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Construction of fractional framelets in $L^2(\mathbb{R})$

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Abstract. Framelets generalize orthogonal wavelets by adding the desired properties of redundancy in their systems and flexibility in their construction. These extra features greatly improve their performance over orthogonal wavelets in applications such as image denoising and data processing. The main objective of this paper is to study fractional framelets associted with the fractional refinable functions that are obtained via unitary extension principles. Furthermore all the possible solutions of the matrix equations that arise in the study are obtained. Towards the end it is shown that the problem of extension has always a solution with two fractional framelets.

1. Introduction.

Fourier transform is one of the most valuable and frequently used tools in signal processing and analysis. For Fourier transform, a signal can be represented either in the time or in the frequency domain, and it can be viewed as the time-frequency representation of a signal. In 1980, Victor Namias [23] introduced the concept of fractional Fourier transform (FrFT) as a generalization of the conventional Fourier transform to solve certain problems arising in quantum mechanics. It is also referred as *rotational Fourier transform* or *angular Fourier transform* since it depends on a parameter α which is interpreted as a rotation by an angle α in the time-frequency plane. Like the ordinary Fourier transform corresponds to a rotation in the time frequency plane over an angle $\alpha = 1 \times \pi/2$, the FrFT corresponds to a rotation over an arbitrary angle $\alpha = \rho \times \pi/2$ with $\rho \in \mathbb{R}$.It has applications in different fields like quantum mechanics [23], optics [25, 26], signal processing [17, 22, 30, 34, 37], and image processing [18, 35, 36]. Although the FrFT has a number of attractive properties, the fractional Fourier representation of a signal only provides overall FrFD- frequency content with no indication about the occurrence of the FrFD spectral component at a particular time. Since the FrFT uses a global kernel like Fourier transform, it fails in locating the FrFD spectral contents which is required in some applications. The concept of FrWT was initially proposed in [21], where FrFT is firstly used to derive the fractional spectrum of a signal and wavelet transform is then performed on the obtained fractional spectrum. Since the fractional spectrum derived by the FrFT only represents the FrFD-frequency over the entire duration of the signal, the FrWT defined in [21] actually fails in obtaining the information of the local property of the signal. In [16], a fractional wave packet transform was developed and the basic

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idea is to introduce the wavelet basis function to FrFT. More recently, a new FrWT was proposed in [32] based on the concept of fractional convolution. In [31], the notion of fractional wavepacket systems in $L^2(\mathbb{R})$ is introduced and the correponding frames are characterized.

Multiresolution analysis is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2\left(\mathbb{R}^d\right)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. All these concepts are developed on regular lattices, that is the translation set is always a group. In the heart of any MRA, there lies the concept of scaling functions. Cifuentes et al.[11] characterized the scaling function of MRA in a general settings .The multiresoltion analysis whose scaling functions are characteristic functions some elementary properties of MRA of $L^2(\mathbb{R}^n)$ are established by Madych [19]. Zhang [38] studied scaling functions of standard MRA and wavelets. Zhang [38] characterized support of the Fourier transform of scaling functions. The multiresolution analysis (MRA) associated with corresponding to FrWT [32] was then given in [33]. Since this kind of FrWT analyze the signal in time-frequency-FrFD domain, its physical meaning requires deeper interpretation. Another kind of FrWT which was developed in [28] solves the issue in [32] since the analysis only involves time-FrFD domain. For more about frames and framelets , we refer to [1–7, 12, 13, 15, 29]. Recently Ahmad et al [8] established the theory of fractional biorthogonal wavelets in $L^2(\mathbb{R})$ and in [12] established the characterization of scaling functions associated with fractional MRA. Motivated ans inspired by the work of Petukhov[27], we in this paper study fractional framelets associted with the fractional refinable functions that are obtained via unitary extension principles. Furthermore all the possible solutions of the matrix equations that arise in the study are obtained. Towards the end it is shown that the problem of extension has always a solutuon with two fractional framelets.

The paper is structured as follows. In section 2, we discuss the preliminaries about the fractional Fourier transform, fractional refinable functions and the corresponding fractional MRA and discuss the main problem of the extension. Section 3 is devoted to showthat the problem of extension has always a solutuon with two fractional framelets.

2. Preliminaries

This section gives the basic background to the theory of fractional Fourier and wavelet transforms which is as follows.

The fractional Fourier transform with parameter α of function $f(t)$ is defined by

$$\mathcal{F}_\alpha\left\{f(t)\right\}(\xi) = \hat{f}^\alpha(\xi) = \int_{-\infty}^{\infty} \mathcal{K}_\alpha(t,\xi) f(t)\,dt, \tag{2.1}$$

where $\mathcal{K}_\alpha(t,\xi)$ is called kernel of the FrFT given by

$$\mathcal{K}_\alpha(t,\xi) = \begin{cases} C_\alpha \exp\left\{i(t^2+\xi^2)\dfrac{\cot\alpha}{2} - it\xi \csc\,\alpha\right\}, & \alpha \neq n\pi, \\ \delta(t-\xi), & \alpha = 2n\pi, \\ \delta(t+\xi), & \alpha = (2n\pm 1)\pi, \end{cases} \tag{2.2}$$

