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The structure of the 2-factor transfer digraph common for thin cylinder, torus and Klein bottle grid graphs

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Abstract. We prove that the transfer digraph $\mathcal{D}^*_{C,m}$ needed for the enumeration of 2-factors in the thin cylinder $TnC_m(n)$, torus $TG_m(n)$ and Klein bottle $KB_m(n)$ (all grid graphs of the fixed width m and with $m \cdot n$ vertices), when m is odd, has only two components of order 2^{m-1} which are isomorphic. When m is even, $\mathcal{D}^*_{C,m}$ has $\left\lfloor \dfrac{m}{2} \right\rfloor + 1$ components which orders can be expressed via binomial coefficients and all but one of the components are bipartite digraphs. The proof is based on the application of recently obtained results concerning the related transfer digraph for linear grid graphs (rectangular, thick cylinder and Moebius strip).

1. Introduction

Interest in Hamiltonian paths (or cycles) stems from a variety of applications in chemistry, condensed matter (statistical) physics and biophysics related to polymer melting dynamics, protein folding and study of magnetic systems with $O(n)$ symmetry [4, 8, 9, 12]. Studying Hamiltonian cycles can lead to an understanding of a variety of other models in a given system, and for this reason a great deal of work has been put into their study - primarily in the simplest context of periodic regular lattices [13]. The path planning problem for robots and machine tools [11], as well as security and intellectual property protection by using the microelectrode dot array (MEDA) biochips [10] make the problems of generating and enumerating Hamiltonian paths in grid graphs important in engineering and bioinformatics.

Although the research related to the enumeration of Hamiltonian cycles on special classes of grid graphs of fixed width, such as Cartesian products of paths and/or cycles, was initiated more than thirty years ago [14], there still remain many open questions whose answers should be sought in the structure of so-called *transfer digraphs* - auxiliary digraphs using which the counting of the required objects is performed. For more details see [1–3].

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2-factors are the natural generalization of the concept of Hamiltonian cycles. For a graph G, *2-factor* is defined as a spanning subgraph of G where each vertex has exactly two neighbors. Obviously, it represents the spanning union of cycles. In the special case when we have just one cycle, this 2-factor is called Hamiltonian cycle. The systematic study on 2-factors of the grid graphs with fixed width has recently begun [5–7] in order to help in solving the above mentioned open questions. The first results [5, 6] were obtained for so-called *linear grid graphs* of width $m \in N$ (rectangular grid graphs, thick cylinders and Moebius strips) - the grid graphs whose any subgraph induced by the vertices from the same columns is the path P_m (see Figure 1 (a)-(c)). The subject of interest in this paper are the grid graphs whose any subgraph induced by the vertices from the same columns is the cycle C_m. These are the grid graphs from the title and we refer to them as *circular grid graphs* (see Figure 1 (d)-(f)). Since in the process of forming a torus grid or Klein bottle grid graph, before gluing the ends of the initial tube (thin cylinder grid), one of these ends can be twisted, we obtain more types of such grid graphs. In Figure 1 (e)-(f) the value p $(0 \le p \le m-1)$ determines the type of the grid graph.

Definition 1.
*The **rectangular (grid) graph** $RG_m(n)$, **thin (grid) cylinder** $TnC_m(n)$ and **thick (grid) cylinder** $TkC_m(n)$ $(m, n \in N)$ are $P_m \times P_n$, $C_m \times P_n$ and $P_m \times C_n$, respectively.*

*The **Moebius strip** $MS_m(n)$ is obtained from $RG_m(n+1) = P_m \times P_{n+1}$ by identification of corresponding vertices from the first and last column in the opposite direction without duplicating edges.*

*The **torus (grid)** $TG_m^{(p)}(n)$ $(0 \le p \le m-1)$ is the graph obtained from $TnC_m(n+1)$ by identification of the vertices B_i and D_{i+p}, $i = 1, \ldots, m$, without duplicating edges, where B_i and D_i denote the vertices belonging to the i-th row $(1 \le i \le m)$ from the first and last column, respectively (the sign + in subscript is addition modulo m).*

*The **Klein bottle** $KB_m^{(p)}(n)$ is the graph obtained from $TnC_m(n+1)$ by identification of the vertices B_i and $D_{m-i+p+1}$, $i = 1, \ldots, m$, without duplicating edges.*

*The value $m \in N$ is called the **width** of the grid graph. The grid graphs $RG_m(n)$, $TkC_m(n)$ and $MS_m(n)$ are uniformly called the **linear grid graphs**, whereas $TnC_m(n)$, $TG_m^{(p)}(n)$ and $KB_m^{(p)}(n)$ are called the **circular grid graphs**.*

The 2-factor of the Klein bottle $KB_4^{(1)}(3)$ depicted in Figure 2 (a) consists of 2 cycles, while the one of the torus grid $TG_4^{(0)}(3)$ in Figure 2 (b) has only one cycle and hence it is Hamiltonian cycle.

Observe one of the above defined grid graphs, G and one of its 2-factors. Since any vertex $v \in V(G)$ is incident with exactly two edges of the 2-factor, all the possible arrangements of these edges around v are shown in Figure 3 (the edges in bold belong to the 2-factor). The letter assigned to any arrangement (situation) is called *code letter*.

