda x)) \ln(\lambda x) = \lim_{t \rightarrow +\infty} (f(\mu t) - f(t)) \ln(t) = 0, \tag{33}$$

and for $\lambda > 1$, $\ln(x) < \ln(\lambda x)$, so

$$|(f(\lambda\mu x) - f(\lambda x)) \ln(x)| \leq |(f(\lambda\mu x) - f(\lambda x)) \ln(\lambda x)|, \tag{34}$$

so $\lim_{x \rightarrow +\infty} |(f(\lambda\mu x) - f(\lambda x)) \ln(x)| = 0$. By (32) it follows

$$\lim_{x \rightarrow +\infty} (f(\lambda\mu x) - f(x)) \ln(x) = 0 \tag{35}$$

so, $\lambda\mu \in S$. Hence, S is closed under multiplication, therefore, by Steinhaus lemma [17], there are $a, b \in \mathcal{R}^+$, $a < b$, so that $[a, b] \subseteq S$. \square

In the next lemma we show that the convergence interval $[a, b]$ can be expanded to $(0, +\infty)$.

Lemma 3.8. *Let $f(x)$ be as in the theorem and suppose (30) for all $\lambda \in [a, b]$ for some $0 < a < b$. Then (30) holds for all $\lambda \in \mathcal{R}^+$.*

Proof of Lemma Let $\lambda \in [a, b]$ and $\mu \in \mathcal{R}^+$ such that $a \leq \lambda/\mu \leq b$. Further,

$$\begin{aligned} (f(\lambda x) - f(x)) \ln(x) &= \left(f\left(\mu \frac{\lambda x}{\mu}\right) - f\left(\frac{\lambda x}{\mu}\right) \right) \ln(x) + \\ &\quad \left(f\left(\frac{\lambda x}{\mu}\right) - f(x) \right) \ln(x). \end{aligned} \tag{36}$$

By assumptions on λ and μ , we have

$$\lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0, \quad \lim_{x \rightarrow +\infty} \left(f\left(\frac{\lambda x}{\mu}\right) - f(x) \right) \ln(x) = 0, \tag{37}$$

hence, by (36)

$$\lim_{x \rightarrow +\infty} \left(f\left(\mu \frac{\lambda x}{\mu}\right) - f\left(\frac{\lambda x}{\mu}\right) \right) \ln(x) = 0 \tag{38}$$

For $x > \lambda/\mu$ we have

$$\left| \left(f\left(\mu \frac{\lambda x}{\mu}\right) - f\left(\frac{\lambda x}{\mu}\right) \right) \ln(x) \right| \geq \left| \left(f\left(\mu \frac{\lambda x}{\mu}\right) - f\left(\frac{\lambda x}{\mu}\right) \right) \ln(\lambda/\mu) \right|, \tag{39}$$

so $\lim_{x \rightarrow +\infty} (f(\mu \frac{\lambda x}{\mu}) - f(\frac{\lambda x}{\mu})) \ln(\lambda/\mu) = 0$. But

$$\begin{aligned} \lim_{t \rightarrow +\infty} (f(\mu t) - f(t)) \ln(t) &= \lim_{x \rightarrow +\infty} (f(\mu \frac{\lambda x}{\mu}) - f(\frac{\lambda x}{\mu})) \ln(\frac{\lambda x}{\mu}) = \\ &= \lim_{x \rightarrow +\infty} (f(\mu \frac{\lambda x}{\mu}) - f(\frac{\lambda x}{\mu})) \ln(x) + \\ &= \lim_{x \rightarrow +\infty} (f(\mu \frac{\lambda x}{\mu}) - f(\frac{\lambda x}{\mu})) \ln(\frac{\lambda}{\mu}) = 0 \end{aligned} \tag{40}$$

i.e. $\lim_{t \rightarrow +\infty} (f(\mu t) - f(t)) \ln(t) = 0$ for $a/b \leq \mu \leq b/a$. Hence we proved

$$\begin{aligned} \bigwedge_{a \leq \lambda \leq b} \lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0 \text{ implies} \\ \bigwedge_{a/b \leq \lambda \leq b/a} \lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0 \end{aligned} \tag{41}$$

Iterating (41) n times, we obtain for arbitrary positive integer n

$$\bigwedge_{(\frac{b}{a})^n \leq \lambda \leq (\frac{a}{b})^n} \lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0. \tag{42}$$

As $a/b < 1$ and $b/a > 1$ and so $\lim_{n \rightarrow +\infty} (a/b)^n = 0$ and $\lim_{n \rightarrow +\infty} (b/a)^n = +\infty$, we infer

$$\bigwedge_{\lambda \in \mathcal{R}^+} \lim_{x \rightarrow +\infty} (f(\lambda x) - f(x)) \ln(x) = 0. \tag{43}$$

□

Combining lemmas 3.7 and 3.8, we obtain a proof of Theorem 3.6.