$\alpha = \rho\pi/2$ denotes the rotation angle of the transformed signal for FrFT, the FrFT operator is designated by $\mathcal{F}_\alpha$ and

$$C_\alpha = (2\pi i \sin\alpha)^{-1/2} e^{i\alpha/2} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}. \tag{2.3}$$

The corresponding inversion formula is given by

$$f(t) = \int_{-\infty}^{\infty} \overline{\mathcal{K}_\alpha(t,\xi)}\, \hat{f}^\alpha(\xi)\, d\xi, \tag{2.4}$$

where

$$\begin{aligned}\mathcal{K}_\alpha(t,\xi) &= \frac{(2\pi i \sin\alpha)^{1/2}\, e^{-i\alpha/2}}{\sin\alpha} \cdot \exp\left\{\frac{-i(t^2+\xi^2)\cot\alpha}{2} + it\xi \csc\,\alpha\right\}\\ &= \overline{C_\alpha} \exp\left\{\frac{-i(t^2+\xi^2)\cot\alpha}{2} + it\xi \csc\,\alpha\right\}\\ &= \mathcal{K}_{-\alpha}(t,\xi)\end{aligned} \tag{2.5}$$

and

$$C_\alpha = \frac{(2\pi i \sin\alpha)^{1/2} e^{-i\alpha/2}}{2\pi \sin\alpha} = \sqrt{\frac{1+i\cot\alpha}{2\pi}} = C_{-\alpha}. \tag{2.6}$$

Definition 2.1 Let $\varphi \in L^2(\mathbb{R})$ be a real valued function satisfying

(i) $\Theta_\alpha(2u) = \Lambda_\alpha(u)\Theta_\alpha(u)$, where Λ_α is essentially $2\pi\sin\alpha$- periodic function and Θ_α is the FrFT of φ.

(ii) $\lim_{u\to 0} \Theta_\alpha(u) = \dfrac{1}{\sqrt{2\pi\sin\alpha}}$,

then the function φ is called *fractional refinable or fractional scaling*, Λ_α is called *symbol* of φ and the relation (i) is called *fractional refinement equation*.

Every fractional refinable function generates a fractional multiresolution analysis of the space $L^2(\mathbb{R})$ i.e, a sequence of closed subspaces $\{V_j^\alpha\} \in L^2(\mathbb{R})$ such that

(a) $V_j^\alpha \subseteq V_{j+1}^\alpha,\;\; j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j^\alpha$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j^\alpha = \{0\}$;

(d) $f(t) \in V_j^\alpha$ if and only if $f(2t)\exp\left\{\frac{i}{2}\left[(2t)^2 - t^2\right]\cot\alpha\right\} \in V_{j+1}^\alpha,\;\; j \in \mathbb{Z}$.

The most popular method for the design of orthogonal and biorthogonal wavelets is based on the construction of fractional MRA of the space $L^2(\mathbb{R})$, generated with a given fractional refinable function. It is well known fact that if the system $\left\{\varphi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\} : n \in \mathbb{Z}\right\}$ constitute a Reisz basis of the space V_0^α, then there exists a fractional refinable function $\phi \in V_0^\alpha$ with the symbol $\Lambda_{\alpha,\phi}$ such that the system $\left\{\phi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\} : n \in \mathbb{Z}\right\}$ forms an orthonormal basis of the space V_0^α.

If we use the notation W_j^α to denote the orthogonal complement of the space V_j^α in the space V_{j+1}^α then the wavelet function ψ, defined by the relation

$$\mathcal{F}_\alpha\{\psi\}(2u) = \Lambda_{\alpha,\psi}(u)\mathcal{F}_\alpha\{\phi\}(u),$$

where $\Lambda_{\alpha,\psi}(u) = \exp\{-iu\csc\alpha\}\,\Lambda_{\alpha,\phi}\,(u + \pi\sin\alpha)$, generates orthonormal basis

$$\left\{\psi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\} : n \in \mathbb{Z}\right\}$$

of the space W_0^α. Thus the system

$$\left\{ 2^{\frac{k}{2}}\phi(2^k t-n)\exp\left\{\frac{-i}{2}\left[t^2-(2^{-k}n)^2-(2^k t-n)^2\right]\cot\alpha\right\}\right\} \tag{1}$$

constitutes an orthonormal basis of the space $L^2(\mathbb{R})$.

We see that if we have a fractional refinable function, generating a Reisz basis then we have an explicit formula for the wavelets associated with the given fractional refinable function. This assures a simple method for constructing wavelets. Generally, any orthonormal basis of $L^2(\mathbb{R})$ of the form (1) is called a *fractional wavelet system*. However, the construction of wavelets based on fractional MRA has an advantage from the point of view effectiveness of computational algorithm and reconstruction.