Definition 2. *[5, 7] For a given 2-factor of a linear or circular grid graph G of width m and with $m \cdot n$ vertices $(m, n \in N)$, the **code matrix** $\left[\alpha_{i,j}\right]_{m \times n}$ is a matrix of order $m \times n$ with entries from $\{a, b, c, d, e, f\}$ where $\alpha_{i,j}$ is the code letter for the i-th vertex in j-th column of G.*

By reading each column of the code matrix from top to down, we obtain a word over alphabet $\{a, b, c, d, e, f\}$ of length m, named *alpha-word*. When G is circular grid graph, then we treat these words as circular ones and the letter $\alpha_{m+1,j} \overset{\text{def}}{=} \alpha_{1,j}$ follows the letter $\alpha_{m,j}$. The possibility that two vertices are adjacent in the assigned 2-factor is expressed through the two auxiliary digraphs $\mathcal{D}_{ud}$ and $\mathcal{D}_{lr}$ depicted in Figure 4.

For each alpha-letter α, we denote by $\overline{\alpha}$ the alpha-letter of the situation from Figure 3 obtained by applying reflection symmetry with the horizontal axis as its line of symmetry. Precisely, $\overline{a} \overset{\text{def}}{=} c, \overline{b} \overset{\text{def}}{=} b, \overline{c} \overset{\text{def}}{=} a$, $\overline{d} \overset{\text{def}}{=} f, \overline{e} \overset{\text{def}}{=} e$ and $\overline{f} \overset{\text{def}}{=} d$. Further, for any alpha-word $v = \alpha_1\alpha_2 \ldots \alpha_m$, we introduce a new alpha-word $\overline{v} \overset{\text{def}}{=} \overline{\alpha}_m\overline{\alpha}_{m-1} \ldots \overline{\alpha}_1$.

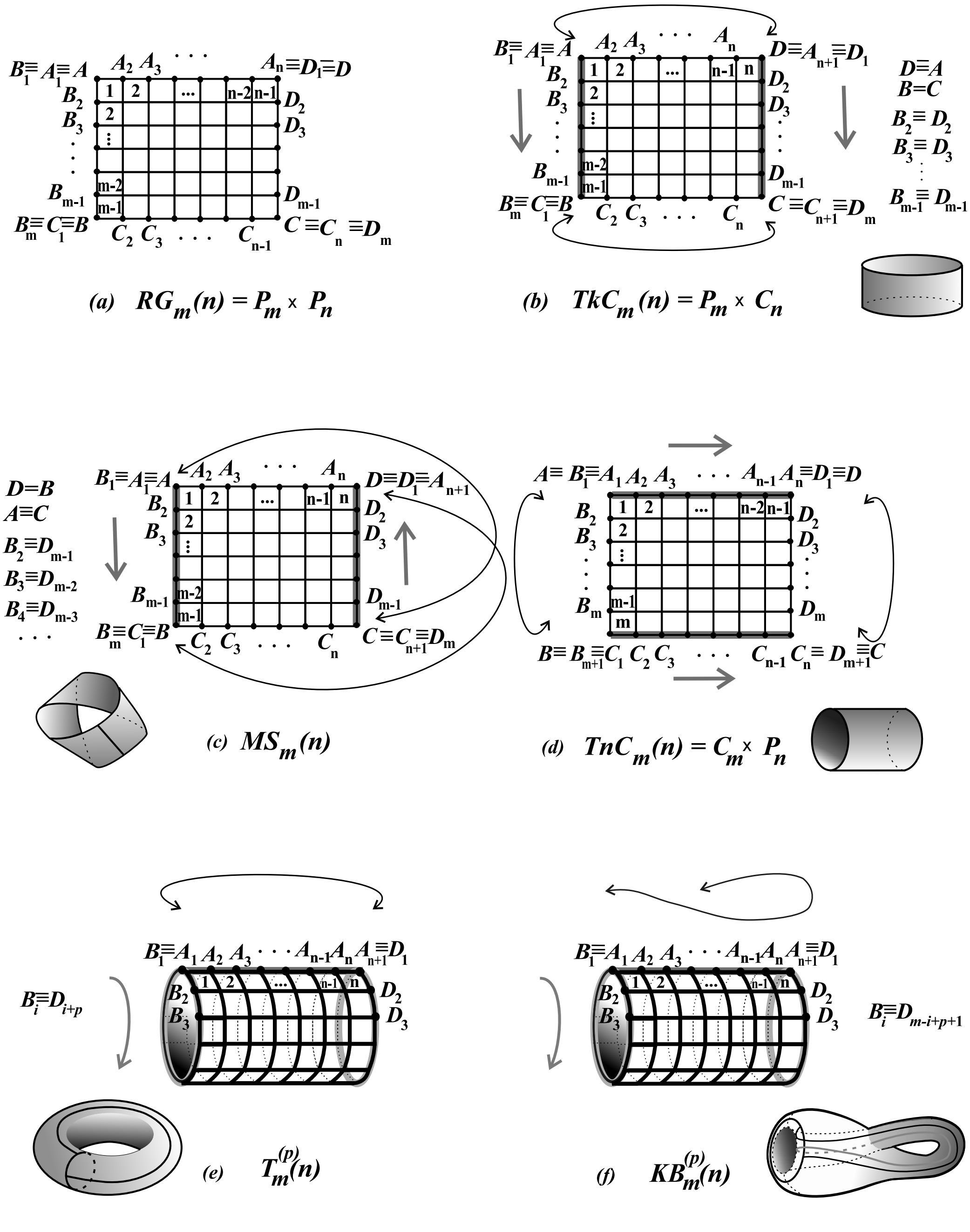

Figure 1: **(a)** The rectangular grid $RG_m(n) = P_m \times P_n$; **(b)** The thick cylinder $TkC_m(n) = P_m \times C_n$; **(c)** The Moebius strip $MS_m(n)$; **(d)** The thin cylinder $TnC_m(n) = P_m \times C_n$; **(e)** The torus grid $TG_m^{(p)}(n)$; **(f)** The Klein bottle $KB_m^{(p)}(n)$

