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Estimation of approximation error of a function having its derivatives belonging to Lipschitz class of order - α by extended Legendre wavelet (ELW) method and its applications

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Abstract. In this paper, the approximation errors of a function having its derivatives belonging to the Lipschitz class of order - α by the ELW method have been obtained. We have solved various differential equations like the Hermite differential equation, Riccati differential equation, Blasius equation, radioactive decay problem by using this method. Also, we have approximated the sum of some specific series that is not convergent by traditional methods.

1. Introduction

There is the significant importance of analyzing the functions using the Fourier Series, Fourier transform, and other allied developments. Its interpretation is very useful for studying the number of problems emerging in different fields of science and technology. But due to certain restrictions on the Fourier techniques, we cannot apply these techniques to discuss every such problems. One of the reasons behind it is that the Fourier method depends on exponential functions. Wavelets[5] fit better in dealing such type of problems, due to its well localized behaviour.

In recent years, wavelet analysis has played a very elegant role in the solution of differential, and integral equations, and some other problems of mathematical analysis. Wavelets are an excellent alternative to the present traditional methods. There are different types of wavelets available in the literature and the choice of a particular wavelet in approximating the function depends on the numerical accuracy.

Daubechies[6] proposed the compactly supported wavelets having properties like orthogonality, orthonormality, localization property, and dilation property etc. J. Ma. et al. [16], Daz et al. [7], Wells et al. [24] have discussed the wavelet method given by Daubechies to solve some special types of differential equations. Vampa et al. [23] presented a hybrid wavelet method for the solution of the boundary value problem. Han et al. [9], Xiang et al. [25] explained a special wavelet method to solve some problems of physical science. Some authors like Chang et al. [2], Chen C. F. et al. [3], Hwang et al. [11], Paraskevopoulos

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et al. [17] have applied the shifted Legendre polynomials, Walsh function, shifted Chebyshev polynomials, Bessel functions respectively for solving various problems of mathematical analysis.

Lal[13] investigated the numerical approximation based upon the ELW to get the solution of some nonlinear differential equations. Lal and Kumar[14] developed the hybrid Legendre polynomial method to solve some differential equations associated with real-world problems. Lal and Kumar[15] proposed a wavelet method depending on the Legendre wavelet for solving the integral equations. Working in the same direction, in this paper approximation of functions having derivative belonging to Lipschitz class of order - α have been obtained by the ELW method. These approximations are used as follows :

1. To solve radioactive decay problem [18].
2. To solve the Riccati differential equation [13].
3. To approximate the sum of some specific series[10] that is not convergent by traditional methods.
4. To solve the Blasius equation [1].

This paper is groomed as follows :

Section 1 is introductory. In section 2, the extended Legendre wavelet, Lipschitz class of order - α, wavelet approximation have been introduced. In section 3, the approximation errors of a function having derivative in the Lipschitz class of order - α have been estimated. In Section 4, the detailed proofs of approximation theorems are given. In section 5, some remarks have been given in the form of theorems. Section 6 contains the explanation of numerical approximation. Section 7 is concerned with the applications of the approximation methods. In section 8, conclusions have been given.

2. Definitions and Preliminaries

2.1. The Extended Legendre Wavelets (ELWs)

The ELW on the interval $[0,1)$ are defined by

$$\psi_{n,m}^{(\nu)}(t) = \begin{cases} \sqrt{2m+1}\,\nu^{\frac{k}{2}} L_m(2\nu^k t - 2n + 1), & \frac{n-1}{\nu^k} \le t < \frac{n}{\nu^k}; \\ 0, & \text{otherwise} \end{cases}$$

where $n = 1, 2, \ldots, \nu^k$ and m is order of the Legendre polynomial, $k = 1, 2, 3, \ldots$ and t is normalized time. In the above definition, the polynomials $L_m(t)$ are the Legendre polynomials of degree m over the interval $[-1, 1]$ which are defined in [15].

2.2. Extended Legendre Wavelet Expansion

A function $g \in L^2(\mathbb{R})$ defined over $[0,1)$ is expanded in terms of ELW series as

$$g(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}^{(\nu)} \psi_{n,m}^{(\nu)}(t), \tag{1}$$

where $c_{n,m}^{(\nu)} = \langle g, \psi_{n,m}^{(\nu)} \rangle$ on $L^2\,[0,1]$ (Razzaghi [19]).
If the above infinite series is truncated then (1) is written as

$$S_{\nu^k,M}(t) = \sum_{n=1}^{\nu^k} \sum_{m=0}^{M-1} c_{n,m}^{(\nu)} \psi_{n,m}^{(\nu)}(t) = C^T \psi^{(\nu)}(t), \tag{2}$$

where $C = \left[c_{1,0}^{(\nu)}, c_{1,1}^{(\nu)}, ..., c_{1,M-1}^{(\nu)}, c_{2,0}^{(\nu)}, ..., c_{2,M-1}^{(\nu)}, ..., c_{\nu^k,0}^{(\nu)}, ..., c_{\nu^k,M-1}^{(\nu)}\right]^T$
and $\psi^{(\nu)}(t) = \left[\psi_{1,0}^{(\nu)}, \psi_{1,1}^{(\nu)}, ..., \psi_{1,M-1}^{(\nu)}, \psi_{2,0}^{(\nu)}, ..., \psi_{2,M-1}^{(\nu)}, ..., \psi_{\nu^k,0}^{(\nu)}, ..., \psi_{\nu^k,M-1}^{(\nu)}\right]^T$.

2.3. Wavelet Approximation

We define

$$\|g\|_1 = \int_0^1 |g(t)|\, dt$$

and

$$\|g\|_2 = \left\{ \int_0^1 |g(t)|^2\, dt \right\}^{\frac{1}{2}}.$$

The Wavelet Approximation $E_{\nu^k,M}$ of function g by $S_{\nu^k,M}$ of its ELW expansion under the norm $\|.\|_2$ is defined by

$$E_{\nu^k,M}(g) = inf \left\|g - S_{\nu^k,M}\right\|_2 \quad (Zygmund[24]).$$

If $E_{\nu^k,M}(g) \to 0$ as $k \to \infty, M \to \infty$ then $E_{\nu^k,M}(g)$ is called the best wavelet approximation of function g. In the similarly manner we can define the wavelet approximation under norm $\|.\|_1$.

2.4. Lip_α[0,1] Class

A function $g \in Lip_\alpha\,[0,1]$, for any $0 < \alpha \leq 1$, if

$$\left|g(x) - g(y)\right| \leq k_1 \left|x - y\right|^\alpha, \quad \forall\ x, y \in [0,1].$$

3. Theorems

In this paper following new approximation theorems have been proved :

Theorem 3.1. *Let g be a differentiable function on the interval $[0,1]$ such that $g' \in Lip_\alpha\,[0,1]$ and its ELW series is given by equation (1) with $(\nu^k, M)^{th}$ partial sums given by equation (2). Then the ELW approximation $E_{\nu^k,M}$ of function g by its $(\nu^k, M)^{th}$ partial sums $S_{\nu^k,M}$ of ELW series under norm $\|.\|_2$ is given by*

$$E_{\nu^k,M}(g) = \begin{cases} O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right), & M = 1; \\ O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-1)^{\frac{1}{2}}}\right), & M \geq 2. \end{cases}$$

with $0 < \alpha \leq 1$.

Theorem 3.2. *Let g be a function on the interval $[0,1]$ such that $g'' \in Lip_\alpha\,[0,1]$ and its ELW series is given by equation (1) with $(\nu^k, M)^{th}$ partial sums given by equation (2). Then the ELW approximation $E_{\nu^k,M}$ of g by its $(\nu^k, M)^{th}$ partial sums under norm $\|.\|_2$ is given by*

$$E_{\nu^k,M}(g) = \begin{cases} O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^k}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right)\right), & M = 1,2; \\ O\left(\frac{1}{\nu^{2k}}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-3)^{\frac{3}{2}}}\right), & M \geq 3. \end{cases}$$

$0 < \alpha \leq 1$.

Theorem 3.3. *Let g be a function on the interval $[0,1]$ such that $g''' \in Lip_\alpha\,[0,1]$ and its ELW series is given by equation (1) with $(\nu^k, M)^{th}$ partial sums is given by equation (2). Then the ELW approximation $E_{\nu^k,M}$ of g by its $(\nu^k, M)^{th}$ partial sums under norm $\|.\|_2$ is given by*

$$E_{\nu^k,M}(g) = \begin{cases} O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^k} + \frac{1}{\nu^{2k}}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right)\right), & M = 1,2,3; \\ O\left(\frac{1}{\nu^{3k}}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-5)^{\frac{5}{2}}}\right), & M \geq 4. \end{cases}$$

$0 < \alpha \leq 1$.