The following theorem gives us one interesting property of slowly varying functions.

Theorem 3.9. Assume $L(x)$ is a slowly varying function. Then there are measurable functions $\xi(x) \in \mathcal{Z}_0$ and $g(x)$, so that $L(x) = g(x)x^{\xi(x)}$, where $g(x) \rightarrow g_0$ as $x \rightarrow +\infty$, g_0 is a real positive constant.

Proof. Suppose that $L(x)$ is a slowly varying function. By integral representation theorem for SV functions, it follows that there are measurable functions $g(x)$, $\varepsilon \in \mathcal{Z}_0$ and $b \in \mathcal{R}$ so that

$$L(x) = g(x)e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geq b, \tag{44}$$

and $g(x) \rightarrow g_0$ as $x \rightarrow +\infty$, g_0 is a real positive constant, wherefrom for $b = 1$ we directly infer

$$L(x) = g(x)x^{\mathcal{L}(\varepsilon)(x)}. \tag{45}$$

Since $\varepsilon(x) \in \mathcal{Z}_0$, by Proposition 3.4 follows that $\mathcal{L}(\varepsilon)(x) \in \mathcal{Z}_0$, what proves the theorem. □

The converse does not hold. If we suppose that $f(x) = g(x)x^{\xi(x)}$ is a SV function, where $g(x) \rightarrow g_0$ as $x \rightarrow +\infty$, g_0 is a real positive constant, then for all $\lambda > 0$ we have

$$1 = \lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{g(\lambda x)(\lambda x)^{\xi(\lambda x)}}{g(x)x^{\xi(x)}} = \lim_{x \rightarrow +\infty} \frac{g(\lambda x)\lambda^{\xi(\lambda x)}x^{\xi(\lambda x)}}{g(x)x^{\xi(x)}}. \tag{46}$$

Since $\lambda > 0$, then $\lim_{x \rightarrow +\infty} g(\lambda x) = \lim_{x \rightarrow +\infty} g(x) = g_0$, and $\lim_{x \rightarrow +\infty} \xi(\lambda x) = \lim_{x \rightarrow +\infty} \xi(x) = 0$. Therefore, the limit value in (46) depends only on the limit $\lim_{x \rightarrow +\infty} \frac{x^{\xi(\lambda x)}}{x^{\xi(x)}}$. Furthermore we have

$$1 = \lim_{x \rightarrow +\infty} \frac{x^{\xi(\lambda x)}}{x^{\xi(x)}} = \lim_{x \rightarrow +\infty} x^{\xi(\lambda x) - \xi(x)} = \lim_{x \rightarrow +\infty} \exp((\xi(\lambda x) - \xi(x)) \ln(x)), \tag{47}$$

wherefrom for every $\lambda > 0$

$$\lim_{x \rightarrow +\infty} (\xi(\lambda x) - \xi(x)) \ln(x) = 0. \tag{48}$$

However, that does not need to be the case, as the following example shows. Let $\xi(x) = \sin(x)/\ln(x)$ and $\lambda = \pi$. Taking the limit (48) over positive integers n , we obtain

$$\lim_{n \rightarrow +\infty} (\sin(\pi n)/\ln(\pi n) - \sin(n)/\ln(n)) \ln(n) = \lim_{n \rightarrow +\infty} (-\sin(n)), \tag{49}$$

which does not exist, contradicting (48). Therefore, not all functions $f(x) = g(x)x^{\xi(x)}$, where $g(x) \rightarrow g_0$ as $x \rightarrow +\infty$, g_0 is a real positive constant, have to be slowly varying. Next proposition gives a sufficient and necessary condition for representation of normalized SV functions in Theorem 3.9.

Theorem 3.10. *Let $\xi \in \mathcal{Z}_0$, $0 < a < b$ and $I = [a, b]$. Then $F(x) = x^{\xi(x)}$ is a SV function if and only if*

$$\lim_{x \rightarrow +\infty} (\xi(\lambda x) - \xi(x)) \ln(x) = 0 \tag{50}$$

uniformly with respect to $\lambda \in I$.

Proof. First we prove Claim: Suppose $H(x, u)$ is a real function and $\lim_{x \rightarrow +\infty} H(x, u) = 1$ uniformly with respect to $u \in I \subseteq \mathcal{R}$. Then

$$\lim_{x \rightarrow +\infty} \ln(H(x, u)) = 0 \tag{51}$$

uniformly with respect to $u \in I$.