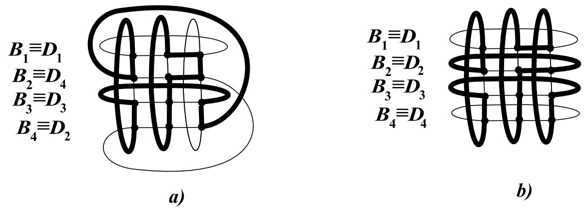

Figure 2: **(a)** Klein bottle $KB_4^{(1)}(3)$ with a 2-factor; **(b)** Torus grid $TG_4^{(0)}(3)$ with a Hamiltonian cycle

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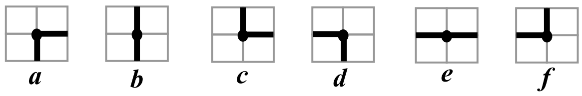

Figure 3: The six possible arrangements of the two edges around any vertex with assigned code letters.

Theorem 1. *(The characterization of a 2-factor [5, 7])*
The code matrix $\left[\alpha_{i,j}\right]_{m\times n}$ for a given 2-factor of a grid graph G of width m ($m \in N$) has the following properties:

1. ***Column conditions:***
   For every fixed j ($1 \leq j \leq n$),
   (a) *if G is linear (circular) grid graph, than the ordered pairs $(\alpha_{i,j}, \alpha_{i+1,j})$ must be arcs in the digraph $\mathcal{D}_{ud}$ for $1 \leq i \leq m-1$ ($1 \leq i \leq m$).*
   (b) *if G is linear grid graph, than $\alpha_{1,j} \in \{a, d, e\}$ and $\alpha_{m,j} \in \{c, e, f\}$.*
2. ***Adjacency of column condition:***
   For every fixed j ($1 \leq j \leq n-1$), the ordered pairs $(\alpha_{i,j}, \alpha_{i,j+1})$ must be arcs in the digraph $\mathcal{D}_{lr}$ for $1 \leq i \leq m$.
3. ***First and Last Column conditions:***
   (a) *If $G = RG_m(n)$ or $G = TnC_m(n)$, then the alpha-word of the first column consists of the letters from the set $\{a, b, c\}$ and of the last column of the letters from the set $\{b, d, f\}$.*
   (b) *If $G = TkC_m(n)$, then the ordered pairs $(\alpha_{i,n}, \alpha_{i,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*
   (c) *If $G = MS_m(n)$, then the ordered pairs $(\overline{\alpha}_{i,n}, \alpha_{m-i+1,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*
   (d) *If $G = TG_m^{(p)}(n)$, then the ordered pairs $(\alpha_{i+p,n}, \alpha_{i,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*
   (e) *If $G = KB_m^{(p)}(n)$, then the ordered pairs $(\alpha_{m+p+1-i,n}, \overline{\alpha}_{i,1})$, where $1 \leq i \leq m$, must be arcs in the digraph $\mathcal{D}_{lr}$.*

The converse, for every matrix $[\alpha_{i,j}]_{m\times n}$ with entries from $\{a, b, c, d, e, f\}$ that satisfies conditions 1–3 there is a unique 2-factor on the considered grid graph G.

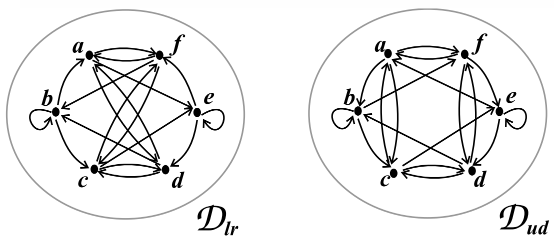

Figure 4: The digraphs $\mathcal{D}_{lr}$ (from left to right) and $\mathcal{D}_{ud}$ (from up to down).

This assertion enables that for considered grid graph G and for fixed m ($m \in N$), the counting of such code matrices (in fact all 2-factors of G) is reduced to the counting of some directed walks in an auxiliary digraph. If G is a linear grid graph, then we label this digraph by $\mathcal{D}_{L,m} \overset{\text{def}}{=} (V(\mathcal{D}_{L,m}), E(\mathcal{D}_{L,m}))$, otherwise, if G is a circular one by $\mathcal{D}_{C,m} \overset{\text{def}}{=} (V(\mathcal{D}_{C,m}), E(\mathcal{D}_{C,m}))$. By agreement, in what follows, when the class to which G belongs is not specified, we label this digraph and corresponding sets without the letter L or C in subscript. The set of its vertices $V(\mathcal{D}_m)$ consists of all possible alpha-words, i.e. the words $\alpha_{1,j}\alpha_{2,j}\ldots\alpha_{m,j}$ over alphabet $\{a, b, c, d, e, f\}$ which fulfill *Column conditions*. An arc $(v, u) \in E(\mathcal{D}_m)$ joins $v = \alpha_{1,j}\alpha_{2,j}\ldots\alpha_{m,j}$ to $u = \alpha_{1,j+1}\alpha_{2,j+1}\ldots\alpha_{m,j+1}$, i.e. $v \to u$ if and only if the *Adjacency of column condition* is satisfied for the ordered pair (v, u) (i.e. the vertex v can be the previous column for the vertex u in the code matrix $\left[\alpha_{i,j}\right]_{m\times n}$ for a 2-factor of G).