Theorem 3.4. *Let g be a differentiable function such that $g'' \in Lip_\alpha[0,1]$ and its ELW approximation $E_{\nu^k,M}$ of g by $S_{\nu^k,M}$ under norm $\|.\|_1$ is given by*

$$\begin{aligned} E_{\nu^k,M}(g) &= inf\left\|g - S_{\nu^k,M}\right\|_1 \\ &= O\left(\frac{1}{\nu^{\frac{3k}{2}}}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-3)}\right), \end{aligned}$$

$0 < \alpha \leq 1,\ M \geq 3.$

Theorem 3.5. *Let g be a function on the interval $[0,1]$ such that $g''' \in Lip_\alpha[0,1]$ and its ELW approximation $E_{\nu^k,M}$ of g by $S_{\nu^k,M}$ under norm $\|.\|_1$ is given by*

$$\begin{aligned} E_{\nu^k,M}(g) &= inf\left\|g - S_{\nu^k,M}\right\|_1 \\ &= O\left(\frac{1}{\nu^{\frac{5k}{2}}}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-5)^2}\right), \end{aligned}$$

$0 < \alpha \leq 1,\ M \geq 4.$

4. Proofs

Proof of Theorem(3.1) (For M=1)

The ELW expansion of a function $g \in L^2(\mathbb{R})$ for $M = 1$ is

$$g(t) = \sum_{n=1}^{\infty} c_{n,0}^{(\nu)}\psi_{n,o}^{(\nu)}(t).$$

Let

$$\begin{aligned} e_n^{(\nu)}(t) &= c_{n,0}^{(\nu)}\psi_{n,0}^{(\nu)}(t) - g(t)\chi_{\left[\frac{n-1}{\nu^k},\frac{n}{\nu^k}\right)}, \quad \frac{n-1}{\nu^k} \leq t < \frac{n}{\nu^k}. \\ \sum_{n=1}^{\nu^k} e_n^{(\nu)}(t) &= \sum_{n=1}^{\nu^k} c_{n,0}^{(\nu)}\ \psi_{n,0}^{(\nu)}(t) - \sum_{n=1}^{\nu^k} g(t)\chi_{\left[\frac{n-1}{\nu^k},\frac{n}{\nu^k}\right)} \\ &= S_{\nu^k,1}(t) - g(t). \end{aligned} \tag{3}$$

$$\begin{aligned} \left\|e_n^{(\nu)}\right\|_2^2 &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} |e_n^{(\nu)}(t)|^2\,dt \\ &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} |c_{n,0}^{(\nu)}\ \psi_{n,0}^{(\nu)}(t) - g(t)|^2\,dt \\ &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\,dt - (c_{n,0}^{(\nu)})^2. \end{aligned} \tag{4}$$

Next,

$$\begin{aligned} c_{n,0}^{(\nu)} &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\, \psi_{n,0}^{(\nu)}(t)\, dt \\ &= \nu^{\frac{k}{2}} \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\, dt \\ &= \nu^{\frac{k}{2}} \left\{ \frac{1}{\nu^k} g\left(\frac{n-1}{\nu^k}\right) + \int_0^{\frac{1}{\nu^k}} u g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right) du \right\}. \qquad (5) \\ &\quad \left(where, \quad 0 < \theta < 1 \;\; and \;\; t = u + \frac{n-1}{\nu^k} \right). \end{aligned}$$

Since

$$\int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\, dt = \int_0^{\frac{1}{\nu^k}} g^2\left(u + \frac{n-1}{\nu^k}\right) du$$

By using Taylor's theorem, we get

$$= \frac{g^2\left(\frac{n-1}{\nu^k}\right)}{\nu^k} + \int_0^{\frac{1}{\nu^k}} u^2 \left(g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right)\right)^2 du + 2g\left(\frac{n-1}{\nu^k}\right) \int_0^{\frac{1}{\nu^k}} u g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right) du. \qquad (6)$$

using equations(4)-(6), we have

$$\begin{aligned} \left\| e_n^{(\nu)} \right\|_2^2 &= \int_0^{\frac{1}{\nu^k}} u^2 \left(g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right)\right)^2 du - \nu^k \left(\int_0^{\frac{1}{\nu^k}} u g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right) du \right)^2 \\ &\le |I_1| + \nu^k\, |I_2|^2. \qquad (7) \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{\nu^k}} u^2 \left(g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right)\right)^2 du, \\ I_2 &= \int_0^{\frac{1}{\nu^k}} u g^{'}\left(\frac{n-1}{\nu^k} + \theta u\right) du. \end{aligned}$$

Now, taking modulus on both side of I_1 and simlifying, we get

$$
\begin{aligned}
|I_1| &\le \int_0^{\frac{1}{\nu^k}} k_1^2 u^2 \, |\theta u|^{2\alpha} \, du + \frac{M_1^2}{3\nu^{3k}} + 2M_1 \int_0^{\frac{1}{\nu^k}} k_1 u^2 \, |\theta u|^{\alpha} \, du, \\
&\quad \left(say, \quad \left| g' \left(\frac{n-1}{\nu^k} \right) \right| \le M_1 \right). \\
&\le \frac{k_1^2}{(2\alpha+3)\nu^{k(2\alpha+3)}} + \frac{M_1^2}{3\nu^{3k}} + \frac{2M_1 k_1}{(\alpha+3)\nu^{k(\alpha+3)}} \\
&\le \frac{M'}{\nu^{3k}} \left\{ \frac{1}{\nu^{2k\alpha}} + 1 + \frac{2}{\nu^{k\alpha}} \right\}, \\
&\quad \left(say, \ M' = max \left\{ k_1^2, M_1^2, M_1 k_1 \right\} \ also, \ \frac{1}{5} \le \frac{1}{2\alpha+3} < \frac{1}{3} < 1 \right). \\
&\le \frac{M'}{\nu^{3k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\le \frac{M''}{\nu^{2k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}, \\
&\quad \left(say, \ M'' = max \left\{ k_1, M_1 \right\} \ also, \ \frac{1}{3} \le \frac{1}{\alpha+2} < \frac{1}{2} < 1 \right).
\end{aligned}
$$

Putting the values of $|I_1|$ and $|I_2|$ in equation (7), we get

$$
\begin{aligned}
\left\| e_n^{(\nu)} \right\|_2^2 &\le \frac{M'}{\nu^{3k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2 + \nu^k \frac{M''^2}{\nu^{4k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2, \\
&= \frac{M' + M''^2}{\nu^{3k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2 \\
&= \frac{M'''}{\nu^{3k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2, \quad (say, \ M''' = M' + M''^2).
\end{aligned}
$$

Next, taking the norm of both side of equation (3) and using above equation, we have

$$
\begin{aligned}
\left\| g - S_{\nu^k,1} \right\|_2^2 &\le \sum_{n=1}^{\nu^k} \left\| e_n^{(\nu)} \right\|_2^2 \\
&\le \sum_{n=1}^{\nu^k} \frac{M'''}{\nu^{3k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2 \\
&= \frac{M'''}{\nu^{2k}} \left\{ 1 + \frac{1}{\nu^{k\alpha}} \right\}^2, \qquad 0 < \alpha \le 1. \\
\therefore \quad \left\| g - S_{\nu^k,1} \right\|_2 &= O\left(\frac{1}{\nu^k} \left(1 + \frac{1}{\nu^{k\alpha}} \right) \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E_{\nu^k,1}(g) &= inf \left\| g - S_{\nu^k,1} \right\|_2 \\
&= O\left(\frac{1}{\nu^k} \left(1 + \frac{1}{\nu^{k\alpha}} \right) \right), \qquad 0 < \alpha \le 1.
\end{aligned}
$$

Proof of Theorem (3.1)(For M ≥ 2)

Now,

$$\begin{aligned}
c_{n,m}^{(\nu)} &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\ \psi_{n,m}^{(\nu)}\,dt \\
&= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\ \sqrt{2m+1}\nu^{\frac{k}{2}} L_m(2\nu^k t-2n+1)\,dt, \quad (take,\ \ 2\nu^k t-2n+1=u) \\
&= \frac{\sqrt{2m+1}}{2\nu^{\frac{k}{2}}}\int_{-1}^{1} g\left(\frac{u+2n-1}{2\nu^k}\right)L_m(u)\,du \qquad \left(\because \quad L_m(u)=\frac{L'_{m+1}(u)-L'_{m-1}(u)}{2m+1}\right) \\
&= \frac{1}{4\nu^{\frac{3k}{2}}\ \sqrt{2m+1}}\left\{\int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m-1}(u)\,du - \int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m+1}(u)\,du\right\}. \qquad (8)
\end{aligned}$$

Thus,

$$\left|c_{n,m}^{(\nu)}\right| \le \frac{1}{4\nu^{\frac{3k}{2}}\ \sqrt{2m+1}}\left\{|J_1|+|J_2|\right\}. \qquad (9)$$

Where,

$$\begin{aligned}
J_1 &= \int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m-1}(u)\,du \\
J_2 &= \int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m+1}(u)\,du.
\end{aligned}$$

Taking modulus on both side of J_1 and using Cauchy - Schwarz inequality, we get

$$\begin{aligned}
|J_1| &\le \frac{k_1}{2^\alpha \nu^{k\alpha}}\left(\frac{2}{2\alpha+1}\right)^{\frac{1}{2}}\left(\frac{2}{2m-1}\right)^{\frac{1}{2}} + N_1\left(\frac{4}{2m-1}\right)^{\frac{1}{2}} \\
&= \frac{2k_1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}\sqrt{2m-1}} + \frac{2N_1}{\sqrt{2m-1}}. \qquad (10) \\
&\left(say,\quad \left|g'\left(\frac{2n-1}{2\nu^k}\right)\right| \le N_1\right).
\end{aligned}$$

Similarly,

$$|J_2| \le \frac{2k_1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}\sqrt{2m+3}} + \frac{2N_1}{\sqrt{2m+3}}. \qquad (11)$$

By using equations (9) - (11), we have

$$\begin{aligned}
\left|c_{n,m}^{(\nu)}\right| &\le \frac{N_2}{\nu^{\frac{3k}{2}}\ (2m-1)}\left\{\frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}, \qquad (12) \\
&(N_2 = max\left\{k_1, N_1\right\}).
\end{aligned}$$

Since

$$g(t) - S_{\nu^k,M}(t) = \sum_{n=1}^{\nu^k} \sum_{m=M}^{\infty} c_{n,m}^{(\nu)} \psi_{n,m}^{(\nu)}(t). \tag{13}$$

So, taking norm on both sides of equation (13) and using the equation (12), we have

$$\begin{aligned} \left\| g - S_{\nu^k,M} \right\|_2^2 &\leq N_2^2 \sum_{n=1}^{\nu^k} \frac{1}{\nu^{3k}(2M-1)} \left\{ \frac{1}{2^\alpha \nu^{k\alpha} \sqrt{2\alpha+1}} + 1 \right\}^2 \\ &\quad \left(\because \sum_{m=M}^{\infty} \frac{1}{(2m-1)^2} \leq \frac{1}{(2M-1)}, \quad M > 1 \right) \\ &= \frac{N_2^2}{\nu^{3k}(2M-1)} \left\{ \frac{1}{2^\alpha \nu^{k\alpha} \sqrt{2\alpha+1}} + 1 \right\}^2 \nu^k. \end{aligned}$$