The problem of finding orthonormal wavelet basis, generated by the fractional scaling function is equivalent to solving the matrix equation

$$M_\alpha(u)M_\alpha^*(u)=I \tag{2}$$

where

$$M_\alpha(u)=\begin{bmatrix}\Lambda_\alpha(u) & \Gamma_\alpha(u)\\ \Lambda_\alpha(u+\pi\sin\alpha) & \Gamma_\alpha(u+2\pi\sin\alpha)\end{bmatrix}$$

and $\Lambda_\alpha(u),\Gamma_\alpha(u)$ are essentially bounded functions $\Lambda_\alpha(-u)=\overline{\Lambda_\alpha(u)}$. It is clear that for any fractional scaling function $\varphi(t)$ and the associated fractional wavelet $\psi(t)$, generating an orthogonal wavelet basis with the corresponding symbols $\Lambda_\alpha(u),\Gamma_\alpha(u)$ satisfying the matrix equation (2). Any fractional refinable function φ, whose symbol Λ_α is a solution of (2), generates a tight frame. we cannot look independently for the functions Λ_α and Γ_α. In fact, we find a solution of the equation

$$|\Lambda_\alpha(u)|^2+|\Lambda_\alpha(u+\pi\sin\alpha)|^2=1, \tag{3}$$

and then all possible functions Γ_α can be represented in the form

$$\Gamma_\alpha(u)=\mathcal{J}_\alpha(u)\exp\{-inu\csc\alpha\}\,\overline{\Lambda_\alpha(u+\imath\sin\alpha)} \tag{4}$$

where $\mathcal{J}_\alpha(u)$ is an arbitrary $\pi\sin\alpha$ - periodic function satisfying $|\mathcal{J}_\alpha(u)|=1$, $\mathcal{J}_\alpha(-u)=\overline{\mathcal{J}_\alpha(u)}$.

Now suppose we have an arbitrary fractional refinable function φ with the symbol Λ_α which does not satisfy (3). Then the set $\left\{\varphi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\}: n\in\mathbb{Z}\right\}$ does not constitute an orthonormal basis of V_0^α. If this set forms a Reisz basis, then we can use orthogonalization. However, in this case when the function φ has a compact support, this property fails for the orthogonalized basis. This argument argues for construction other systems under support compactness.

It is easy to design a fractional refinable function such that the MRA associated with it does not allow othogonalization. If we introduce a fractional refinable function $\varphi(t)=\dfrac{\sin\pi a(t-n)}{\pi(t-n)}\exp\left\{-i\dfrac{(t^2-n^2)}{2}\cot\alpha\right\}$, where $0<a<1$. It generates the space V_0^α which is the space of functions in $L^2(\mathbb{R})$ with Fourier transform supported on $[-2a\pi\sin\alpha,2a\pi\sin\alpha]$. Therefore, for any function $f\in V_0^\alpha$ the function $\sum_{k\in\mathbb{Z}}\left|\mathcal{F}_\alpha\{f\}(u+2k\pi\sin\alpha)\right|^2$ vanishes on the set $[-2\pi\sin\alpha,2\pi\sin\alpha]\setminus[-2a\pi\sin\alpha,2a\pi\sin\alpha]$. Hence, its integral translates do not form an orthonormal bases . Thus in this case the classical procedure for orthonormal basis construction cannot be used. By the same argument we can say that with this MRA a biorthogonal pair cannot be constructed.

In the context when the symbol Λ_α of a fractional refinable function φ does not satisfy (3) we cannot construct an orthonormal bases of V_1^α of the form

$$\left\{\varphi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\},\psi(t-n)\exp\left\{-j(tn+n^2)\cot\alpha\right\}\right\}.$$

However, we can expect that there exists a collection of fractional framelets $\psi^1, \psi^2, \psi^3, ..., \psi^n \in V_1^\alpha$, satisfying the following conditions :

(i) the functions $\left\{\{\psi^\ell_{\alpha,j,k}\}_{j,k\in\mathbb{Z}}\right\}_{\ell=1}^n$, where

$$\psi^\ell_{\alpha,j,k}(t) = 2^{\frac{j}{2}}\psi^\ell(2^jt-n)\exp\left\{\frac{-i}{2}\left[t^2-(2^{-j}n)^2-(2^jt-n)^2\right]\cot\alpha\right\},$$

form a tight frame of the space $L^2(\mathbb{R})$.

(ii) for any $f \in L^2(\mathbb{R})$, decomposition and reconstruction algorithm recurrent formulae are :

$$\left\langle \varphi_{\alpha,j,k}, f\right\rangle = c_{\alpha,j,\ell} = \sum_{k\in\mathbb{Z}} c_{\alpha,j+1,k}\overline{h}_{k-2\ell},$$

$$\left\langle \varphi^g_{\alpha,j,k}, f\right\rangle = d^q_{\alpha,j,\ell} = \sum_{k\in\mathbb{Z}} c_{\alpha,j+1,k}\overline{g}^q_{k-2\ell}, \qquad 1\le q\le n, \tag{5}$$

and

$$c_{\alpha,j+1,\ell} = \sum_{k\in\mathbb{Z}} c_{\alpha,j,k}h_{\ell-k} + \sum_{q=1}^{n}\sum_{k\in\mathbb{Z}} d^q_{\alpha,j,k}g^q_{\ell-k}, \tag{6}$$

where h_k, g^q_k are coefficients of the expansions

$$\Lambda_\alpha(u) = 2^{-1/2}\sum_{k\in\mathbb{Z}} h_k e^{-ik\csc\alpha}$$

and

$$\Lambda^q_\alpha(u) = 2^{-1/2}\sum_{k\in\mathbb{Z}} g^q_k e^{-ik\csc\alpha}, \;\; 1\le q\le n$$

take place.