Example 1. *For the Hamiltonian cycle depicted in Figure 2 (a), the columns of the code matrix (reading from left to right) are the words $bfdb$, $cabb$ and $dfac$. Similarly, the columns in Figure 2 (b) are $bfdb$, $cabb$ and $feab$. Both graphs $KB_4^{(1)}(3)$ and $TG_4^{(0)}(3)$ in this figure has the same transfer digraph $\mathcal{D}_{C,4}$. In it, the first 2-factor corresponds to the directed walk of length two: $\alpha_1\alpha_2\alpha_3\alpha_4 \overset{\text{def}}{=} bfdb \to cabb \to \beta_1\beta_2\beta_3\beta_4 \overset{\text{def}}{=} dfac$, where starting and finishing vertices fulfill $\beta_1\beta_2\beta_3\beta_4 = dfac \to \overline{\alpha}_1\overline{\alpha}_4\overline{\alpha}_3\overline{\alpha}_2 = bbfd$. The second one corresponds to the closed directed walk of length 3: $bfdb \to cabb \to feab \to bfdb$.*

Clearly, $\mathcal{D}_{L,m}$ is subdigraph of $\mathcal{D}_{C,m}$ treating each alpha-word from $V(\mathcal{D}_{L,m})$ as circular one in $V(\mathcal{D}_{C,m})$. For these digraphs we have that $| V(\mathcal{D}_{C,m}) |= 2 | V(\mathcal{D}_{L,m}) |= 3^m + (-1)^m$ and that both digraphs $\mathcal{D}_{C,m}$ and $\mathcal{D}_{L,m}$ are disconnected where $m \geq 2$ [5, 7].

Definition 3. *[5, 7] The **outlet (inlet) word** of a vertex $\alpha \equiv \alpha_1\alpha_2\ldots\alpha_m \in V(\mathcal{D}_m)$ is the binary word $o(\alpha) \equiv o_1o_2\ldots o_m$ ($i(\alpha) \equiv i_1i_2\ldots i_m$), where*

$$o_j \overset{\text{def}}{=} \begin{cases} 0, & if \ \ \alpha_j \in \{b, d, f\} \\ 1, & if \ \ \alpha_j \in \{a, c, e\} \end{cases} \quad and \quad i_j \overset{\text{def}}{=} \begin{cases} 0, & if \ \ \alpha_j \in \{a, b, c\} \\ 1, & if \ \ \alpha_j \in \{d, e, f\} \end{cases}, \quad 1 \leq j \leq m.$$

For a binary word $v \equiv b_1b_2\ldots b_{m-1}b_m \in \{0, 1\}^m$,

$$\overline{v} \overset{\text{def}}{=} b_mb_{m-1}\ldots b_2b_1 \ \ and \ \ \rho(v) \overset{\text{def}}{=} b_2\ldots b_{m-1}b_mb_1.$$

Example 2. *For the 2-factor of $KB_4^{(1)}(3)$ depicted in Figure 2 (a) the outlet words for the first three columns are 0^4, 1100 and 0011, respectively. Similarly, for the Hamiltonian cycle of $TG_4^{(0)}(3)$ in Figure 2 (b) they are 0^4, 1100 and 0110, respectively.*

The digraph $\mathcal{D}^*_m \overset{\text{def}}{=} (V(\mathcal{D}^*_m), E(\mathcal{D}^*_m))$ is obtained by gluing all the vertices from $V(\mathcal{D}_m)$ having the same corresponding outlet word (this word becomes the vertex in new digraph) and replacing all the arcs from $E(\mathcal{D}_m)$ starting from these glued vertices and ending at the same vertex with only one arc. It is proved [5] that every binary word from $\{0, 1\}^m$ except the word $(01)^k0$ when $m = 2k + 1$ ($k \in N$) belongs to $V(\mathcal{D}^*_{L,m})$. For $\mathcal{D}^*_{C,m} \overset{\text{def}}{=} (V(\mathcal{D}^*_{C,m}), E(\mathcal{D}^*_{C,m}))$, the set $V(\mathcal{D}^*_{C,m})$ consists of all binary words of length m [7]. Both digraphs $\mathcal{D}^*_{L,m}$ and $\mathcal{D}^*_{C,m}$ are disconnected for $m \geq 2$ (the adjacent vertices must have the numbers of 1s of the same parity). Each component of $\mathcal{D}^*_{L,m}$ or $\mathcal{D}^*_{C,m}$ is a strongly connected digraph, i.e. their adjacency matrices $\mathcal{T}^*_{L,m}$ and $\mathcal{T}^*_{C,m}$ are symmetric matrices. While the elements of the first matrix are from the set $\{0, 1\}$, in the second they are from the set $\{0, 1, 2\}$.

Theorem 2. *([5]) If $f_m^{RG}(n)$, $f_m^{TkC}(n)$ and $f_m^{MS}(n)$ ($m \geq 2$) denote the number of 2-factors of $RG_m(n)$, $TkC_m(n)$ and $MS_m(n)$, respectively, then*

$$f_m^{RG}(n) = a_{1,1}^{(n)},$$

$$f_m^{TkC}(n) = \sum_{v_i \in V(\mathcal{D}^*_{L,m})} a_{i,i}^{(n)} \quad and \quad f_m^{MS}(n) = \sum_{\substack{v_i, v_j \in V(\mathcal{D}^*_{L,m}) \\ \overline{v_i} = v_j}} a_{i,j}^{(n)},$$

where $v_1 \equiv 0^m$ *(corresponding to the first row and first column of* $\mathcal{T}^*_{L,m}$*) and* $a_{i,j}^{(n)}$ *denotes the* (i, j)*-entry of n-th power of* $\mathcal{T}^*_{L,m}$*.*