Suppose $\lim_{x \rightarrow +\infty} H(x, u) = 1$ uniformly with respect to $u \in I$. Then for positive infinite x and $u \in I^*$, $H^*(x, u) = 1 + \varepsilon$, $\varepsilon \in \mu(0)$, and so $\ln(H^*(x, u)) = \varepsilon'$ for some $\varepsilon' \in \mu(0)$. Therefore, by Lemma 2.1 the convergence in question of $\ln(H(x, u))$ is uniform. \square

Now we proceed to the proof of Theorem 3.10.

(\Rightarrow) Assume $F(x) = x^{\xi(x)}$ is a SV function. Then

$$\lim_{x \rightarrow +\infty} F(\lambda x)/F(x) = 1 \quad \text{for all } \lambda \in \mathcal{R}^+, \tag{52}$$

i.e. $\lim_{x \rightarrow +\infty} \lambda^{\xi(\lambda x)} e^{(\xi(\lambda x) - \xi(x)) \ln(x)} = 1$. By Theorem 1.1 (Karamata Uniform Convergence Theorem) it follows $\lim_{x \rightarrow +\infty} e^{(\xi(\lambda x) - \xi(x)) \ln(x)} = 1$ uniformly with respect to $\lambda \in I$. By Claim it follows that $\lim_{x \rightarrow +\infty} (\xi(\lambda x) - \xi(x)) \ln(x) = 0$ uniformly with respect to $\lambda \in I$.

(\Leftarrow) Assume (50) uniformly with respect to $\lambda \in I$. Then for positive infinite x and $\lambda \in I^*$, $\xi(\lambda x) = \eta$ and due to the uniform convergence by Lemma 2.1, $(\xi(\lambda x) - \xi(x)) \ln(x) = \varepsilon$ are infinitesimals, so

$$\frac{F(\lambda x)}{F(x)} = \lambda^{\xi(\lambda x)} e^{(\xi(\lambda x) - \xi(x)) \ln(x)} = \lambda^\eta e^\varepsilon \approx 1. \tag{53}$$

Hence $\lim_{x \rightarrow +\infty} F(\lambda x)/F(x) = 1$ and in fact, by Theorem 1.1, this convergence is uniform with respect to $\lambda \in I$. \square

With a simple modification of above proof, one can prove a variant of the previous theorem for functions of the form $F(x) = g(x)x^{\xi(x)}$, where $g(x) \rightarrow 1$ as $x \rightarrow +\infty$ and $\xi \in \mathcal{Z}_0$. We also note that the previous theorem is still valid if the uniformity condition is omitted. Then Theorem 3.10 would follow from Karamata Uniform Convergence Theorem.

Example. We illustrate the methods presented in the paper and a use of Theorem 3.10 for construction of a SV function which varies unboundedly between zero and positive infinity. Erdős gave an example of such

kind, $f(x) = \exp((\ln x)^{\frac{1}{3}} \cos((\ln x)^{\frac{1}{3}}))$, see [1]. As far as we know this is the only example of such functions appearing in the literature.

Before we proceed to the construction let us note few facts about the function $F(x) = x^{\xi(x)}$. If $\xi(x)$ is measurable on $[a, +\infty]$ for some $a > 0$, then obviously $g(x) = \ln(x)\xi(x)$ is also measurable. Then we see that the composition $F(x) = e^{\ln(x)\xi(x)}$ of the exponential function and the measurable function $g(x)$ is also measurable. Further, $F(x)$ is positive for $x > 0$ so the positivity and measurability assumptions for $F(x)$ in definition of regularly varying functions are fulfilled. The form $F(x) = x^{\xi(x)}$ gives us more freedom in the construction that follows, as $\xi(x)$ may take also the negative values. We also note that if $F(x)$ is SV function, then $\lim_{x \rightarrow +\infty} \xi(x) = 0$. This follows from $\xi(x) = \ln(F(x))/\ln(x)$ and (18). Therefore, the next theorem, a variant of Theorem 3.10, is meaningful and we shall use it in the construction that follows:

Theorem 3.11. *Let $\xi(x)$ be a real-valued function measurable on $[a, +\infty]$ for some $a > 0$. Then $F(x) = x^{\xi(x)}$ is a SV function if and only if*

$$\lim_{x \rightarrow +\infty} (\ln(\lambda x)\xi(\lambda x) - \ln(x)\xi(x)) = 0, \quad \lambda \in R^+. \tag{54}$$

The theorem immediately follows from the limit condition for SV functions. Also, we shall use the following properties of trigonometric and logarithmic functions:

