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Module derivations into iterated duals of triangular Banach algebras

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Abstract. Let $\mathfrak{A}$ be a Banach algebra, A and B be Banach $\mathfrak{A}$-module with compatible actions and X be a Banach left A-$\mathfrak{A}$-module and Banach right B-$\mathfrak{A}$-module. Then the corresponding triangular Banach algebra $\mathrm{Tri}(A, X, B)$ is a Banach $\mathfrak{A}$-module with compatible actions. In this paper, we study n-weak module amenability of module extension Banach algebras to provide necessary and sufficient conditions for n-weak module amenability (as an $\mathfrak{A}$-module) of $\mathrm{Tri}(A, X, B)$, when A and B are not necessarily unital and not have bounded approximate identity. This not only fixes the gaps in some known results in the literature but also extends that results and gives a direct proof for them. Furthermore, we characterize n-weak module amenability of triangular matrix algebras related to inverse semigroups and some triangular Banach algebra related to locally compact groups.

1. Introduction and some Preliminaries

A Banach algebra A is amenable if $H^1(A, X^*) = \{0\}$, for every Banach A-bimodule X, where $H^1(A, X^*)$ is the first Hochschild cohomology group of A with coefficients in X^*. It is n-weakly amenable ($n \geq 0$) if $H^1(A, A^{(n)}) = \{0\}$, where $A^{(n)}$ is the n^{th}-dual space of A and $A^{(0)} = A$. When A is 1-weakly amenable, it is called weakly amenable. A Banach algebra is called permanently weakly amenable if it is n-weakly amenable for each $n \in \mathbb{N}$. These concepts were introduced and studied by Johnson [14], and Dales et al. [10], respectively. See the monograph [9], for more background.

For a locally compact group G, the famous Johnson's theorem assert that the convolution algebra $L^1(G)$ is amenable if and only if G is amenable [14]. Moreover, it is well known that $L^1(G)$ is always n-weakly amenable for every $n \in \mathbb{N}$ (for a proof see [8], [10] and [20]). Both of these facts are not true for inverse semigroups in general, [7]. Amini in [1] and Amini et al. in [2] and [4], introduced and studied the concepts of module amenability and n-weak module amenability for Banach algebras which are Banach module over another Banach algebra with compatible actions. These notions could be considered as a generalization of the notions amenability and n-weak amenability of Banach algebras. They extended the classical results on (n-weak) amenability of $L^1(G)$ and showed that the inverse semigroup algebra $l^1(S)$ is module amenable, as an $l^1(E)$-module, if and only if S is amenable [1, Theorem 3.1], and that it is always n-weakly module amenable, when n is odd and $l^1(E)$ acts trivially on $l^1(S)$ from left and by multiplication from the right [4, Theorem 3.15]. This result for even number $n \in \mathbb{N}$ was proved in [11, Theorem 2.2]. Moreover, it is

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shown in [5, Theorem 2.7, Corollary 2.8] that $l^1(S)$ is n-weakly module amenable for all $n \geq 0$, when S is a commutative and $l^1(E)$ acts on $l^1(S)$ by usual multiplication from both sides.

Forrest and Marcoux [13], studied the n-weak amenability of triangular Banach algebra $\mathrm{Tri}(A, X, B)$ for the case where A and B are unital Banach algebras and X is a unital Banach (A, B)-module. They showed that $\mathrm{Tri}(A, X, B)$ is weakly amenable if and only if both A and B are weakly amenable. The module version of this result was proved in [18]. The n-weak amenability of $\mathrm{Tri}(A, X, B)$, for the case that A and B are not necessarily unital, was investigated by Medghalchi et al. in [16]. Bodaghi and Jabbari [6], extended the results of [16] and studied n-weak module amenability of $\mathrm{Tri}(A, X, B)$. As a main result, they showed in [6, Theorem 4.3] that, if A and B have bounded approximate identity and X is a non-degenerate (A, B)-module, then for $n \geq 0$, $(2n + 1)$-weak module amenability of $\mathrm{Tri}(A, X, B)$ and that of corner Banach algebras A and B are equivalent. They use [6, Proposition 4.2] in their proof, but the assumptions of this proposition do not appear in [6, Theorem 4.3]. Thus, the result will be valid, if $A^{(2n-1)}$, $B^{(2n-1)}$ and $X^{(2n-1)}$ are also non-degenerate modules.

This paper is designed to improve and fix gaps in the main results of [6] on n-weak module amenability of $\mathrm{Tri}(A, X, B)$ and extend the results of [16]. For this purpose, we first study n-weak module amenability (as an $\mathfrak{A}$-module) of the module extension Banach algebra $A \oplus X$, which can be seen as a generalization of triangular Banach algebras. We then, employ our results for $\mathrm{Tri}(A, X, B)$ to not only improve and extend the main results of [6] and [16], but also give necessary and sufficient conditions for $\mathrm{Tri}(A, X, B)$ to be n-weakly module amenable (as an $\mathfrak{A}$-module).