Theorem 3. *([7]) If* $f_m^{TnC}(n)$, $f_{m,p}^{TG}(n)$ *and* $f_{m,p}^{KB}(n)$ $(m \geq 2)$ *denote the number of 2-factors of* $TnC_m(n)$, $TG_m^{(p)}(n)$ *and* $KB_m^{(p)}(n)$*, respectively, then*

$$f_m^{TnC}(n) = a_{1,1}^{(n)},$$

$$f_{m,p}^{TG}(n) = \sum_{\substack{v_i, v_j \in V(\mathcal{D}^*_{C,m}) \\ v_i = \rho^p(v_j)}} a_{i,j}^{(n)} \quad and \quad f_{m,p}^{KB}(n) = \sum_{\substack{v_i, v_j \in V(\mathcal{D}^*_{C,m}) \\ \overline{v_i} = \rho^p(v_j)}} a_{i,j}^{(n)},$$

where $v_1 \equiv 0^m$ *(corresponding to the first row and first column of* $\mathcal{T}^*_{C,m}$*) and* $a_{i,j}^{(n)}$ *denotes the* (i, j)*-entry of n-th power of* $\mathcal{T}^*_{C,m}$*.*

In [5] the data concerning the digraphs $\mathcal{D}^*_{L,m}$, for $m \leq 12$, has been gathered upon implementation of the above described algorithm. They provided a suggestion as to the structure of $\mathcal{D}^*_{L,m}$, which was later proven to be correct [6] (see the theorem below).

Theorem 4. *[6] For each* $m \geq 2$*, the digraph* $\mathcal{D}^*_{L,m}$ *has exactly* $\left\lfloor \frac{m}{2} \right\rfloor + 1$ *components, i.e.* $\mathcal{D}^*_{L,m} = \mathcal{A}^*_{L,m} \cup \left(\bigcup_{s=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}^{*(s)}_{L,m} \right)$*, where* $| V(\mathcal{B}^{*(1)}_{L,m}) | \geq | V(\mathcal{B}^{*(2)}_{L,m}) | \geq \ \ldots \ \geq | V(\mathcal{B}^{*(\lfloor m/2 \rfloor)}_{L,m}) |$ *and* $\mathcal{A}^*_{L,m}$ *is the one containing* 1^m*. All the components* $\mathcal{B}^{*(s)}_{L,m}$ $(1 \leq s \leq \left\lfloor \frac{m}{2} \right\rfloor)$ *are bipartite digraphs.*
If m is odd, then $| V(\mathcal{B}^{*(s)}_{L,m}) | = \binom{m+1}{(m+1)/2 - s}$ *and* $| V(\mathcal{A}^*_{L,m}) | = \binom{m}{(m-1)/2}$*.*
If m is even, then $| V(\mathcal{B}^{*(s)}_{L,m}) | = 2\binom{m}{m/2 - s}$ *and* $| V(\mathcal{A}^*_{L,m}) | = \binom{m}{m/2}$*.*
The vertices v *and* $\overline{v}$ *belong to the same component. When the component is bipartite they are placed in the same class if and only if m is odd.*

For the digraph $\mathcal{D}^*_{C,m}$, the component which contains 1^m is marked by $\mathcal{A}^*_{C,m}$, while the one containing 0^m by $\mathcal{N}^*_m$ (the one responsible for counting 2-factors for thin cylinder grid graphs). The former is produced from the component $\mathcal{A}_{C,m}$ of $\mathcal{D}_{C,m}$ which contains the vertex e^m, and the latter one from the component $\mathcal{N}_m$ which contains the vertex b^m. Data for $m \leq 10$ gathered in [7] suggested the structure of $\mathcal{D}^*_{C,m}$ expressed in the following theorem.

Theorem 5. *(MAIN THEOREM) [7] For each even* $m \geq 2$*, the digraph* $\mathcal{D}^*_{C,m}$ *has exactly* $\left\lfloor \frac{m}{2} \right\rfloor + 1$ *components, i.e.* $\mathcal{D}^*_{C,m} = \mathcal{A}^*_{C,m} \cup \left(\bigcup_{s=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}^{*(s)}_{C,m} \right)$*, where* $\mathcal{A}^*_{C,m}$ *contains both* 1^m *and* 0^m*, all the components* $\mathcal{B}^{*(s)}_{C,m}$ $(1 \leq s \leq \left\lfloor \frac{m}{2} \right\rfloor)$ *are bipartite digraphs,* $| V(\mathcal{B}^{*(s)}_{C,m}) | = 2\binom{m}{m/2 - s}$ *and* $| V(\mathcal{A}^*_{C,m}) | = \binom{m}{m/2}$*.*
For each odd $m \geq 1$*, the digraph* $\mathcal{D}^*_{C,m}$ *has exactly two components, i.e.* $\mathcal{D}^*_{C,m} = \mathcal{A}^*_{C,m} \cup \mathcal{N}^*_m$*, which are mutually isomorphic and with* 2^{m-1} *vertices.*

The aim of this paper is the first proof of Theorem 5. In the next section, we prove the main theorem.

2. Proof of the main theorem

Definition 4. *[6] For an arbitrary binary word x of length m ($m \in N$) the total number of 0's at odd (even) positions is denoted by $odd(x)$ ($even(x)$) and $Z(x) \stackrel{\text{def}}{=} odd(x) - even(x)$.*
The set $S_m^{(0)}$ ($m \in N$) consists of all the binary m-words whose number of 0's at odd positions is equal to the number of 0's at even positions.
For $1 \le s \le \lfloor m/2 \rfloor$, $S_m^{(s)} \stackrel{\text{def}}{=} R_m^{(s)} \cup G_m^{(s)}$ where the sets $R_m^{(s)}$ and $G_m^{(s)}$ consist of all the binary words x of the length m for which $Z(x) = s$ and $Z(x) = -s$, respectively. Additionally, if m is odd, then $R_m^{(\lceil m/2 \rceil)} \stackrel{\text{def}}{=} \{0(10)^{\lfloor m/2 \rfloor}\}$.