Hence,

$$\begin{aligned} E_{\nu^k,M}(g) &= inf \left\| g - S_{\nu^k,M} \right\|_2 \\ &= O\left(\frac{1}{\nu^k} \left(1 + \frac{1}{2^\alpha \nu^{k\alpha} \sqrt{2\alpha+1}} \right) \frac{1}{(2M-1)^{\frac{1}{2}}} \right), \ M \geq 2. \end{aligned}$$

Proof of Theorem (3.2)(For M=1)

Now, following the proof of theorem (3.1, For M=1), we get

$$\begin{aligned} c_{n,0}^{(\nu)} &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\, \psi_{n,0}^{(\nu)}(t)\, dt \\ &= \frac{g\left(\frac{n-1}{\nu^k}\right)}{\nu^{\frac{k}{2}}} + \frac{g'\left(\frac{n-1}{\nu^k}\right)}{2\nu^{\frac{3k}{2}}} + \frac{\nu^{\frac{k}{2}}}{2} \int_0^{\frac{1}{\nu^k}} u^2 g''\left(\frac{n-1}{\nu^k} + \theta u\right) du. \end{aligned} \tag{14}$$

By eq^n.(4), we get

$$\begin{aligned} \left\| e_n^{(\nu)} \right\|_2^2 &= \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\, dt - (c_{n,0}^{(\nu)})^2 \\ &\leq \frac{M_1^2}{12\nu^{3k}} + \frac{1}{4}|T_1| + \frac{\nu^k}{4}|T_2|^2 + M_1|T_3| + \frac{M_1}{2\nu^k}|T_2|. \end{aligned} \tag{15}$$

Where,

$$
\begin{aligned}
T_1 &= \int_0^{\frac{1}{\nu^k}} u^4 g''^2\left(\frac{n-1}{\nu^k}+\theta u\right) du,\\
T_2 &= \int_0^{\frac{1}{\nu^k}} u^2 g''\left(\frac{n-1}{\nu^k}+\theta u\right) du,\\
T_3 &= \int_0^{\frac{1}{\nu^k}} u^3 g''\left(\frac{n-1}{\nu^k}+\theta u\right) du.
\end{aligned}
$$

Now, taking the modulus of both side of T_1 and solving, we get

$$
\begin{aligned}
|T_1| &\leq k_1^2\int_0^{\frac{1}{\nu^k}} |u|^{2\alpha+4}\,du + M_2^2\int_0^{\frac{1}{\nu^k}} |u|^4\,du + 2M_2k_1\int_0^{\frac{1}{\nu^k}} |u|^{\alpha+4}\,du,\\
&= \frac{k_1^2}{(2\alpha+5)\nu^{k(2\alpha+5)}} + \frac{M_2^2}{5\nu^{5k}} + \frac{2M_2k_1}{(\alpha+5)\nu^{k(\alpha+5)}}\\
&\leq \frac{M_2'}{\nu^{5k}}\left\{\frac{1}{\nu^{2k\alpha}}+1+\frac{1}{\nu^{k\alpha}}\right\}, \quad \left(where,\ M_2' = max\left\{k_1^2, M_2^2, 2M_2k_1\right\},\ \left|g''\left(\frac{n-1}{\nu^k}\right)\right| \leq M_2\right).\\
&\leq \frac{M_2'}{\nu^{5k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}^2.
\end{aligned}
$$

Similarly,

$$
|T_2| \leq \frac{M_2''}{\nu^{3k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}, \quad (say,\ M_2'' = max\{k_1, M_2\}).
$$

and,

$$
|T_3| \leq \frac{M_2''}{\nu^{4k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}.
$$

Putting the values of $|T_1|, |T_2|, |T_3|$ in equation (15), we get

$$
\begin{aligned}
\left\|e_n^{(\nu)}\right\|_2^2 &\leq \frac{M_1^2}{12\nu^{3k}} + \frac{1}{4}\frac{M_2'}{\nu^{5k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}^2 + \frac{\nu^k}{4}\frac{M_2''^2}{\nu^{6k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}^2 + \frac{M_1M_2''}{\nu^{4k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\} + \frac{M_1M_2''}{2\nu^{4k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}\\
&\leq \frac{M_1^2}{12\nu^{3k}} + \frac{M_2'+M_2''^2}{4\nu^{5k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}^2 + \frac{3M_1M_2''}{2\nu^{4k}}\left\{1+\frac{1}{\nu^{k\alpha}}\right\}\\
&\leq \frac{N_3}{\nu^{3k}}\left\{1+\frac{1}{\nu^{2k}}\left(1+\frac{1}{\nu^{k\alpha}}\right)^2 + \frac{2}{\nu^k}\left(1+\frac{1}{\nu^{k\alpha}}\right)\right\}\\
&\leq \frac{N_3}{\nu^{3k}}\left\{1+\frac{1}{\nu^k}\left(1+\frac{1}{\nu^{k\alpha}}\right)\right\}^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E_{\nu^k,1}(g) &= inf\left\|g - S_{\nu^k,M}\right\|_2\\
&= O\left(\frac{1}{\nu^k}\left(1+\frac{1}{\nu^k}\left(1+\frac{1}{\nu^{k\alpha}}\right)\right)\right).
\end{aligned}
$$

This completes the proof of the theorem.

Proof of Theorem (3.2)(For M=2)

Since

$$\left\|e_n^{(\nu)}\right\|_2^2 = \int_{\frac{n-1}{n^k}}^{\frac{n}{\nu^k}} g^2(t)\,dt - (c_{n,0}^{(\nu)})^2 - (c_{n,1}^{(\nu)})^2.$$

Now,

$$c_{n,1}^{(\nu)} = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\,\psi_{n,1}^{(\nu)}(t)\,dt.$$

On simplifying the above integrals and using the theorem (3.2), we obtain

$$E_{\nu^k,2}(g) = O\left(\frac{1}{\nu^k}\left(1+\frac{1}{\nu^k}\left(1+\frac{1}{\nu^{k\alpha}}\right)\right)\right).$$

Proof of Theorem (3.2)(For M ≥ 3)

Following the equation (8), we get

$$c_{n,m}^{(\nu)} = \frac{1}{4\nu^{\frac{3k}{2}}\sqrt{(2m+1)}}\left\{\int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m-1}(u)\,du - \int_{-1}^{1} g'\left(\frac{u+2n-1}{2\nu^k}\right)L_{m+1}(u)\,du\right\}. \tag{16}$$

Next, integrating by part the above integral and after simplifying, we get

$$\begin{aligned}
\left|c_{n,m}^{(\nu)}\right| &\le \frac{1}{8\nu^{\frac{5k}{2}}\sqrt{(2m+1)}}\int_{-1}^{1}\left|g''\left(\frac{u+2n-1}{2\nu^k}\right)-g''\left(\frac{2n-1}{2\nu^k}\right)+g''\left(\frac{2n-1}{2\nu^k}\right)\right| \\
&\quad\times\left|\frac{(2m-1)L_{m+2}(u)-(4m+2)L_m(u)+(2m+3)L_{m-2}(u)}{(2m+3)(2m-1)}\right|du, \\
\left|c_{n,m}^{(\nu)}\right|^2 &\le \frac{6S_2^2}{16\nu^{5k}(2m+1)}\left\{\frac{1}{2^\alpha\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{1}{(2m-1)^2(2m-3)}, \\
&\le \frac{S_2^2}{\nu^{5k}(2m-3)^4}\left\{\frac{1}{2^\alpha\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2. \qquad (17)\\
&\left(where,\ \left|g''\left(\frac{2n-1}{2\nu^k}\right)\right|\le S_1,\ \ S_2 = max\,\{k_1,S_1\}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\left\|g(t)-S_{\nu^k,M}(t)\right\|_2^2 &= \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty}\left|c_{n,m}^{(\nu)}\right|^2 \\
&\le \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty}\frac{S_2^2}{\nu^{5k}(2m-3)^4}\left\{\frac{1}{2^\alpha\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2 \\
&\le \frac{S_2^2}{(2M-3)^3}\sum_{n=1}^{\nu^k}\left\{\frac{1}{2^\alpha\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{1}{\nu^{5k}} \\
&= \frac{S_2^2}{(2M-3)^3}\left\{\frac{1}{2^\alpha\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{1}{\nu^{4k}}.
\end{aligned}$$

Hence,

$$E_{\nu^k,M}(g) = inf\left\|g - S_{\nu^k,M}\right\|_2 = O\left(\frac{1}{\nu^{2k}}\left(1 + \frac{1}{2^\alpha \nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-3)^{\frac{3}{2}}}\right), \ M \geq 3.$$

Proof of Theorem (3.3)(For M = 1)

Since

$$\left\|e_n^{(\nu)}\right\|_2^2 = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\,dt - (c_{n,0}^{(\nu)})^2.$$

Now,

$$c_{n,0}^{(\nu)} = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\,\psi_{n,0}^{(\nu)}(t)\,dt.$$

On solving the above integrals and using the theorem (3.2) for $M = 1$, we get

$$E_{\nu^k,1}(g) = O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^k} + \frac{1}{\nu^{2k}}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right)\right).$$

Proof of Theorem (3.3)(For M = 2)

Since

$$\left\|e_n^{(\nu)}\right\|_2^2 = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\,dt - (c_{n,0}^{(\nu)})^2 - (c_{n,1}^{(\nu)})^2.$$

Now,

$$c_{n,1}^{(\nu)} = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\,\psi_{n,1}^{(\nu)}(t)\,dt.$$

Simplifying the above integrals and using the theorem (3.2) for $M = 1$, we have

$$E_{\nu^k,2}(g) = O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^k} + \frac{1}{\nu^{2k}}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right)\right).$$

Proof of Theorem (3.3)For M = 3)

Since

$$\left\|e_n^{(\nu)}\right\|_2^2 = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g^2(t)\,dt - (c_{n,0}^{(\nu)})^2 - (c_{n,1}^{(\nu)})^2 - (c_{n,2}^{(\nu)})^2.$$