The main purpose of the next section is to ascertain that this problem can be solved with at most two fractional framelets and to propose explicit formulae for the symbols of fractional framelets.

3. Explicit Formula for Fractional Framelets

Let φ be a fractional refinable function with the symbol $\Lambda_\alpha, \mathcal{F}_\alpha\{\psi^k\}(u) = \Lambda^k_\alpha(u/2)\mathcal{F}_\alpha\{\phi\}(u/2) \in V_1^\alpha$, where each symbol Λ^k_α is a $2\pi\sin\alpha$ - periodic and essentially bounded function for $1\le k\le n$.It is clear that, the following matrix plays an important role for constructing tight frames

$$\mathcal{M}_\alpha(u) = \begin{bmatrix} \Lambda^0_\alpha(u) & \Lambda^1_\alpha(u) & ... & \Lambda^n_\alpha(u) \\ \Lambda^0_\alpha(u+\pi\sin\alpha) & \Lambda^1_\alpha(u+\pi\sin\alpha) & ... & \Lambda^n_\alpha(u+\pi\sin\alpha) \end{bmatrix}$$

It can be easily seen that the matrix equality

$$\mathcal{M}_\alpha(u)\mathcal{M}^*_\alpha(u) = I \tag{7}$$

is equivalent to (5) and (6) and it also implies the tightness of the corresponding frame.

Lemma 3.1. Let the symbols $\left\{\Lambda^k_\alpha\right\}_{k=0}^n$ satisfy equation (7). Then we have

$$\left|\Lambda^\ell_\alpha(u)\right|^2 + \left|\Lambda^\ell_\alpha(u+\pi\sin\alpha)\right|^2 \le 1, \;\; 0\le\ell\le n. \tag{8}$$

Proof. Without loss of generality, it is sufficient to prove inequality (8) only for $\ell = 0$. Here we rewrite the equation (7) in the following form

$$\mathfrak{M}_\alpha(u) := \mathcal{M}_{\alpha,\psi}(u)\mathcal{M}^*_{\alpha,\psi}(u) = \begin{bmatrix} 1 - |\Lambda_\alpha(u)|^2 & -\Lambda_\alpha(u)\overline{\Lambda_\alpha(u + \pi \sin\alpha)} \\ -\overline{\Lambda_\alpha(u)}\Lambda_\alpha(u + \pi\sin\alpha) & 1 - |\Lambda_\alpha(u + \pi\sin\alpha)|^2 \end{bmatrix} \tag{9}$$

where

$$\mathcal{M}_{\alpha,\psi}(u) = \begin{bmatrix} \Lambda^1_\alpha(u) & \Lambda^2_\alpha(u) & \dots & \Lambda^n_\alpha(u) \\ \Lambda^1_\alpha(u + \pi\sin\alpha) & \Lambda^2_\alpha(u + \pi\sin\alpha) & \dots & \Lambda^n_\alpha(u + \pi\sin\alpha) \end{bmatrix}.$$

The above Hermitian matrix $\mathfrak{M}_\alpha(u)$ has eigenvalues $\lambda_1(u) = 1$ and $\lambda_2(u) = 1 - |\Lambda_\alpha(u)|^2 - |\Lambda_\alpha(u + \pi\sin\alpha)|^2$. By using (9), $\mathfrak{M}_\alpha(u)$ is positive definite matrix. Hence $\lambda_2(u) \geq \mathbf{0}$, which is the equation (8) for $\ell = 0$. □

Lemma 3.2. If $\Phi \in L^2(\mathbb{R})$ is a fractional refinable function with a symbol $\Lambda_\alpha(u)$ that satisfies the condition

$$|\Lambda_\alpha(u)|^2 + |\Lambda_\alpha(u + \pi\sin\alpha)|^2 \leq 1, \;\; a.\ e., \tag{10}$$

then $\mathfrak{S}^\alpha_j := \sum_{k\in\mathbb{Z}} \left|\left\langle f, \Phi_{\alpha,j,k}\right\rangle\right|^2 < \infty$ for any function $f \in L^2(\mathbb{R})$ and

$$(i)\ \lim_{j\to\infty} \mathfrak{S}^\alpha_j = \|f\|^2; \qquad (ii)\ \lim_{j\to-\infty} \mathfrak{S}^\alpha_j = 0,$$

where $\Phi_{\alpha,j,k} = 2^{\frac{j}{2}}\Phi(2^jt - n)\exp\left\{\dfrac{-i}{2}\left[t^2 - (2^{-j}n)^2 - (2^jt - n)^2\right]\cot\alpha\right\}$.