Note that $\bigcup_{s=0}^{\lfloor m/2 \rfloor} S_m^{(s)} = V(\mathcal{D}^*_{L,m})$, where $V(\mathcal{D}^*_{L,m}) = \{0,1\}^m$ for m-even, and $V(\mathcal{D}^*_{L,m}) = \{0,1\}^m \backslash R_m^{(\lceil m/2 \rceil)} = \{0,1\}^m \backslash \{0(10)^{\lfloor m/2 \rfloor}\}$ for m-odd. In [6], it is proved that the subdigraphs of $\mathcal{D}^*_{L,m}$ induced by the sets $S_m^{(s)}$, $0 \le s \le \lfloor m/2 \rfloor$ are its components, i.e. $\langle S_m^{(0)} \rangle_{\mathcal{D}^*_{L,m}} = \mathcal{A}^*_{L,m}$ and $\mathcal{B}^{*(s)}_{L,m} = \langle S_m^{(s)} \rangle_{\mathcal{D}^*_{L,m}}$, where $s = 1, 2, \ldots, \lfloor m/2 \rfloor$. Next, each component $\mathcal{B}^{*(s)}_{L,m}$ is a bipartite digraph and the sets $R_m^{(s)}$ and $G_m^{(s)}$ are its classes. We call the vertices from $R_m^{(s)}$ and $G_m^{(s)}$ **red** and **green** vertices, respectively. Some of the representatives for these sets are introduced in the following way.

Definition 5. *[6] For even m, the zero-word $Q_m^{(0)} \stackrel{\text{def}}{=} 0^m \in S_m^{(0)}$ and the words $Q_m^{(s)} \stackrel{\text{def}}{=} (01)^s 0^{m-2s} \in R_m^{(s)}$ ($1 \le s \le m/2$) are called the* ***queens****.*
For odd m, the words $Q_m^{(s)} \stackrel{\text{def}}{=} (01)^{s-1} 0^{m-2s+2} \in R_m^{(s)}$ ($1 \le s \le \lfloor m/2 \rfloor + 1$) are called the ***queens****, while the words $L_m^{(s)} \stackrel{\text{def}}{=} (10)^{s+1} 0^{m-2s-2} \in S_m^{(s)}$ ($0 \le s < \lfloor m/2 \rfloor$) and the word $L_m^{(\lfloor m/2 \rfloor)} \stackrel{\text{def}}{=} (10)^{\lfloor m/2 \rfloor} 1 \in S_m^{(\lfloor m/2 \rfloor)}$ are called the* ***court ladies****.*

Lemma 1. *For each even $m \ge 2$, the digraph $\mathcal{D}^*_{C,m}$ has exactly $\frac{m}{2} + 1$ components, i.e. $\mathcal{D}^*_{C,m} = \mathcal{A}^*_{C,m} \cup \left(\bigcup_{s=1}^{m/2} \mathcal{B}^{*(s)}_{C,m} \right)$, where the component $\mathcal{A}^*_{C,m}$ is the one containing 1^m,*

$$| V(\mathcal{A}^*_{C,m}) | = \binom{m}{m/2} \text{ and } | V(\mathcal{B}^{*(s)}_{C,m}) | = 2\binom{m}{m/2 - s}, \quad 1 \le s \le m/2.$$

Proof. Note that $\mathcal{D}^*_{L,m}$ is subdigraph of $\mathcal{D}^*_{C,m}$. The only difference is in added (new) arcs. Recall that the statement of this lemma is valid if $\mathcal{D}^*_{C,m}$ is replaced with $\mathcal{D}^*_{L,m}$ (Theorem 4). Therefore, it is sufficient to prove that two different queens $Q_m^{(s_1)} = (01)^{s_1} 0^{m-2s_1}$ and $Q_m^{(s_2)} = (01)^{s_2} 0^{m-2s_2}$ ($0 \le s_1 < s_2$) with the same parity of the number 1's ($s_1 \equiv s_2 (mod 2)$) are not connected in $\mathcal{D}^*_{C,m}$.

For that purpose, we suppose the opposite, i.e. that in $\mathcal{D}^*_{C,m}$ there exists a directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ of length $k \in N$, where $v_0 = Q_m^{(s_1)}$ and $v_k = Q_m^{(s_2)}$. For this directed walk, the corresponding part of the grid induced by $m \cdot k$ vertices (the thin cylinder grid graph) is bipartite because m is even. The directed walk $v_0 \to v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ determines a spanning union of paths (open paths and cycles) in this grid. The ends of these open paths belong to the first or/and the last column of the cylinder grid. Each cycle (if exists) in this union has the same number of vertices of both colors (say gray and black). However, for the union of the open paths it is not valid. Namely, if k is even (see Figure 5 (a)), the difference of the numbers of open paths with both ends in gray vertices and the ones in black vertices is exactly $| \frac{s_1 - s_2}{2} | > 0$. If k is odd (see Figure 5 (b)), all $| \frac{s_1 + s_2}{2} |$ open paths have end vertices in the same color. In both cases we come in contradiction with the fact that in the considered part of the grid the numbers of vertices of both colors are equal.
In this way, we obtain that $\mathcal{A}^*_{C,m} = \langle S_m^{(0)} \rangle_{\mathcal{D}^*_{C,m}}$ and $\mathcal{B}^{*(s)}_{C,m} = \langle S_m^{(s)} \rangle_{\mathcal{D}^*_{C,m}}$, where $s = 1, 2, \ldots, \frac{m}{2}$. Consequently, $| V(\mathcal{A}^*_{C,m}) | = | S_m^{(0)} | = | V(\mathcal{A}^*_{L,m}) | = \binom{m}{m/2}$ and $| V(\mathcal{B}^{*(s)}_{C,m}) | = | S_m^{(s)} | = | V(\mathcal{B}^{*(s)}_{L,m}) | = 2\binom{m}{m/2 - s}$, $1 \le s \le m/2$.