Now,

$$c_{n,2}^{(\nu)} = \int_{\frac{n-1}{\nu^k}}^{\frac{n}{\nu^k}} g(t)\, \psi_{n,2}^{(\nu)}(t)\, dt.$$

On simplifying the above integrals and using the theorem (3.2) for $M = 1$, we get

$$E_{\nu^k,3}(g) = O\left(\frac{1}{\nu^k}\left(1 + \frac{1}{\nu^k} + \frac{1}{\nu^{2k}}\left(1 + \frac{1}{\nu^{k\alpha}}\right)\right)\right).$$

Proof of Theorem(3.3)(For M ≥ 4)

Following the equation (16), we have

$$\begin{aligned} c_{n,m}^{(\nu)} &= \frac{1}{8\nu^{\frac{5k}{2}}\sqrt{(2m+1)}} \left\{ \frac{-1}{(2m-1)} \int_{-1}^{1} g''\left(\frac{u+2n-1}{2\nu^k}\right)(L_m(u) - L_{m-2}(u))\, du \right. \\ &\quad \left. + \frac{1}{(2m+3)} \int_{-1}^{1} g''\left(\frac{u+2n-1}{2\nu^k}\right)(L_{m+2}(u) - L_m(u))\, du \right\} \\ \left|c_{n,m}^{(\nu)}\right| &\leq \frac{1}{16\nu^{\frac{7k}{2}}\sqrt{(2m+1)}} \int_{-1}^{1} \left|g'''\left(\frac{u+2n-1}{2\nu^k}\right) - g'''\left(\frac{2n-1}{2\nu^k}\right) + g'''\left(\frac{2n-1}{2\nu^k}\right)\right| \left|\frac{L_{m+1} - L_{m-1}}{(2m+1)(2m-1)}\right. \\ &\quad \left. - \frac{L_{m-1} - L_{m-3}}{(2m-1)(2m-3)} - \frac{L_{m+3} - L_{m+1}}{(2m+5)(2m+3)} + \frac{L_{m+1} - L_{m-1}}{(2m+1)(2m+3)}\right| du \\ &\leq \frac{1}{16\nu^{\frac{7k}{2}}\sqrt{(2m+1)}} \left(P_1 + P_2\right). \end{aligned} \tag{18}$$

Where,

$$\begin{aligned} P_1 &= \int_{-1}^{1} \left|g'''\left(\frac{u+2n-1}{2\nu^k}\right) - g'''\left(\frac{2n-1}{2\nu^k}\right)\right| \left|\frac{L_{m+1} - L_{m-1}}{(2m+1)(2m-1)} - \frac{L_{m-1} - L_{m-3}}{(2m-1)(2m-3)}\right. \\ &\quad \left. - \frac{L_{m+3} - L_{m+1}}{(2m+5)(2m+3)} + \frac{L_{m+1} - L_{m-1}}{(2m+1)(2m+3)}\right| du. \\ P_2 &= \int_{-1}^{1} \left|g'''\left(\frac{2n-1}{2\nu^k}\right)\right| \left|\frac{L_{m+1} - L_{m-1}}{(2m+1)(2m-1)} - \frac{L_{m-1} - L_{m-3}}{(2m-1)(2m-3)} - \frac{L_{m+3} - L_{m+1}}{(2m+5)(2m+3)} + \frac{L_{m+1} - L_{m-1}}{(2m+1)(2m+3)}\right| du. \end{aligned}$$

By using Cauchy - Schwarz inequality and simlifying the integral P_1, we have

$$\begin{aligned} P_1 &\leq \frac{\sqrt{2}k_1}{(2\nu^k)^\alpha \sqrt{2\alpha+1}} \left\{ \frac{1}{(2m+1)^2(2m-1)^2}\left(\frac{2}{2m+3} + \frac{2}{2m-1}\right) + \frac{1}{(2m-1)^2(2m-3)^2}\left(\frac{2}{2m-1} + \frac{2}{2m-5}\right) \right. \\ &\quad \left. + \frac{1}{(2m+3)^2(2m+5)^2}\left(\frac{2}{2m+7} + \frac{2}{2m+3}\right) + \frac{1}{(2m+3)^2(2m+1)^2}\left(\frac{2}{2m+3} + \frac{2}{2m-1}\right) \right\}^{\frac{1}{2}}. \end{aligned}$$

and

$$\begin{aligned} P_2 \quad \leq \quad & S_1^{'}\sqrt{2}\Bigg\{\frac{1}{(2m+1)^2(2m-1)^2}\left(\frac{2}{2m+3}+\frac{2}{2m-1}\right)+\frac{1}{(2m-1)^2(2m-3)^2}\left(\frac{2}{2m-1}+\frac{2}{2m-5}\right) \\ & +\frac{1}{(2m+3)^2(2m+5)^2}\left(\frac{2}{2m+7}+\frac{2}{2m+3}\right)+\frac{1}{(2m+3)^2(2m+1)^2}\left(\frac{2}{2m+3}+\frac{2}{2m-1}\right)\Bigg\}^{\frac{1}{2}}. \\ & \left(say,\quad \left|g^{'''}\left(\frac{2n-1}{2\nu^k}\right)\right|\leq S_1^{'}\right). \end{aligned}$$

Next, putting the values of P_1 *and* P_2 in equation (18), we get

$$\begin{aligned} \left|c_{n,m}^{(\nu)}\right|^2 \quad \leq \quad & \frac{2(S_2^{'})^2}{256\nu^{7k}(2m+1)}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{16}{(2m-5)^5} \\ & \left(say,\quad S_2^{'}=max(k_1,S_1^{'})\right) \\ \leq \quad & \frac{32(S_2^{'})^2}{256\nu^{7k}(2m-5)^6}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2,\ \ m\geq 3 \\ \leq \quad & \frac{(S_2^{'})^2}{\nu^{7k}(2m-5)^6}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2,\ \ m\geq 3. \end{aligned} \tag{19}$$

Since

$$\begin{aligned} \left\|g-S_{\nu^k,M}\right\|_2^2 \quad = \quad & \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty}\left|c_{n,m}^{(\nu)}\right|^2 \\ \leq \quad & \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty}\frac{(S_2^{'})^2}{\nu^{7k}(2m-5)^6}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2 \\ \leq \quad & \frac{(S_2^{'})^2}{(2M-5)^5}\sum_{n=1}^{\nu^k}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{1}{\nu^{7k}} \\ = \quad & \frac{(S_2^{'})^2}{(2M-5)^5}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}^2\frac{1}{\nu^{6k}}. \end{aligned}$$

Hence,

$$\begin{aligned} E_{\nu^k,M}(g) \quad = \quad & inf\left\|g-S_{\nu^k,M}\right\|_2 \\ = \quad & O\left(\frac{1}{\nu^{3k}}\left(1+\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-5)^{\frac{5}{2}}}\right). \end{aligned}$$

This completes the proof.

Proof of Theorem (3.4)

From equation (13), we have

$$\left\|g-S_{\nu^k,M}\right\|_1 \quad \leq \quad \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty}\left|c_{n,m}^{(\nu)}\right|. \tag{20}$$

Now, using equation (17) in above equation , we get

$$\begin{aligned}
\left\|g - S_{\nu^k,M}\right\|_1 &\leq \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty} \frac{S_2}{\nu^{\frac{5k}{2}}(2m-3)^2}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\} \\
&\leq S_2\sum_{n=1}^{\nu^k}\frac{1}{\nu^{\frac{5k}{2}}(2M-3)}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\} \\
&= \frac{S_2}{\nu^{\frac{5k}{2}}(2M-3)}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}\nu^k. \\
\therefore \quad \left\|g - S_{\nu^k,M}\right\|_1 &\leq \frac{1}{\nu^{\frac{3k}{2}}}\left(1+\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}\frac{1}{(2M-3)}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
E_{\nu^k,M}(g) &= inf\left\|g - S_{\nu^k,M}\right\|_1 \\
&= O\left(\frac{1}{\nu^{\frac{3k}{2}}}\left(1+\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-3)}\right).
\end{aligned}$$

Proof of Theorem (3.5)

From equations(19)) & (20), we have

$$\begin{aligned}
\left\|g - S_{\nu^k,M}\right\|_1 &\leq \sum_{n=1}^{\nu^k}\sum_{m=M}^{\infty} \frac{S_2'}{\nu^{\frac{7k}{2}}(2m-5)^3}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\} \\
&\leq S_2'\sum_{n=1}^{\nu^k}\frac{1}{\nu^{\frac{7k}{2}}(2M-5)^2}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\} \\
&= \frac{S_2'}{\nu^{\frac{7k}{2}}(2M-5)^2}\left\{\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right\}\nu^k. \\
\therefore \left\|g - S_{\nu^k,M}\right\|_1 &\leq \frac{1}{\nu^{\frac{5k}{2}}}\left(\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}+1\right)\frac{1}{(2M-5)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
E_{\nu^k,M}(g) &= inf\left\|g - S_{\nu^k,M}\right\|_1 \\
&= O\left(\frac{1}{\nu^{\frac{5k}{2}}}\left(1+\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-5)^2}\right).
\end{aligned}$$

This completes proof of the theorem.