Proposition 3.12. *There are real functions $k_1(t), k_2(t), k_3(t), t \in [1, +\infty]_R$, such that $|k_1(t)|, |k_2(t)|, |k_3(t)| \leq 2$ and the following identities hold for all infinitesimals ε :*

$$\begin{aligned} \sin(x + \varepsilon) &= \sin(x) + k_1(\varepsilon)\varepsilon, & x \in R^* \\ \cos(x + \varepsilon) &= \cos(x) + k_2(\varepsilon)\varepsilon, & x \in R^* \\ \ln(x + \varepsilon) &= \ln(x) + k_3(\varepsilon)\varepsilon/x, & x \in [1, +\infty]^* \end{aligned} \tag{55}$$

For nonstandard extensions $k_i(x)^*$ we left the original notation $k_i(x)$. These identities follow from Lagrange Remainder Theorem. For example,

$$\begin{aligned} \sin(x + \varepsilon) &= \sin(x) \cos(\varepsilon) + \cos(x) \sin(\varepsilon) \\ &= \sin(x)(1 + \eta_1\varepsilon) + \cos(x)(\varepsilon + \eta_2\varepsilon) \quad \text{for some } \eta_1, \eta_2 \in \mu(0) \\ &= \sin(x) + (\sin(x)\eta_1 + \cos(x)(1 + \eta_2))\varepsilon. \end{aligned}$$

Obviously $|\sin(x)\eta_1 + \cos(x)(1 + \eta_2)| \leq 2$, so the identity for $\sin(x)$ in (55) holds. Using $\ln(x + \varepsilon) = \ln(x) + \ln(1 + \varepsilon/x)$ we infer the identity for $\ln(x)$ in (55) in the similar way.

We shall use in construction an M -bounded function $b(x), M > 0$, which satisfies

$$b(x + \varepsilon) = b(x) + k_b(\varepsilon)\varepsilon, \quad x \in R^*, \quad \varepsilon \in \mu(0), \tag{56}$$

where $k_b(x)$ is m -bounded for some $m \in R^+$. As $b(x)$ satisfies (56), $b(x)$ is continuous, hence it is measurable, too.

Proposition 3.13. *Let*

$$F(x) = e^{\ln(\ln(x))b(\ln(\ln(x)))}, \quad b(x) \text{ is an } M\text{-bounded function satisfying (56)}. \tag{57}$$

Then $F(x)$ is a SV function.

Proof. To simplify notation we set $u = \ln(x), w = \ln(u) = \ln \ln(x), \varepsilon = \ln(\lambda)/\ln(x)$ and omit $*$ in nonstandard extensions, for example we write $b(x)$ for $b^*(x)$. We see that

$$F(x) = x^{\frac{\ln(\ln(x))b(\ln(\ln(x)))}{\ln(x)}},$$

so

$$\xi(x) = \frac{\ln(\ln(x))b(\ln(\ln(x)))}{\ln(x)}.$$

By Theorem 3.11 it suffices to prove that $\lim_{x \rightarrow +\infty} \sigma(x) = 0$ for

$$\sigma(x) = \ln(\lambda x)\xi(\lambda x) - \ln(x)\xi(x) = \ln(\ln(\lambda x))b(\ln(\ln(\lambda x))) - \ln(\ln(x))b(\ln(\ln(x))),$$

i.e that for each $\lambda \in R^+$ and positive infinite $x \in R^*$, $\sigma(x) \approx 0$. So assume x is positive infinite. Then $\varepsilon \approx 0$, so

$$\ln(\ln(\lambda x)) = \ln(\ln(x)) + \ln(1 + \ln(\lambda)/\ln(x)) = w + \ln(1) + k_3(\varepsilon)\varepsilon = w + k_3(\varepsilon)\varepsilon. \tag{58}$$

As $k_3(\varepsilon)\varepsilon \approx 0$, we have

$$b(\ln(\ln(\lambda x))) = b(w + k_3(\varepsilon)\varepsilon) = b(w) + k_b(k_3(\varepsilon)\varepsilon)k_3(\varepsilon)\varepsilon = b(w) + k_0(\varepsilon)\varepsilon, \tag{59}$$

where $k_0(t) = k_3(t) \cdot k_b(k_3(t)t)$ is a $2m$ -bounded function. Hence

$$\begin{aligned} \sigma(x) &= (w + k_3(\varepsilon)\varepsilon)(b(w) + k_0(\varepsilon)\varepsilon) - wb(w) \\ &= \varepsilon b(w)k_3(\varepsilon) + \varepsilon w k_0(\varepsilon) + \varepsilon^2 k_0(\varepsilon)k_3(\varepsilon) \end{aligned} \tag{60}$$