Proof. First we prove the inequality

$$\sum_{k\in\mathbb{Z}} |\mathcal{F}_\alpha\{\Phi\}(u + 2\pi k\sin\alpha)|^2 \leq \frac{1}{2\pi\sin\alpha}. \tag{11}$$

By virtue of (10) and the continuity of $\mathcal{F}_\alpha\{\Phi\}(u)$ at $u = 0$ we have $|\mathcal{F}_\alpha\{\Phi\}(u)| \leq (2\pi\sin\alpha)^{-1/2}$ a.e. Thus, for any positive $\ell \in \mathbb{Z}$ we obtain

$$\begin{aligned}
\sum_{k=-2^\ell}^{2^\ell-1} |\mathcal{F}_\alpha\{\Phi\}(u + 2\pi k\sin\alpha)|^2 &= \sum_{k=-2^\ell}^{2^\ell-1}\prod_{n=1}^{\ell+1} \left|\Lambda_\alpha\left(2^{-n}(u + 2\pi k\sin\alpha)\right)\right|^2 \left|\mathcal{F}_\alpha\{\Phi\}\left(2^{-\ell-1}(u + 2\pi k\sin\alpha)\right)\right|^2 \\
&\leq \frac{1}{2\pi\sin\alpha}\sum_{k=-2^\ell}^{2^\ell-1}\prod_{n=1}^{\ell+1} \left|\Lambda_\alpha\left(2^{-n}(u + 2\pi k\sin\alpha)\right)\right|^2 \\
&\leq \frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^\ell-1}\prod_{n=1}^{\ell+1} \left|\Lambda_\alpha\left(2^{-n}(u + 2\pi k\sin\alpha)\right)\right|^2 \\
&\qquad + \frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^\ell-1}\prod_{n=1}^{\ell+1} \left|\Lambda_\alpha\left(2^{-n}(u + 2\pi(k - 2^\ell)\sin\alpha)\right)\right|^2 \\
&\leq \frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^\ell-1}\prod_{n=1}^{\ell+1} \left|\Lambda_\alpha\left(2^{-n}(u + 2\pi k\sin\alpha)\right)\right|^2
\end{aligned}$$

$$\leq \frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^{\ell-1}-1}\prod_{n=1}^{\ell+1}\left|\Lambda_\alpha\left(2^{-n}(u+2\pi k\sin\alpha)\right)\right|^2$$

$$+\frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^{\ell-1}-1}\prod_{n=1}^{\ell+1}\left|\Lambda_\alpha\left(2^{-n}(u+2\pi(k-2^{\ell-1})\sin\alpha)\right)\right|^2$$

$$\leq \frac{1}{2\pi\sin\alpha}\sum_{k=0}^{2^{\ell-1}-1}\prod_{n=1}^{\ell+1}\left|\Lambda_\alpha\left(2^{-n}(u+2\pi k\sin\alpha)\right)\right|^2$$

$$\leq\cdots\leq\frac{1}{2\pi\sin\alpha}.$$

On applying the Parseval and Plancherel formulae, we have

$$\sum_{k\in\mathbb{Z}}\left|\left\langle f,\Phi_{\alpha,j,k}\right\rangle\right|^2 = 2\pi\sin\alpha 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\infty}\mathcal{F}_\alpha\{f\}(u)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}u)}e^{i2^{-j}ku\csc\alpha}\,du\right|^2$$

$$= 2\pi\sin\alpha 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-2^j\pi\sin\alpha}^{2^j\pi\sin\alpha}\left\{\sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{f\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}\right\}e^{i2^{-j}ku\csc\alpha}\,du\right|^2$$

$$= (2\pi\sin\alpha)^2\int_{-2^j\pi\sin\alpha}^{2^j\pi\sin\alpha}\left|\sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{f\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}\right|^2 du$$

$$= \left(2\pi\sin\alpha F_{\alpha,j}\right)^2, \tag{12}$$

where $F_{\alpha,j} = \sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{f\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}$. We introduce the following sequences of functions

$$\mathcal{F}_\alpha\{g_j\}(u) = \begin{cases}\mathcal{F}_\alpha\{f\}(u) & |u|<2^j\pi\sin\alpha\\ 0 & |u|\geq 2^j\pi\sin\alpha\end{cases}$$

$$h_j = f-g_j,\quad j=0,1,2,\ldots$$

$$\mathcal{G}_{\alpha,j}(u) = \sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{g_j\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}$$

$$\mathcal{H}_{\alpha,j}(u) = \sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{h_j\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}.$$

It is clear that $\|\mathcal{G}_{\alpha,j}\|\to(2\pi\sin\alpha)^{-1/2}\|f\|$ as $j\to\infty$. Further, in view of the equation (11), we have

$$\|\mathcal{H}_{\alpha,j}\|^2 = \int_{-2^j\pi\sin\alpha}^{2^j\pi\sin\alpha}\left|\sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{h_j\}(u+2\pi\sin\alpha 2^jn)\overline{\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)}\right|^2 du$$

$$= \int_{-2^j\pi\sin\alpha}^{2^j\pi\sin\alpha}\left|\sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{h_j\}(u+2\pi\sin\alpha 2^jn)\right|^2\sum_{n\in\mathbb{Z}}\left|\mathcal{F}_\alpha\{\Phi\}(2^{-j}(u+2\pi\sin\alpha 2^jn)\right|^2 du$$

$$\leq \frac{1}{2\pi\sin\alpha}\int_{-2^j\pi\sin\alpha}^{2^j\pi\sin\alpha}\left|\sum_{n\in\mathbb{Z}}\mathcal{F}_\alpha\{h_j\}(u+2\pi\sin\alpha 2^j n)\right|^2 du$$

$$= \frac{1}{2\pi\sin\alpha}\|\mathcal{F}_\alpha\{h_j\}\|^2 \to 0, \quad \textbf{as} \;\; j\to+\infty. \tag{13}$$