5. Remarks

In the view of above theorems, we observe

Theorem 5.1. *Let g be a differentiable function in $[0,1]$ such that $g^{Nth} \in Lip_{\alpha}[0,1]$ and its ELW approximation $E_{\nu^k,M}$ of function g under norm $\|.\|_2$ is given by*

$$E_{\nu^k,M}(g) = \begin{cases} O\left(\frac{1}{\nu^k}\left(1+\frac{1}{\nu^k}+\cdots+\frac{1}{\nu^{(N-1)k}}\left(1+\frac{1}{\nu^{k\alpha}}\right)\right)\right), & M = 1,2,3,\ldots,N; \\ O\left(\frac{1}{\nu^{Nk}}\left(1+\frac{1}{2^{\alpha}\nu^{k\alpha}\sqrt{2\alpha+1}}\right)\frac{1}{(2M-(2N-1))^{\frac{2N-1}{2}}}\right), & M \geq N+1. \end{cases}$$

6. Numerical verification of Wavelet approximation

This section is designed to see the numerical accuracy of the calculated approximation for the function

$$h(t) = \begin{cases} sin(\pi t), & t \in [0,1] \\ 0, & otherwise. \end{cases}$$

for $M = 1,2$. The value of $S_{\nu^k,M}$ for $\nu = 2,3,5$ and $k = 0,1,2,3$ are calculated and are given as

$$S_{2^0,1}(t) = \begin{cases} 0.6366197723, & 0 \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{2^1,1}(t) = \begin{cases} 0.6366197723, & 0 \leq t < \frac{1}{2}, \\ 0.6366197723, & \frac{1}{2} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{2^2,1}(t) = \begin{cases} 0.37292322857, & 0 \leq t < \frac{1}{4}, \\ 0.900316316157, & \frac{1}{4} \leq t < \frac{2}{4}, \\ 0.900316316157, & \frac{2}{4} \leq t < \frac{3}{4}, \\ 0.37292322857, & \frac{3}{4} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{2^3,1}(t) = \begin{cases} 0.19383917874, & 0 \leq t < \frac{1}{8}, \\ 0.55200727841, & \frac{1}{8} \leq t < \frac{2}{8}, \\ 0.826137273909, & \frac{2}{8} \leq t < \frac{3}{8}, \\ 0.974495358174, & \frac{3}{8} \leq t < \frac{4}{8}, \\ 0.974495358174, & \frac{4}{8} \leq t < \frac{5}{8}, \\ 0.826137273909, & \frac{5}{8} \leq t < \frac{6}{8}, \\ 0.55200727841, & \frac{6}{8} \leq t < \frac{7}{8}, \\ 0.19383917874, & \frac{7}{8} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{3^0,1}(t) = \begin{cases} 0.6366197723, & 0 \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{3^1,1}(t) = \begin{cases} 0.4774648292, & 0 \leq t < \frac{1}{3}, \\ 0.95492965855, & \frac{1}{3} \leq t < \frac{2}{3}, \\ 0.4774648292, & \frac{2}{3} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{3^2,1}(t) = \begin{cases} 0.17276791512312, & 0 \leq t < \frac{1}{9}, \\ 0.497465385022, & \frac{1}{9} \leq t < \frac{2}{9}, \\ 0.7621611876812, & \frac{2}{9} \leq t < \frac{3}{9}, \\ 0.9349291028044, & \frac{3}{9} \leq t < \frac{4}{9}, \\ 0.9949307700452, & \frac{4}{9} \leq t < \frac{5}{9}, \\ 0.9349291028044, & \frac{5}{9} \leq t < \frac{6}{9}, \\ 0.7621611876812, & \frac{6}{9} \leq t < \frac{7}{9}, \\ 0.497465385022, & \frac{7}{9} \leq t < \frac{8}{9}, \\ 0.17276791512312, & \frac{8}{9} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

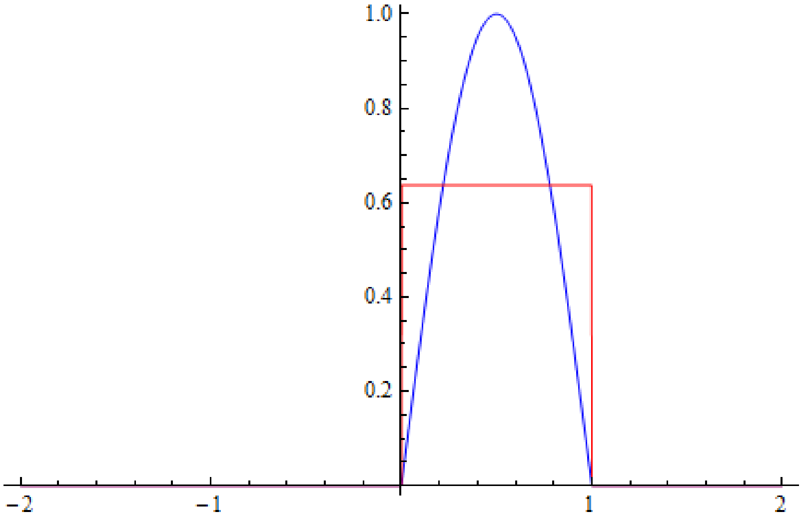

Figure 1: Graphical representation of $S_{2^0,1}$ and $h(t)$

$$S_{5^1,1}(t) = \begin{cases} 0.303958893917 42, & 0 \le t < \frac{1}{5}, \\ 0.79577471545927, & \frac{1}{5} \le t < \frac{2}{5}, \\ 0.98363164308148, & \frac{2}{5} \le t < \frac{3}{5}, \\ 0.79577471545927, & \frac{3}{5} \le t < \frac{4}{5}, \\ 0.30395889391742, & \frac{4}{5} \le t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{5^1,2}(t) = \begin{cases} 0.30395889391742 + 0.07638677816 \times \sqrt{15}(10t-1), & 0 \le t < \frac{1}{5}, \\ 0.79577471545927 + 0.04720962519 \times \sqrt{15}(10t-3), & \frac{1}{5} \le t < \frac{2}{5}, \\ 0.98363164308148, & \frac{2}{5} \le t < \frac{3}{5}, \\ 0.79577471545927 - 0.04720962519 \times \sqrt{15}(10t-7), & \frac{3}{5} \le t < \frac{4}{5}, \\ 0.30395889391742 - 0.07638677816 \times \sqrt{15}(10t-9), & \frac{4}{5} \le t < 1, \\ 0, & otherwise. \end{cases}$$

The graphs of $S_{\nu^k,M}$ and $h(t)$ has been plotted for $\nu = 2,3,5$ $M = 1,2$ and $k = 0,1,2,3$ in figure 1, 2, 3, 4, 5, 6, 7, 8 and 9.