2. *n*-Weak module amenability of module extensions

Throughout this paper, A and $\mathfrak{A}$ are Banach algebras such that A is a Banach $\mathfrak{A}$-module with compatible actions, that is $\alpha \cdot (ab) = (\alpha \cdot a)b$ and $(ab) \cdot \alpha = a(b \cdot \alpha)$ for $a, b \in A, \alpha \in \mathfrak{A}$. Let X be a Banach A-module and a Banach $\mathfrak{A}$-module with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \qquad (a \in A, \alpha \in \mathfrak{A}, x \in X),$$

and the same for the right or two-sided actions. Then, we say that X is a Banach $\mathfrak{A}$-A-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for $\alpha \in \mathfrak{A}, x \in X$, then X is called a commutative $\mathfrak{A}$-A-module. If X is a (commutative) Banach $\mathfrak{A}$-A-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$\langle \alpha \cdot f, x \rangle = f(x \cdot \alpha), \quad \langle a \cdot f, x \rangle = f(x \cdot a) \quad (a \in A, \alpha \in \mathfrak{A}, f \in X^*, x \in X),$$

and the same for the other side actions. So, $X^{(n)}$ is a (commutative) Banach $\mathfrak{A}$-A-module.

Let A and $\mathfrak{A}$ be as above and X and Y be Banach $\mathfrak{A}$-A-modules. A map $T : X \to Y$ is called an $\mathfrak{A}$-module map if

$$T(x \pm z) = T(x) \pm T(z), \quad T(\alpha \cdot x) = \alpha \cdot T(x), \quad T(x \cdot \alpha) = T(x) \cdot \alpha,$$

for $x, z \in X$ and $\alpha \in \mathfrak{A}$. If moreover, $T(a \cdot x) = a \cdot T(x)$ and $T(x \cdot a) = T(x) \cdot a$ for $x \in X$ and $a \in A$, then T is called an $\mathfrak{A}$-A-module map. Although T is not necessary linear, but still its boundedness implies its norm continuity.

Let X be a Banach $\mathfrak{A}$-A-module. A bounded $\mathfrak{A}$-module map $D : A \to X$ is called a module derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for $a, b \in A$. When X is commutative, each $x \in X$ defines a module derivation $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for $a \in A$, which is called an inner $\mathfrak{A}$-module derivation.

Note that when A acts on itself by algebra multiplication, it is not in general a Banach $\mathfrak{A}$-A-module, as we have not assumed the compatibility condition $a(\alpha \cdot b) = (a \cdot \alpha)b$ for $\alpha \in \mathfrak{A}, a, b \in A$. Let J be the closed ideal of A generated by $\{(a \cdot \alpha)b - a(\alpha \cdot b) \, ; \, a, b \in A, \alpha \in \mathfrak{A}\}$. Then, J is an $\mathfrak{A}$-submodule of A. So, the quotient Banach algebra A/J is a Banach $\mathfrak{A}$-module with compatible action. We say that A is n-weakly module amenable, as an $\mathfrak{A}$-module, if A/J is a commutative Banach $\mathfrak{A}$-A-module, and each $\mathfrak{A}$-module derivation $D : A \to (A/J)^{(n)}$ is inner; that is $H^1_{\mathfrak{A}}(A, (A/J)^{(n)}) = \{0\}$. Also A is called permanently weakly module amenable if A is n-weakly module amenable for each $n \in \mathbb{N}$; see [4] and [5] for more details.

Let A be a Banach algebra and let X be an A-module. Then, the module extension Banach algebra corresponding to A and X is $A \oplus X$, the ℓ^1-direct sum $A \times X$ with the algebra multiplication defined by

$$(a, x) \cdot (b, y) = (ab, a \cdot y + x \cdot b) \qquad (a, b \in A, x, y \in X).$$

Following [19], we take $A^{(n)} \times X^{(n)}$ as the underlying space of $(A \oplus X)^{(n)}$. One can directly check that the $A \oplus X$-module actions on $(A \oplus X)^{(n)}$ for $(a, x) \in A \oplus X$ and $(a^{(n)}, x^{(n)}) \in A^{(n)} \times X^{(n)}$ are formulated as follows:

$$\begin{aligned}(a, x) \cdot (a^{(2n)}, x^{(2n)}) &= (a \cdot a^{(2n)}, a \cdot x^{(2n)} + x \cdot a^{(2n)})\\ (a, x) \cdot (a^{(2n+1)}, x^{(2n+1)}) &= (a \cdot a^{(2n+1)} + x \cdot x^{(2n+1)}, a \cdot x^{(2n+1)}),\end{aligned}$$

where $x \cdot a^{(2n)} \in X^{(2n)}$ and $x \cdot x^{(2n+1)} \in A^{(2n+1)}$ are defined by

$$\langle x \cdot a^{(2n)}, x^{(2n-1)} \rangle = \langle a^{(2n)}, x^{(2n-1)} \cdot x \rangle, \quad \langle x \cdot x^{(2n+1)}, a^{(2n)} \rangle = \langle x^{(2n+1)}, a^{(2n)} \cdot x \rangle.$$

And similarly for the right module actions.