Since $\|\mathcal{G}_{\alpha,j}\| - \|\mathcal{H}_{\alpha,j}\| \leq \|\mathcal{G}_{\alpha,j} + \mathcal{H}_{\alpha,j}\| \leq \|\mathcal{G}_{\alpha,j}\| - \|\mathcal{H}_{\alpha,j}\|$, therefore it follows from (12) and (13) that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\left\langle f,\Phi_{\alpha,j,k}\right\rangle\right|^2 &= \left(2\pi\sin\alpha F_{\alpha,j}\right)^2\\ &\to 2\pi\sin\alpha\|\mathcal{F}_\alpha\{f\}\|^2 \;\; \textbf{as} \;\; j\to\infty\\ &= \|f\|^2.\end{aligned}$$

Thus, relation (i) is proved.

Now we shall proceed to establish (ii). We use the notation χ_T the characteristic function of a segment $[-T,T]$ and f_T the function $f\chi_T$. For a fixed $\epsilon>0$, we choose $T>0$ such that $\|f-f_T\|<\epsilon$. Since

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\left\langle f,\Phi_{\alpha,j,k}\right\rangle\right|^2 &\leq \sum_{k\in\mathbb{Z}}\left|\left\langle f_T,\Phi_{\alpha,j,k}\right\rangle\right|^2 + 2\sum_{k\in\mathbb{Z}}\left|\left\langle f-f_T,\Phi_{\alpha,j,k}\right\rangle\right|^2\\ &\leq 2\sum_{k\in\mathbb{Z}}\left|\left\langle f_T,\Phi_{\alpha,j,k}\right\rangle\right|^2 + \frac{\|f-f_T\|}{\pi}\\ &\leq \sum_{k\in\mathbb{Z}}\left|\left\langle f_T,\Phi_{\alpha,j,k}\right\rangle\right|^2 + \epsilon/\pi.\end{aligned}$$

Here we need only to prove that $\lim_{j\to-\infty} 2\sum_{k\in\mathbb{Z}}\left|\left\langle f_T,\Phi_{\alpha,j,k}\right\rangle\right|^2 = 0$. If we assume that $2^jT \leq 1/2$, then the above relation follows from the following chain of arguments

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\left\langle f,\Phi_{\alpha,j,k}\right\rangle\right|^2 &= \sum_{k\in\mathbb{Z}}\left\{\int_{|t|\leq T} f(t)\Phi_{\alpha,j,k}(t)\,dt\right\}^2\\ &\leq \|f\|^2\sum_{k\in\mathbb{Z}}\int_{|t|\leq T}\Phi^2_{\alpha,j,k}(t)\,dt\\ &= \|f\|^2\sum_{k\in\mathbb{Z}}\int_{|t+k|\leq 2^jT}\Phi^2(t)\,dt\\ &= \|f\|^2\sum_{k\in\mathbb{Z}}\int_{\cup_{k\in\mathbb{Z}}[-2^jT+k,2^jT+k]}\Phi^2(t)\,dt\\ &\to 0 \;\; \textbf{as} \;\; j\to-\infty.\Box\end{aligned}$$

Lemma 3.3. If the equation (7) holds, then for any $f\in L^2(\mathbb{R})$ and $J\in\mathbb{Z}$

$$\sum_{k=1}^{n}\sum_{j,\ell\in\mathbb{Z}}\left|\left\langle f,\psi^k_{\alpha,j,\ell}\right\rangle\right|^2 = \sum_{\ell\in\mathbb{Z}}\left|\left\langle f,\phi_{\alpha,J,\ell}\right\rangle\right|^2 + \sum_{k=1}^{n}\sum_{j\geq J}\sum_{\ell\in\mathbb{Z}}\left|\left\langle f,\psi^k_{\alpha,j,\ell}\right\rangle\right|^2 < \infty.$$

Proof. It follows from equation (7) that

$$\sum_{\ell=1}^{k} |\Lambda_\alpha^\ell(u)|^2 = 1,$$

$$\sum_{\ell=1}^{n} \Lambda_\alpha^\ell(u)\overline{\Lambda_\alpha^\ell(u+\pi\sin\alpha)} = 0.$$

We introduce the notations

$$\Delta_1^\alpha = \sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^{L+1}\ell\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi\}\,(2^{-L-1}u+2\pi\ell\sin\alpha)},$$

$$\Delta_2^\alpha = \sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^{L+1}\ell\sin\alpha+2\pi 2^L\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi\}\,(2^{-L-1}u+2\pi\ell\sin\alpha+\pi\sin\alpha)},$$