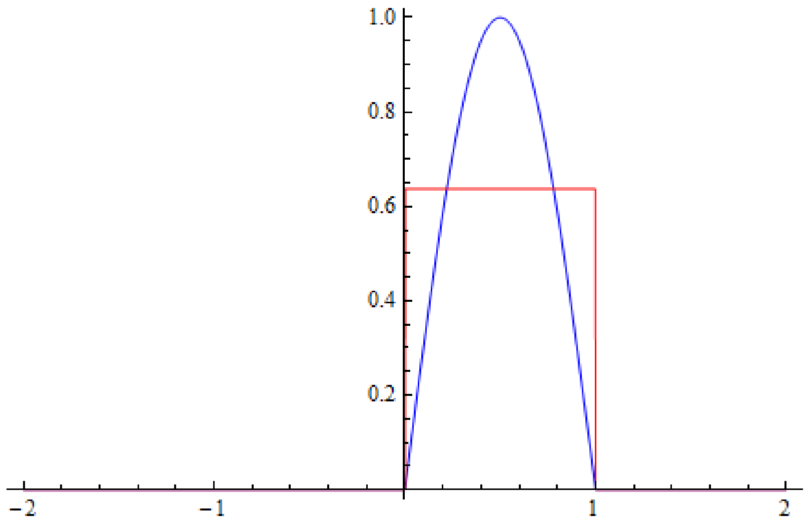

Figure 2: Graphical representation of $S_{2^1,1}$ and $h(t)$

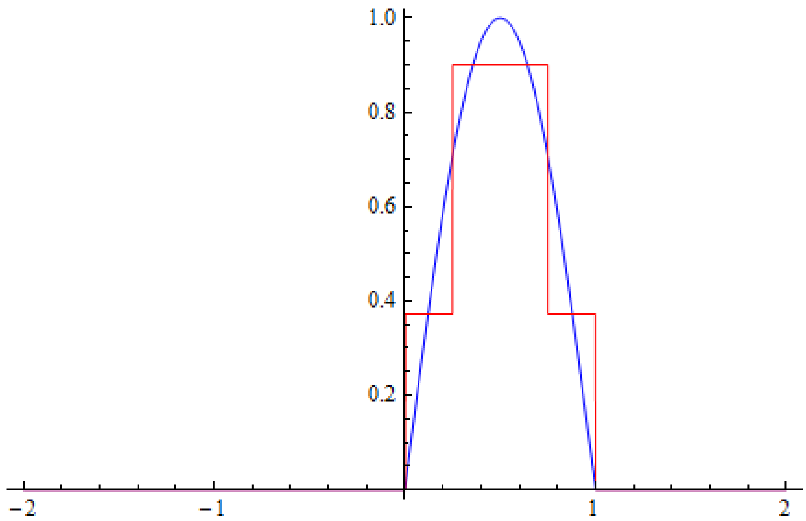

Figure 3: Graphical representation of $S_{2^2,1}$ and $h(t)$

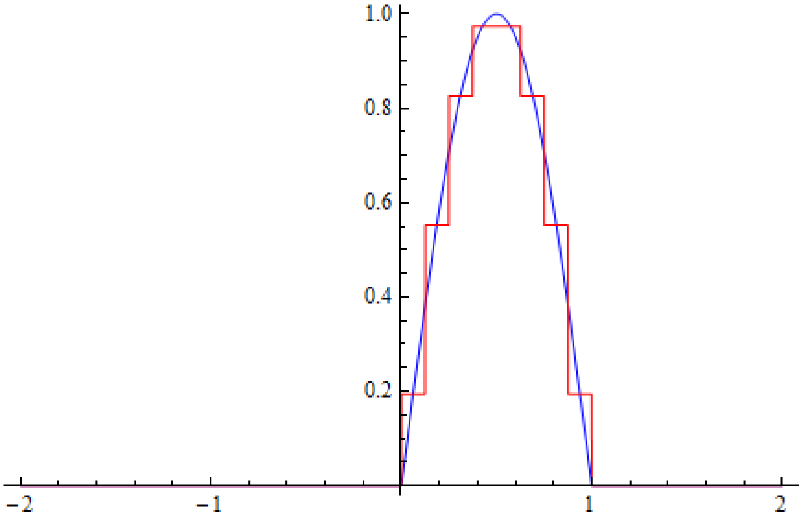

Figure 4: Graphical representation of $S_{2^3,1}$ and $h(t)$

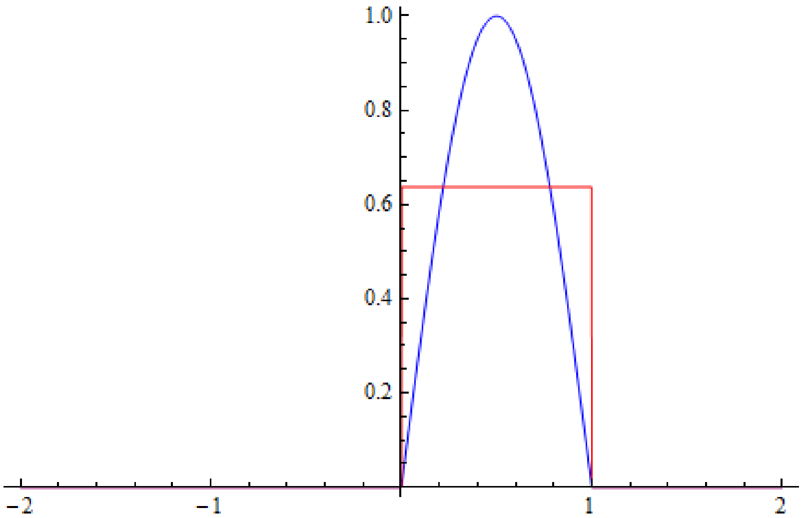

Figure 5: Graphical representation of $S_{3^0,1}$ and $h(t)$

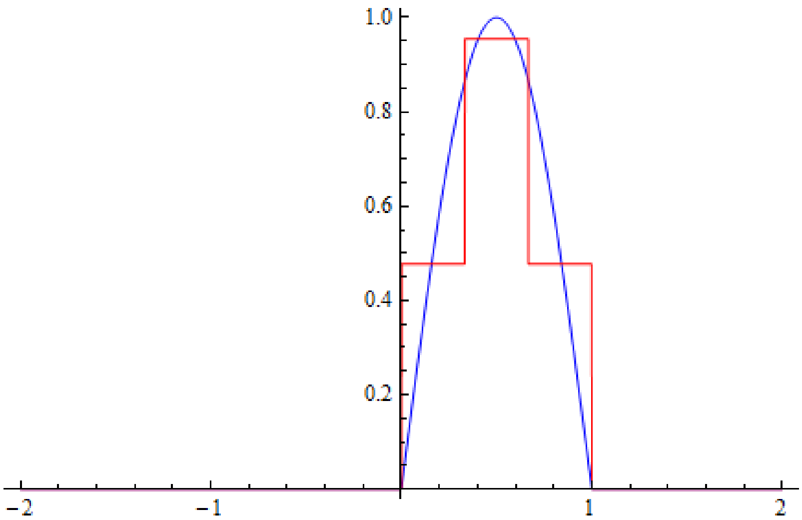

Figure 6: Graphical representation of $S_{3^1,1}$ and $h(t)$

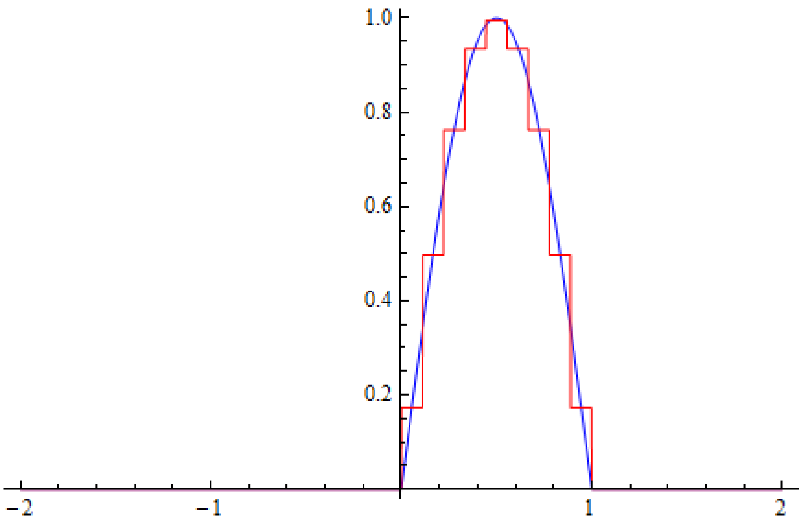

Figure 7: Graphical representation of $S_{3^2,1}$ and $h(t)$

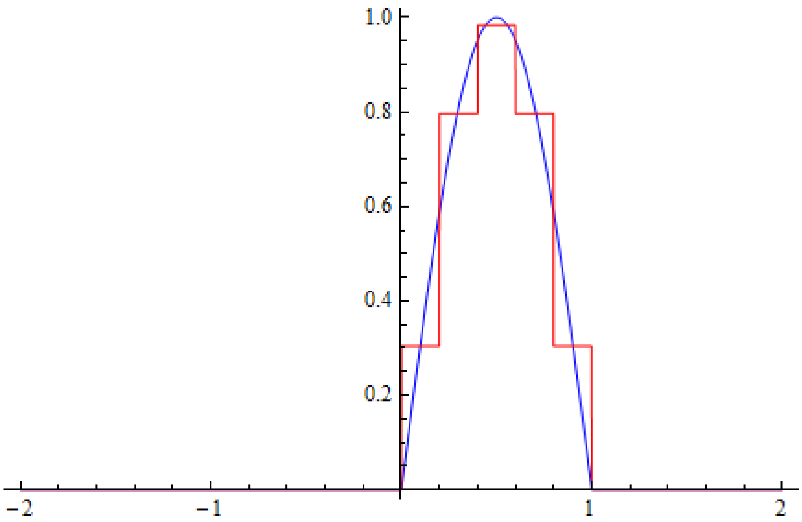

Figure 8: Graphical representation of $S_{5^1,1}$ and $h(t)$

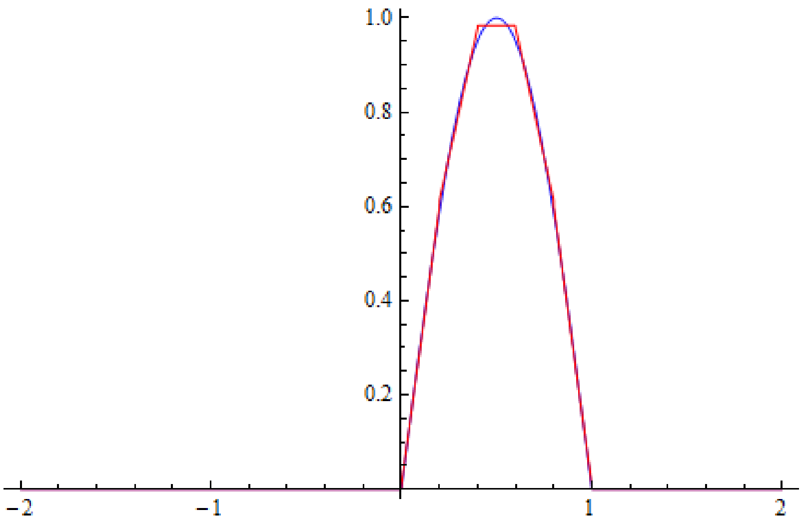

Figure 9: Graphical representation of $S_{5^1,2}$ and $h(t)$

7. Applications

7.1. Solution of Hermite differential equation (HDE) of order-1 by using ELW method

Consider the HDE of order-1

$$y'' - 2ty' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2. \tag{21}$$

Now, we have solved the equation (21) by the ELW operational matrix of integration for $\nu = 2, k = 2, M = 4$ as below :

Let

$$\psi^{(\nu)}(t) = \left[\psi^{(\nu)}_{1,0}, \psi^{(\nu)}_{1,1}, \ldots, \psi^{(\nu)}_{4,2}, \psi^{(\nu)}_{4,3}\right]^T.$$

Here $\psi^{(\nu)}(t)$ be $(\nu^k \times M) \times 1$ column vector, the integration of above vector $\psi^{(\nu)}(t)$ is given as :

$$\int_0^t \psi^{(\nu)}(x)\, dx = P\psi^{(\nu)}(t). \tag{22}$$

Where P is (16×16) the ELW operational matrix of integration and it is given as :

$$P = \begin{bmatrix} A & B & B & B \\ O & A & B & B \\ O & O & A & B \\ O & O & O & A \end{bmatrix}. \tag{23}$$

Here,

$$A = \begin{bmatrix} \frac{1}{8} & \frac{\sqrt{3}}{24} & 0 & 0 \\ -\frac{\sqrt{3}}{24} & 0 & \frac{\sqrt{15}}{120} & 0 \\ 0 & -\frac{\sqrt{15}}{120} & 0 & \frac{\sqrt{35}}{280} \\ 0 & 0 & -\frac{\sqrt{35}}{280} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similar for $\nu = 3, k = 2, m = 4$ the operational matrix of integration is

$$P = \begin{bmatrix} C & D & D & D & D & D & D & D & D \\ O & C & D & D & D & D & D & D & D \\ O & O & C & D & D & D & D & D & D \\ O & O & O & C & D & D & D & D & D \\ O & O & O & O & C & D & D & D & D \\ O & O & O & O & O & C & D & D & D \\ O & O & O & O & O & O & C & D & D \\ O & O & O & O & O & O & O & C & D \\ O & O & O & O & O & O & O & O & C \end{bmatrix}. \tag{24}$$

Here,

$$C = \begin{bmatrix} \frac{1}{18} & \frac{\sqrt{3}}{54} & 0 & 0 \\ -\frac{\sqrt{3}}{54} & 0 & \frac{\sqrt{15}}{270} & 0 \\ 0 & -\frac{\sqrt{15}}{270} & 0 & \frac{\sqrt{35}}{630} \\ 0 & 0 & -\frac{\sqrt{35}}{630} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$y^{''} = C^T\psi(t). \tag{25}$$