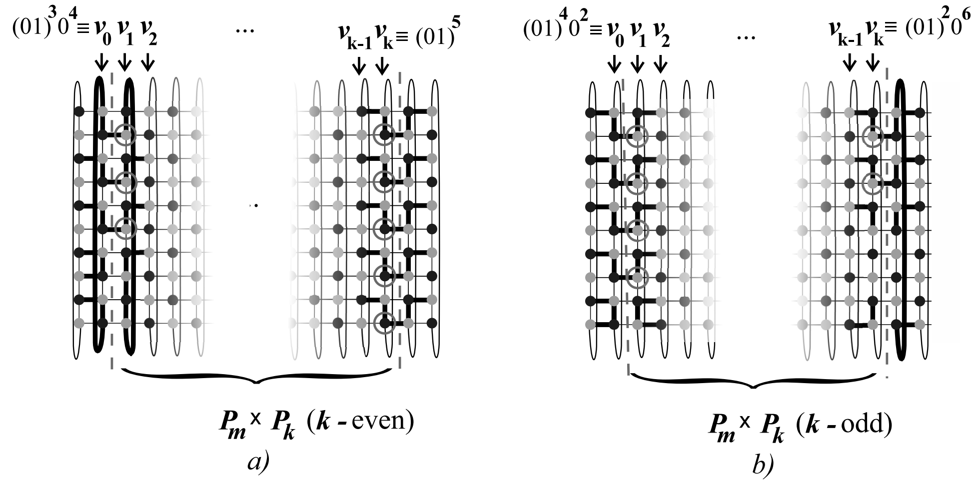

Figure 5: Two different queens $Q_m^{(s_1)} = (01)^{s_1}0^{m-2s_1}$ and $Q_m^{(s_2)} = (01)^{s_2}0^{m-2s_2}$ where m is even, $s_1 \equiv s_2(mod\ 2)$ and $0 \leq s_1 < s_2$ are not connected in $\mathcal{D}^*_{C,m}$.

□

Lemma 2. *When m is even, all the components $\mathcal{B}^{*(s)}_{C,m}$ $(1 \leq s \leq \frac{m}{2})$ are bipartite digraphs. The vertices v and $\overline{v}$ belong to the same component but in different classes (colors).*

Proof. In order to prove that the component $\mathcal{B}^{*(s)}_{C,m}$ $(1 \leq s \leq \frac{m}{2})$ is the bipartite digraph $(R_m^{(s)}, G_m^{(s)})$, it is sufficient to prove that every arc from the set $E(\mathcal{B}^{*(s)}_{C,m}) \backslash E(\mathcal{B}^{*(s)}_{L,m})$ has the vertices in different colors (red and green). Consider such an arc $vw \in E(\mathcal{B}^{*(s)}_{C,m}) \backslash E(\mathcal{B}^{*(s)}_{L,m})$. Then, there exists $\alpha = \alpha_1\alpha_2 \ldots \alpha_m \in V(\mathcal{D}_{C,m})$ for which $i(\alpha) = v = v_1v_2 \ldots v_m$, $o(\alpha) = w = w_1w_2 \ldots w_m$, $\alpha_1 \in \{b, c, f\}$ and $\alpha_m \in \{a, b, d\}$. Since $\alpha \neq b^m$ $(o(b^m) = 0^m \in V(\mathcal{A}^*_{C,m}))$, then there exists $J = \min\{j \in N \mid 1 \leq j \leq m-1 \wedge (v_j = 1 \vee w_j = 1)\}$. This implies that there exists an arc $\rho^J(v) \to \rho^J(w)$ in $\mathcal{D}^*_{L,m}$ where $\rho^j(x_1x_2 \ldots x_m) \overset{\text{def}}{=} x_{j+1}x_{j+2} \ldots x_m x_1 x_2 \ldots x_j$ for any word $x_1x_2 \ldots x_m$ of length m and $1 \leq j \leq m-1$. Namely, $\rho^J(\alpha) \in V(\mathcal{D}_{L,m})$, $i(\rho^J(\alpha)) = \rho^J(v)$ and $o(\rho^J(\alpha)) = \rho^J(w)$. Note that for an arbitrary binary word $x = x_1x_2 \ldots x_m$ and $1 \leq j \leq m-1$, we have that $Z(\rho^j(x)) = (-1)^j Z(x)$. Consequently, both v and w belong to the same set $S_m^{(s)}$ $(1 \leq s \leq m/2)$ which contains $\rho^J(v)$ and $\rho^J(w)$. Since the vertices $\rho^J(v)$ and $\rho^J(w)$ belong to the different sets $R_m^{(s)}$ and $G_m^{(s)}$ (they belong to different classes of the component $\mathcal{B}^{*(s)}_{L,m}$ of $\mathcal{D}^*_{L,m}$), it is valid for v and w, too.
The second statement of this lemma is a simple consequence from the linear case (Theorem 4). □

Lemma 3. *For each odd $m \geq 1$, the digraph $\mathcal{D}^*_{C,m}$ has exactly two components $\mathcal{A}^*_{C,m}$ and $\mathcal{N}^*_m$ $(\mathcal{D}^*_{C,m} = \mathcal{A}^*_{C,m} \cup \mathcal{N}^*_m)$ which are isomorphic and each of them has 2^{m-1} vertices.*