By analogy with (12), for any $L\in\mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}} \left|\left\langle f,\varphi_{\alpha,L,\ell}\right\rangle\right|^2 + \sum_{k=1}^{n}\sum_{\ell\in\mathbb{Z}} \left|\left\langle f.\psi^k_{\alpha,L,\ell}\right\rangle\right|^2 \\
&= (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \left|\sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^L\ell\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi\}\,(2^{-L}(u+2\pi 2^L\ell\sin\alpha)}\right|^2 du \\
&\quad + (2\pi\sin\alpha)^2 \sum_{k=1}^{n} \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \left|\sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^L\ell\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi^k\}\,(2^{-L}(u+2\pi 2^L\ell\sin\alpha)}\right|^2 du \\
&= (2\pi\sin\alpha)^2 \sum_{k=0}^{n} \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \Bigg|\sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^L\ell\sin\alpha) \\
&\qquad\qquad \times \overline{\Lambda_\alpha^k\,(2^{-L-1}(u+2\pi 2^L\ell\sin\alpha)\,\mathcal{F}_\alpha\{\phi^k\}\,(2^{-L-1}(u+2\pi 2^L\ell\sin\alpha)}\Bigg|^2 du \\
&= (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \left|\Delta_1^\alpha(u)\overline{\Lambda_\alpha^k(2^{-L-1}u)}\right|^2 du \\
&\quad + (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \left|\Delta_2^\alpha(u)\overline{\Lambda_\alpha^k(2^{-L-1}u+\pi\sin\alpha)}\right|^2 du \\
&\quad + (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \Delta_1^\alpha(u)\overline{\Lambda_\alpha^k(2^{-L-1}u)\Delta_2^\alpha(u)}\Lambda_\alpha^k(2^{-L-1}u+\pi\sin\alpha)\,du \\
&\quad + (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \Delta_1^\alpha(u)\overline{\Lambda_\alpha^k(2^{-L-1}u+\pi\sin\alpha)\Delta_2^\alpha(u)}\Lambda_\alpha^k(2^{-L-1}u)\,du \\
&= (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \left|\sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^{L+1}\ell\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi\}\,(2^{-L-1}u+2\pi\ell\sin\alpha)}\right|^2 du \\
&\quad + (2\pi\sin\alpha)^2 \int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha} \Bigg|\sum_{\ell\in\mathbb{Z}} \mathcal{F}_\alpha\{f\}(u+2\pi 2^{L+1}\ell\sin\alpha+2\pi 2^L\sin\alpha)
\end{aligned}$$

$$\times \left.\overline{\mathcal{F}_\alpha\{\phi\}\left(2^{-L-1}u+2\pi\ell\sin\alpha+\pi\sin\alpha\right)}\right|^2 du$$

$$= (2\pi\sin\alpha)^2\int_{-2^L\pi\sin\alpha}^{2^L\pi\sin\alpha}\left|\sum_{\ell\in\mathbb{Z}}\mathcal{F}_\alpha\{f\}(u+2\pi 2^{L+1}\ell\sin\alpha)\overline{\mathcal{F}_\alpha\{\phi\}\left(2^{-L-1}u+2\pi\ell\sin\alpha\right)}\right|^2 du$$

$$= \sum_{\ell\in\mathbb{Z}}\left|\langle f,\varphi_{\alpha,L+1,\ell}\rangle\right|^2<\infty.$$

By invoking Lemma 2.2. we obtain Lemma 2.3.□

As an easy consequence of Lemmas 2.1.-2.3., we have the following theorem.

Theorem 3.1. If the equation (7) holds, then the functions $\{\psi^k\}_{k=1}^n$ generate a tight frame of $L^2(\mathbb{R})$.

Thus, the problem of constructing tight frames, generated by a fractional refinable function can be reduced to finding Λ_α^k, that satisfy the equation (7). Here we shall describe all possible solutions to (7).

Let the symbol Λ_α^0 satisfy (10). The unit vectors of the matrix $\mathfrak{M}(u)$ can be represented in the form

$$\overrightarrow{\nu}_1(u)=\begin{bmatrix}\dfrac{\overline{e^{iu\csc\alpha}\Lambda_\alpha^0(u+\pi\sin\alpha)}}{\mathcal{B}(u)}\\ \\ -\dfrac{\overline{e^{iu\csc\alpha}\Lambda_\alpha^0(u)}}{\mathcal{B}(u)}\end{bmatrix},\qquad \overrightarrow{\nu}_2(u)=\begin{bmatrix}\dfrac{\Lambda_\alpha^0(u)}{\mathcal{B}(u)}\\ \\ \dfrac{\Lambda_\alpha^0(u+\pi\sin\alpha)}{\mathcal{B}(u)}\end{bmatrix},$$

where $\mathcal{B}(u)\neq 0$ is an arbitrary $\pi\sin\alpha-$ periodic measurable functions, satisfying

$$|\mathcal{B}(u)|^2=|\Lambda_\alpha^0(u)|^2+|\Lambda_\alpha^0(u+\pi\sin\alpha)|^2 \ \textbf{a.e.}$$