Integrating and using the equations (21) & (22), we obtain

$$y^{'}(t) - 2 = C^T P\psi(t).$$

Suppose,

$$1 = d^T\psi(t),$$

$$y^{'}(t) = C^T P\psi(t) + 2d^T\psi(t). \tag{26}$$

Again integrating the equation (26) and using (21) & (22), we get

$$y(t) = C^T P^2\psi(t) + 2d^T P\psi(t). \tag{27}$$

Now using the equations (25), (26) and (27) in equation (21), we have

$$C^T\psi(t) - 2t\left(C^T P\psi(t) + 2d^T\psi(t)\right) + 2\left(C^T P^2\psi(t) + 2d^T P\psi(t)\right) = 0.$$

Let

$$t = e^T\psi(t),$$

then

$$\psi^T C - 2e^T\psi\psi^T\left(P^T C + 2d\right) + 2\psi^T(P^2)^T C + 4\psi^T P^T d = 0 \tag{28}$$

and

$$e^T\psi\psi^T = \psi^T\tilde{E}. \tag{29}$$

By using the equation (29) in (28) and simplifying, we get

$$C = \left\{I - 2\tilde{E}P^T + 2(P^2)^T\right\}^{-1}(4\tilde{E}d - 4P^T d). \tag{30}$$

For $\nu = 2, k = 2, m = 4$, we have

$$\tilde{E} = \begin{bmatrix} X_1 & O & O & O \\ O & X_2 & O & O \\ O & O & X_3 & O \\ O & O & O & X_4 \end{bmatrix}, \quad X_1 = \begin{bmatrix} \frac{1}{8} & \frac{\sqrt{3}}{24} & 0 & 0 \\ \frac{\sqrt{3}}{24} & \frac{1}{8} & \frac{\sqrt{15}}{60} & 0 \\ 0 & \frac{\sqrt{15}}{60} & \frac{1}{8} & \frac{3\sqrt{35}}{280} \\ 0 & 0 & \frac{3\sqrt{35}}{280} & \frac{1}{8} \end{bmatrix}, \tag{31}$$

$$X_2 = X_1 + \frac{1}{4}I, \quad X_3 = X_1 + \frac{2}{4}I, \quad X_4 = X_1 + \frac{3}{4}I.$$

Where I is the 4×4 identity matrix.

Similarly for $\nu = 3, k = 2, M = 4$, we obtain

$$\tilde{E} = \begin{bmatrix} Y_1 & O & O & O & O & O & O & O & O \\ O & Y_2 & O & O & O & O & O & O & O \\ O & O & Y_3 & O & O & O & O & O & O \\ O & O & O & Y_4 & O & O & O & O & O \\ O & O & O & O & Y_5 & O & O & O & O \\ O & O & O & O & O & Y_6 & O & O & O \\ O & O & O & O & O & O & Y_7 & O & O \\ O & O & O & O & O & O & O & Y_8 & O \\ O & O & O & O & O & O & O & O & Y_9 \end{bmatrix}, Y_1 = \begin{bmatrix} \frac{1}{18} & \frac{\sqrt{3}}{54} & 0 & 0 \\ \frac{\sqrt{3}}{54} & \frac{1}{18} & \frac{\sqrt{15}}{135} & 0 \\ 0 & \frac{\sqrt{15}}{135} & \frac{1}{18} & \frac{\sqrt{35}}{210} \\ 0 & 0 & \frac{\sqrt{35}}{210} & \frac{1}{18} \end{bmatrix}, \quad (32)$$

$$Y_2 = Y_1 + \frac{1}{9}I, \quad Y_3 = Y_1 + \frac{2}{9}I, \quad Y_4 = Y_1 + \frac{3}{9}I, \quad Y_5 = Y_1 + \frac{4}{9}I,$$
$$Y_6 = Y_1 + \frac{5}{9}I, \quad Y_7 = Y_1 + \frac{6}{9}I, \quad Y_8 = Y_1 + \frac{7}{9}I, \quad Y_9 = Y_1 + \frac{8}{9}I.$$

t	ELW solution for $\nu = 2$, $k = 2, M = 4$	Exact solution	Absolute error=|Exa.sol.-ELW.sol.|
0.0	0.0000000	0.0000000	0.0000000
0.1	0.1999999	0.2000000	0.0000001
0.2	0.3999955	0.4000000	0.0000045
0.3	0.5999969	0.6000000	0.0000031
0.4	0.7999983	0.8000000	0.0000017
0.5	0.9999965	1.0000000	0.0000035
0.6	1.1999937	1.2000000	0.0000063
0.7	1.3999951	1.4000000	0.0000049
0.8	1.5999964	1.6000000	0.0000036
0.9	1.7999999	1.8000000	0.0000001

Table 1: Comparison table for approx. sol. and exact sol. for $\nu = 2$, $k = 2$, and $M = 4$ of example (7.1)

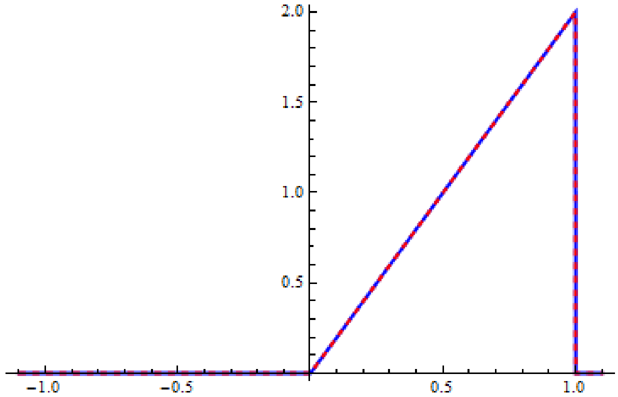

Figure 10: Graphical representation of exact sol.(dark line) and approx. sol. (dashed line) of example (7.1) for $\nu = 2$

t	ELW solution for $\nu = 3$, $k = 2, M = 4$	Exact solution	Absolute error=|Exa.sol.-ELW.sol.|
0.0	0.0000000	0.0000000	0.0000000
0.1	0.2000000	0.2000000	0.0000000
0.2	0.4000000	0.4000000	0.0000000
0.3	0.5999999	0.6000000	0.0000001
0.4	0.8000000	0.8000000	0.0000000
0.5	1.0000000	1.0000000	0.0000000
0.6	1.2000000	1.2000000	0.0000000
0.7	1.4000000	1.4000000	0.0000000
0.8	1.6000000	1.6000000	0.0000000
0.9	1.7999999	1.8000000	0.0000001

Table 2: Comparison table for approx. sol. and exact sol. for $\nu = 3$, $k = 2$, and $M = 4$ of example (7.1)

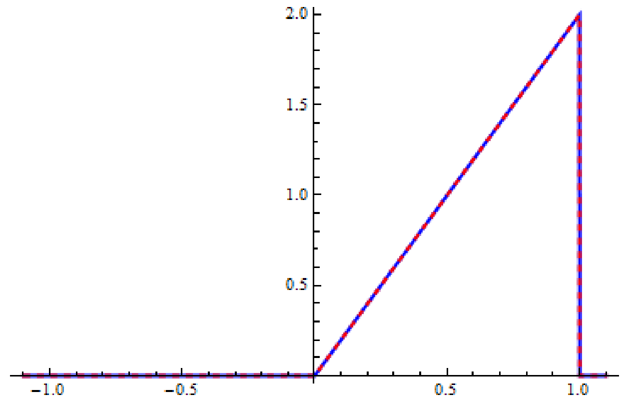

Figure 11: Graphical representation of exact sol.(dark line) and approx. sol.(dashed line) of example (7.1) for $\nu = 3$

7.2. *Application in summability theory*

In this section, we have obtained the sum of some special series by using the ELW method.
In tables (3) and (4), the sum of some special kinds of series by different summability methods and by the

Using function	Series	Sum by summability method	Sum by ELW method
$\frac{1}{1+x}$	1-1+1-1+....	$\frac{1}{2}$ (C,1)	0.49999766
$\frac{1}{1+2x}$	$1-2+2^2-2^3+....$	$\frac{1}{3}$ (E,1)	0.33332786
$\frac{1}{1+3x}$	$1-3+3^2-3^3+....$	$\frac{1}{4}$ (E,2)	0.24999307
$\frac{1}{(1+x)^2}$	1-2+3-4+....	$\frac{1}{4}$ (H,2)	0.24999380

Table 3: Comparison table for approx. sum and exact sum for $\nu = 2$, $k = 2$, and $M = 4$ of example (7.2)

Using function	Series	Sum by summability method	Sum by ELW method
$\frac{1}{1+x}$	1-1+1-1+....	$\frac{1}{2}$ (C,1)	0.49999992
$\frac{1}{1+2x}$	$1-2+2^2-2^3+....$	$\frac{1}{3}$ (E,1)	0.33331633
$\frac{1}{1+3x}$	$1-3+3^2-3^3+....$	$\frac{1}{4}$ (E,2)	0.24999979
$\frac{1}{(1+x)^2}$	1-2+3-4+....	$\frac{1}{4}$ (H,2)	0.24999980

Table 4: Comparison table for approx. sum and exact sum for $\nu = 3$, $k = 2$, and $M = 4$ of example (7.2)

ELW method have been compared.

t	Exact solution	ELW solution for $\nu = 2$, k=2 ,M=4	Absolute error=|Exa.sol.-ELW.sol.|
0.0	3.00000000	2.99960776	0.000393
0.1	2.22245466	2.22235488	0.000100
0.2	1.64643490	1.64659754	0.000163
0.3	1.21970897	1.21978266	0.000074
0.4	0.90358263	0.90354450	0.000038
0.5	0.66939048	0.66930296	0.000088
0.6	0.49589666	0.49587440	0.000022
0.7	0.36736928	0.36740557	0.000036
0.8	0.27215385	0.27217030	0.000017
0.9	0.20161653	0.20160803	0.000008

Table 5: Comparison table for approx. sol. and exact sol. for $\nu = 2$, k=2 and $M = 4$ of example (7.3)

7.3. *Application in Radioactive decay*

The real world problem associated with following differential equation have solved by ELW technique : If $m = m(t)$ is the mass at time t of a radioactive substance, then

$$\frac{dm}{dt} = -Am(t), \quad m(0) = m_0. \tag{33}$$

Where, $A > 0$ known as decay constant and m_0 is the initial mass.
Let us consider A=3 and m_0=3, then above equation becomes

$$\frac{dm}{dt} = -3m(t), \quad m(0) = 3. \tag{34}$$

Now, we have solved above differential equation by the ELW operational matrix of integration by using above matrices.
Suppose

$$m(t) = C^T\psi(t), \tag{35}$$

integrating the equation (34) and using (22), we get

$$m(t) - 3 = -3C^T P\psi(t). \tag{36}$$

Let

$$1 = d^T\psi(t), \tag{37}$$

now using the equations (35),(37) in (36) and simplifying, we have

$$C = 3(I + 3P^T)^{-1}d. \tag{38}$$

Also, the exact solution of equation (34) is,

$$m(t) = 3e^{-3t}.$$

Comparison table for approx. sol. and exact sol. is given in tables (5) & (6).