As $\varepsilon, \varepsilon^2 \approx 0$ and $b(w)k_3(\varepsilon), k_0(\varepsilon)k_3(\varepsilon)$ are finite we have

$$\varepsilon b(w)k_3(\varepsilon), \varepsilon^2 k_0(\varepsilon)k_3(\varepsilon) \approx 0. \tag{61}$$

We have

$$\varepsilon w = \frac{\ln(\lambda) \ln \ln(x)}{\ln(x)} \approx 0,$$

hence $\sigma(x) \approx 0$, and the proof of the proposition is finished. □

If we take $b(x) = \sin(x)$, then

$$F(x) = x^{\frac{\ln \ln(x) \sin(\ln \ln(x))}{\ln(x)}} = e^{\ln \ln(x) \sin(\ln \ln(x))} \tag{62}$$

is a SV function for which obviously holds

$$\liminf_{x \rightarrow +\infty} F(x) = 0, \quad \limsup_{x \rightarrow +\infty} F(x) = +\infty. \tag{63}$$

Hence, $F(x)$ is a SV function varying between zero and positive infinity.

We also note that we in fact defined the whole class of such functions, as for $b(x)$ we can take for example any trigonometric polynomial.

Conclusion

We used methods of nonstandard analysis in order to obtain proof of a generalization of Karamata uniform convergence theorem for slowly varying functions. We introduced operator \mathcal{L} and proved its several properties. Furthermore, the connection between the operator \mathcal{L} and slowly varying functions is derived. Moreover, some properties of slowly varying functions are obtained.

References

- [1] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation* (Encyclopedia of Mathematics and its Applications), Cambridge Univ. Press, Cambridge, 1987.
- [2] E. Bottazzi, *Describing limits of integrable functions as grid functions of nonstandard analysis*, *Partial Differ. Equ. Appl.* **2** (2021) 51.
- [3] C. C. Chang, H. J. Keisler, *Model Theory*, (3rd edition), North Holland, 1990.
- [4] M. Davis, *Applied Nonstandard Analysis*, Dover Publications, 2005.
- [5] M. Di Nasso, I. Goldbring, M. Lupini, *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory*, Lecture Notes in Mathematics, Springer, 2019.
- [6] H. Duanmu, J. S. Rosenthal, W. Weiss, *Ergodicity of Markov Processes via Nonstandard Analysis*, *Mem. Amer. Math. Soc.* **273** (2021).
- [7] J. Karamata, *Sur un mode de croissance régulière des fonctions*, *Math. (Cluj)* **4** (1930), 38–53.
- [8] A. R. Liddle, D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, Cambridge Univ. Press, Cambridge, 2000.
- [9] Ž. Mijajlović, N. Pejović, *Non-Archimedean Methods in Cosmology*, *AIP Conf. Proc.* **895** (2007), 317–320.
- [10] Ž. Mijajlović, N. Pejović, S. Ninković, *Nonstandard Representations of Processes in Dynamical Systems*, *AIP Conf. Proc.* **934** (2007), 151–157.
- [11] Ž. Mijajlović, N. Pejović, S. Šegan, G. Damjanović, *On asymptotic solutions of Friedmann equations*, *Appl. Math. Comput.* **219** (2012), 1273–1286.
- [12] Ž. Mijajlović, D. Arandjelović, M. Rašković, R. Djordjević, *Nestandardna Analiza*, Matematički fakultet Beograd, Beograd, 2014.
- [13] Ž. Mijajlović, N. Pejović, V. Marić, *On the ϵ cosmological parameter*, *Serb. Astron. J.* **190** (2015), 25–31.
- [14] Ž. Mijajlović, N. Pejović, V. Radović, *Asymptotic solution for expanding universe with matter-dominated evolution*, *Int. J. Geom. Methods Mod. Phys.* **16** (2019), 1950063.
- [15] S. Sanders, *The unreasonable effectiveness of nonstandard analysis*, *J. Log. Comput.* **30** (2020), 459–524.
- [16] E. Seneta, *Regularly Varying Functions*, Springer Berlin, Heidelberg, 1976.
- [17] H. Steinhaus, *Sur les distances des points dans les ensembles de mesure positive*, *Fund. Math.* **1** (1920), 93–104.
- [18] I. Stern, *On Fractal Modeling in Astrophysics: The Effect of Lacunarity on the Convergence of Algorithms for Scaling Exponents*, *Astron. Data Anal. Software and Systems VI, ASP Conference Series*, **125** (1997), 222–225, ed. G. Hunt and H.E. Payne.
- [19] K. D. Stroyan, W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.