For definiteness, we can take here the positive root of the right-hand expression. For those u when $\Lambda_\alpha^0(u)=\lambda_\alpha^0(u+\pi\sin\alpha)=0$ the matrix $\mathfrak{M}(u)$ becomes the identity matrix. Therefore, any non-zero vector is its eigenvector. In this case we put $\overrightarrow{\nu}_1(u)=(1,0)^T$, $\overrightarrow{\nu}_2(u)=(0,1)^T$. Thus, we have

$$\mathfrak{M}(u)=\mathcal{P}(u)\mathcal{R}(u)\mathcal{P}^*(u) \tag{14}$$

where

$$\mathcal{P}(u)=\begin{bmatrix}\dfrac{\overline{e^{iu\csc\alpha}\Lambda_\alpha^0(u+\pi\sin\alpha)}}{\mathcal{B}(u)} & \dfrac{\Lambda_\alpha^0(u)}{\mathcal{B}(u)}\\ \\ -\dfrac{\overline{e^{iu\csc\alpha}\Lambda_\alpha^0(u)}}{\mathcal{B}(u)} & \dfrac{\Lambda_\alpha^0(u+\pi\sin\alpha)}{\mathcal{B}(u)}\end{bmatrix}$$

and

$$\mathcal{R}(u)=\begin{bmatrix}1 & 0\\ \\ 0 & 1-|\Lambda_\alpha^0(u)|^2-|\Lambda_\alpha^0(u+\pi\sin\alpha)|^2\end{bmatrix}.$$

We note that eigenvectors are determined up to multiplication by a scalar function of absolute value 1 a.e. we have chosen the normalization convenient for further consideration.

Theorem 3.2. Let a $2\pi\sin\alpha$-periodic function $\Lambda_\alpha^0(u)$ satisfy (10). Then there exists a pair of $2\pi\sin\alpha$ periodic measurable functions $\Lambda_\alpha^1,\Lambda_\alpha^2$ satisfy (7) for $n=2$. Any solution of (7) can be represented in the form of the first row of the matrix

$$\widetilde{\mathcal{M}}_\alpha(u)=\mathcal{P}(u)\sqrt{\mathcal{R}(u)}\mathcal{Q}(u),$$

where $Q(u)$ is an unitary matrix with $\pi \sin\alpha-$ periodic measurable components.

Proof. The matrix $\mathcal{M}_{\alpha,\psi}$ can be represented in the form of its singular decomposition

$$\mathcal{M}_{\alpha,\psi}(u) = \mathcal{U}(u)\mathcal{V}(u)\mathcal{W}(u),$$

where $\mathcal{U}, \mathcal{W}$ are unitary matrices, $\mathcal{V}(u)$ is a non negative diagonal matrix. The representations may differ by multiplication of columns of the matrix U by functions $\gamma_1(u), \gamma_2(u), |\gamma_1(u)| = |\gamma_2(u)| = 1$ and simultaneous multiplication of the rows of the matrix $\mathcal{W}$ by $\gamma_1^{-1}(u)$ and $\gamma_2^{-1}(u)$. Therefore, in view of equations (9) and (14) without loss of generality we can suppose $\mathcal{U}, \mathcal{V} = \sqrt{\mathcal{R}}$. Here we show that we can take any unitary matrix with $\pi \sin\alpha$- periodic elements as above, with $Q(u) = \mathcal{W}(u)$. In fact our choice is restricted to such matrices.

For any 2×2 matrix C, we denote by C_R the matrix with the transposed rows. Further we have

$$\mathcal{M}_{\alpha,\psi}(u + \pi\sin\alpha) = \mathcal{P}(u + \pi\sin\alpha)\mathcal{V}(u + \pi\sin\alpha)\mathcal{W}(u + \pi\sin\alpha)$$

$$= \mathcal{P}_R(u)\mathcal{V}(u)\mathcal{W}(u + \pi\sin\alpha)$$

and

$$\mathcal{M}_{\alpha,\psi\ R} = \mathcal{P}_R(u)\mathcal{V}(u)\mathcal{W}(u).$$

Since $\mathcal{M}_{\alpha,\psi\ R}(u) = \mathcal{M}_{\alpha,\psi}(u + \pi\sin\alpha)$, it means that $\mathcal{W}(u + \pi\sin\alpha) = \mathcal{W}(u)$ for atleast for those u and $u + \pi\sin\alpha$ for which $\lambda_2(u) = \lambda_2(u + \pi\sin\alpha) \neq 0$. If $\lambda_2(u) = \lambda_2(u + \pi\sin\alpha) = 0$, then $\mathcal{M}_{\alpha,\psi}(u)$ does not depend on the choice of the second row of the matrix $\mathcal{W}$, so that we can take an arbitrary value of $\mathcal{W}(u + \pi\sin\alpha)$ and $\mathcal{W}(u)$. In particular, we can assume $\mathcal{W}(u + \pi\sin\alpha) = \mathcal{W}(u)$.□

Declaration of competing interest.

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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