Proof. Recall that when m is odd, the court ladies $L_m^{(s)} = (10)^{s+1}0^{m-2s-2} \in G_m^{(s)} \subseteq S_m^{(s)}$, $1 \leq s \leq \lfloor m/2 \rfloor - 1$ and $L_m^{(0)} = (10)0^{m-2} \in S_m^{(0)}$ fulfill $Z(L_m^{(s)}) = -s$, $0 \leq s \leq \lfloor m/2 \rfloor - 1$. For the queens $Q_m^{(s)} = (01)^{(s-1)}0^{m-2s+2} \in R_m^{(s)} \subseteq S_m^{(s)}$, $1 \leq s \leq \lfloor m/2 \rfloor$ we have $Z(Q_m^{(s)}) = s$. Additionally, the queen $Q_m^{(\lfloor m/2 \rfloor+1)} = (01)^{\lfloor m/2 \rfloor}0 \notin V(\mathcal{D}^*_{L,m})$ now belongs to $V(\mathcal{D}^*_{C,m})$ and $Z(Q_m^{(\lfloor m/2 \rfloor+1)}) = \lfloor m/2 \rfloor + 1$. The digraphs induced by the sets $S_m^{(s)}$, $s = 0, 1, 2, \ldots, \lfloor m/2 \rfloor$ in $\mathcal{D}^*_{L,m}$ are connected digraphs. Clearly, they are subdigraphs of $\mathcal{D}^*_{C,m}$. We prove that all the vertices $v \in V(\mathcal{D}^*_{C,m})$ with even $Z(v)$ belong to $\mathcal{A}^*_{C,m}$ - the component of $\mathcal{D}^*_{C,m}$ containing 1^m $(Z(1^m) = 0)$ while the ones with odd $Z(v)$ to $\mathcal{N}^*_m$ - the component of $\mathcal{D}^*_{C,m}$ containing 0^m ($Z(0^m) = 1$ and the vertices 1^m and 0^m are not connected because their numbers of 1's have opposite parity).

For this purpose, note that the number of 1's in the words $L_m^{(s)}$ and $Q_m^{(s+2)}$ $(0 \leq s \leq \lfloor m/2 \rfloor - 1)$ is equal $(s+1)$. They are directly connected by an arc in $\mathcal{D}^*_{C,m}$ because there exists the alpha word $\alpha = f(af)^s ab^{m-2s-2} \in V(\mathcal{D}_{C,m})$ for which $i(\alpha) = L_m^{(s)}$ and $o(\alpha) = Q_m^{(s+2)}$. In this way we obtain that $V(\mathcal{A}^*_m)$ and $V(\mathcal{N}^*_m)$ consist of all the circular binary words v of length m for which $Z(v)$ is even and odd, respectively.

The isomorphism between $\mathcal{A}^*_{C,m}$ and $\mathcal{N}^*_m$ is the simple consequence of the isomorphism between $\mathcal{A}_{C,m}$ - the component of $\mathcal{D}_{C,m}$ containing e^m and $\mathcal{N}_m$ - the component of $\mathcal{D}_{C,m}$ containing b^m. Let us prove the latter. For this sake, we define the function $f : \{a, b, c, d, e, f\} \longrightarrow \{a, b, c, d, e, f\}$ with $f : \begin{pmatrix} a & b & c & d & e & f \\ f & e & d & c & b & a \end{pmatrix}$. With direct verification we conclude that f is automorphism of the digraph $\mathcal{D}_{ud}$. Consequently, for any word $\alpha = \alpha_1\alpha_2 \ldots \alpha_m \in V(\mathcal{D}_{C,m})$, the word $f(\alpha_1)f(\alpha_2)\ldots f(\alpha_m)$ belongs to $V(\mathcal{D}_{C,m})$, too.

Now, we define the function $F : V(\mathcal{D}_{C,m}) \longrightarrow V(\mathcal{D}_{C,m})$ by $F(\alpha) = \beta$ if and only if $\beta_i = f(\alpha_i)$, for all $i = 1, 2, \ldots, m$ where $\alpha = \alpha_1\alpha_2 \ldots \alpha_m, \beta = \beta_1, \beta_2, \ldots, \beta_m \in V(\mathcal{D}_{C,m})$.
Obviously, F is a bijection and involution because the same is valid for the function f. The function f is also automorphism of the digraph $\mathcal{D}_{lr}$, which implies that

$$(\forall \alpha, \beta \in V(\mathcal{D}_{C,m}))(\alpha \to \beta \Leftrightarrow F(\alpha) \to F(\beta)).$$

Since $F(e^m) = b^m$, we conclude that $\mathcal{A}_{C,m}$ and $\mathcal{N}_m$ are isomorphic. Consequently, $\mathcal{A}^*_{C,m}$ and $\mathcal{N}^*_m$ are isomorphic. □

Additionally, note that if $F(\alpha) = \beta$, where $o(\alpha) = v = v_1v_2 \ldots v_m$ and $o(\beta) = w = w_1w_2 \ldots w_m$, then $v_i = 0$ if and only if $w_i = 1$, for all $i = 1, 2, \ldots, m$. In this way, the isomorphism between $\mathcal{A}^*_{C,m}$ and $\mathcal{N}^*_m$ is determined with the function $c : V(\mathcal{D}^*_{C,m}) \longrightarrow V(\mathcal{D}^*_{C,m})$ defined by $c(v) = w = w_1w_2 \ldots w_m$, where $v = v_1v_2 \ldots v_m$ and $w_i = \begin{cases} 0, & \text{if } v_i = 1 \\ 1, & \text{if } v_i = 0 \end{cases}$. Clearly, $c(c(v)) = v$. We can say that the words v and w are mutually complementary.

Lemma 1, Lemma 2 and Lemma 3 complete the proof of Theorem 5.

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