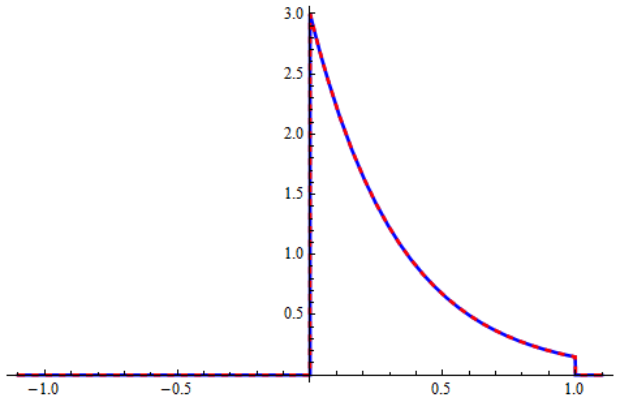

Figure 12: Graphical representation of Exact sol.(dark line) and approx. sol.(dashed line) of example (7.3) for $\nu = 2$

t	Exact solution	ELW solution for $\nu = 3$, k=2 ,M=4	Absolute error=|Exa.sol.-ELW.sol.|
0.0	3.00000000	2.99998130	0.000019
0.1	2.22245466	2.22245884	0.000004
0.2	1.64643490	1.64644042	0.000006
0.3	1.21970897	1.21971018	0.000002
0.4	0.90358263	0.90358111	0.000001
0.5	0.66939048	0.66938863	0.000002
0.6	0.49589666	0.49589580	0.000001
0.7	0.36736928	0.36736953	0.000000
0.8	0.27215385	0.27215459	0.000001
0.9	0.20161653	0.20161685	0.000000

Table 6: Comparison table for approx. sol. and exact sol. for $\nu = 3$, $k = 2$ and $M = 4$ of example (7.3)

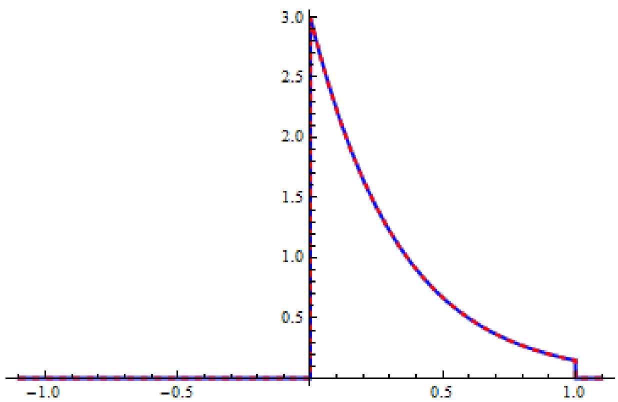

Figure 13: Graphical representation of exact sol.(dark line) and approx. sol. (dashed line) of example (7.3) for $\nu = 3$

t	Exact solution	ELW solution for $\nu = 2$, k=2 ,M=4	Absolute error=|Exa.sol.-ELW.sol.|
0.0	0.00000000	0.00000722	0.000007
0.1	0.09966799	0.09966541	0.000002
0.2	0.19737532	0.19737382	0.000002
0.3	0.29131261	0.29131450	0.000002
0.4	0.37994896	0.37994539	0.000003
0.5	0.46211715	0.46211947	0.000002
0.6	0.53704956	0.53704851	0.000001
0.7	0.60436777	0.60436578	0.000002
0.8	0.66403677	0.66403783	0.000001
0.9	0.71629787	0.71629789	0.000000

Table 7: Comparison table for approx. sol. and exact sol. for $\nu = 2$, k=2 and $M = 4$ of example (7.4)

7.4. *Application of ELW in solving Riccati differential equation*

Let us consider the Riccati nonlinear differential equation,

$$y^{'}(t) = 1 - y^2(t), \qquad y(0) = 0. \tag{39}$$

Here, we have solved above differential equation by the ELW operational matrix of integration.
Let

$$y(t) = C^T\psi(t), \tag{40}$$

then

$$\begin{aligned} y^2(t) &= C^T\psi(t)\psi^T(t)C \\ &= C^T\tilde{C}^T\psi(t), \quad (\psi(t)\psi^T(t)C = \tilde{C}^T\psi(t), \quad say). \end{aligned} \tag{41}$$

Using the equation (41) in (39) and integrating, we obtain

$$y(t) = d^TP\psi(t) - C^T\tilde{C}^TP\psi(t),$$

on simplifying, we get

$$C^T = d^TP(I + \tilde{C}^TP)^{-1}. \tag{42}$$

Also, the exact solution of equation (39) is,

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Comparison table for approx. sol. and exact sol. is given in tables (7) & (8).

t	Exact solution	ELW solution for $\nu = 3$, k=2 ,M=4	Absolute error=|Exa.sol.-ELW.sol.|
0.0	0.00000000	-0.00000439	0.000004
0.1	0.09966799	0.09966723	0.000000
0.2	0.19737532	0.19737702	0.000002
0.3	0.29131261	0.29131642	0.000004
0.4	0.37994896	0.37994676	0.000002
0.5	0.46211715	0.46211848	0.000001
0.6	0.53704956	0.53704767	0.000002
0.7	0.60436777	0.60437094	0.000003
0.8	0.66403677	0.66403811	0.000002
0.9	0.71629787	0.71629721	0.000000

Table 8: Comparison table for approx. sol. and exact sol. for $\nu = 3$, k=2 and $M = 4$ of example (7.4)

7.5. *Application of the ELW in solving Blasius differential equation*

Let us consider nonlinear Blasius differential equation,

$$2y'''(t) + yy''(t) = 0 \quad y(0) = 0, \quad y'(0) = 0, y''(0) = 0.332057 = \alpha(say). \tag{43}$$

Now, we have solved above differential equation by the ELW operational matrix of integration. Let

$$y'''(t) = C^T\psi(t), \tag{44}$$

integrating (44) and using the equations (43) & (22), we have

$$y''(t) = (\alpha d^T + C^T P)\psi(t) \quad (1 = d^T\psi(t) \ \ say). \tag{45}$$

Integrating the equation (45) two times and using (43) & (22), we obtain

$$y(t) = \alpha d^T P^2\psi(t) + C^T P^3\psi(t).$$

Putting the values of $y(t)$, $y''(t)$ and $y'''(t)$ in equation (43), we get

$$2C^T\psi(t) + (C^T P^3\psi(t) + \alpha d^T P^2\psi(t))(\psi^T(t)P^T C + \alpha\psi^T(t)d) = 0. \tag{46}$$

Let

$$\psi\psi^T P^T C = \tilde{E}\psi \quad and \quad \psi\psi^T d = \tilde{d}\psi. \tag{47}$$

Using the equation (47) in (46) and simplifying, we have

$$C^T = -(\alpha d^T P^2\tilde{E} + \alpha^2 d^T P^2\tilde{d})(2I + P^3\tilde{E} + \alpha P^3\tilde{d})^{-1}. \tag{48}$$

Comparison table for approx. sol. and exact sol. is given in tables (9), (10) & (11).

t	Exact solution by Howarth method of $y(t)$	ELW solution for $\nu = 2$, k=1, M=3	ELW solution for $\nu = 3$, k=2, M=4
0.0	0.00000	-0.00000119	0.00000001
0.2	0.00664	0.00663995	0.00664099
0.4	0.02656	0.02655189	0.02655986
0.6	0.05974	0.05973083	0.05973459
0.8	0.10611	0.10612742	0.10610806

Table 9: Comparison table for approx. sol. and exact sol. of example (7.5)

t	Exact solution by Howarth method of $y'(t)$	ELW solution for $\nu = 2$, k=1, M=3	ELW solution for $\nu = 3$, k=2, M=4
0.0	0.00000	-0.0000000	0.00000012
0.2	0.06641	0.0663884	0.06640767
0.4	0.13277	0.1327310	0.13276405
0.6	0.19894	0.1989947	0.19893707
0.8	0.26471	0.2647221	0.26470882

Table 10: Comparison between approx. sol. and exact sol. of example (7.5)

t	Exact solution by Howarth method of $y''(t)$	ELW solution for $\nu = 2$, k=1, M=3	ELW solution for $\nu = 3$, k=2, M=4
0.0	0.33206	0.332057	0.3320569
0.2	0.33198	0.331828	0.3319835
0.4	0.33147	0.331598	0.3314695
0.6	0.33008	0.330688	0.3300787
0.8	0.32739	0.329097	0.3273889

Table 11: Comparison table for approx. sol. and exact sol. of example (7.5)

8. Conclusions

(i) Since the estimated errors in the theorems (3.1) - (3.5) tends to zero as $k, M \to \infty$. Therefore the calculated approximations are best possible in wavelet analysis.

(ii) As we increase the order of derivative of the function then from the theorems it is clear that approximation errors become smaller. This is the significant achievement of this paper.

(iii) We have used these approximations to solve differential equations namely the Riccati differential equation, Hermite differential equation, radioactive decay problem and the Blasius equation. Also, we have approximated the sum of some special types of series by this wavelet method. The significant part of this paper is that we have presented the exact solutions and approximate solutions in the tables for different values of ν and we observe that if we increase the value of ν then the numerical accuracy increases.

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