Zhang in [19], investigated the n-weak amenability of module extension Banach algebras and used them to construct an example of a weakly amenable Banach algebra which is not 3-weakly amenable. In this section, we extend the main results of [19], and characterize n-weak module amenability of module extension Banach algebra $A \oplus X$ in terms of A and X. From now on, we shall assume that $A \oplus X$ is a commutative $\mathfrak{A}$-module with compatible actions. A simple computation shows that this assumption holds if and only if A is a commutative $\mathfrak{A}$-module, and X is a commutative $\mathfrak{A}$-A-module.

We start with the following result which is a module version of [19, Theorem 2.1] and can be proved by a similar argument. However, we bring its proof.

Theorem 2.1. *Let $n \geq 0$. Then $A \oplus X$ is $(2n + 1)$-weakly module amenable if and only if*

(i) A is $(2n + 1)$-weakly module amenable.

(ii) $H^1_{\mathfrak{A}}(A, X^{(2n+1)}) = \{0\}$.

(iii) For every bounded $\mathfrak{A}$-A-module map $T : X \to A^{(2n+1)}$, there is $g \in X^{(2n+1)}$ such that $a \cdot g = g \cdot a$ and $T(x) = x \cdot g - g \cdot x$ for all $a \in A$ and $x \in X$.

(iv) The only bounded $\mathfrak{A}$-A-module map $S : X \to X^{(2n+1)}$ for which $S(x) \cdot y + x \cdot S(y) = 0$ in $A^{(2n+1)}$, for all $x, y \in X$, is zero.

Proof. Suppose that conditions (i)-(iv) hold. Let $D : A \oplus X \to (A \oplus X)^{(2n+1)}$ be a $\mathfrak{A}$-module derivation. Then, a direct verification reveals that $D(a, x) = (D_A(a) + T(x), D_X(a) + S(x))$, where the component mappings $D_A : A \to A^{(2n+1)}$ and $D_X : A \to X^{(2n+1)}$ are $\mathfrak{A}$-module derivations, $T : X \to A^{(2n+1)}$ is a bounded $\mathfrak{A}$-module map such that $T(x \cdot a) = T(x) \cdot a + x \cdot D_X(a)$ and $T(a \cdot x) = a \cdot T(x) + D_X(a) \cdot x$ and $S : X \to X^{(2n+1)}$ is a bounded $\mathfrak{A}$-A-module map satisfying $S(x) \cdot y + x \cdot S(y) = 0$ in $A^{(2n+1)}$. By conditions (i) and (ii), D_A and D_X are inner derivations and by condition (iv), $S = 0$. Thus, there are $f \in A^{(2n+1)}$ and $g_0 \in X^{(2n+1)}$ such that $D_A = \mathrm{ad}_f$ and $D_X = \mathrm{ad}_{g_0}$. Define $T_1 : X \to A^{(2n+1)}$ by

$$T_1(x) = T(x) - x \cdot g_0 + g_0 \cdot x.$$

It simply follows from commutativity $\mathfrak{A}$-module X that, T_1 is a $\mathfrak{A}$-A-module map. Thus, from (iii), there exists $g_1 \in X^{(2n+1)}$ such that $a \cdot g_1 = g_1 \cdot a$ and $T_1(x) = x \cdot g_1 - g_1 \cdot x$. It follows that $T(x) = x \cdot g - g \cdot x$ and $D_X = \mathrm{ad}_g$, where $g = g_0 + g_1$. Consequently,

$$\begin{aligned}D(a, x) &= (D_A(a) + T(x), D_X(a) + S(x))\\ &= (\mathrm{ad}_f(a) + x \cdot g - g \cdot x, \mathrm{ad}_g(a)) \qquad (1)\\ &= \mathrm{ad}_{(f,g)}(a, x),\end{aligned}$$

for all $(a, x) \in A \oplus X$. This complete the proof of sufficiency.

For necessity, suppose that $A\oplus X$ is $(2n+1)$-weakly module amenable, as an $\mathfrak{A}$-module. Let $d : A \to A^{(2n+1)}$ be a $\mathfrak{A}$-module derivation. Then, $D : A \oplus X \to (A \oplus X)^{(2n+1)}$ defined by $D(a, x) = (d(a), 0)$ is a $\mathfrak{A}$-module map. We follow from [19, Lemma 3.5] that D is a $\mathfrak{A}$-module derivation and so it is inner. Now relation (1) implies that d is also inner, so A is $(2n + 1)$-weakly module amenable.
To prove (ii), let $d : A \to X^{(2n+1)}$ be a $\mathfrak{A}$-module derivation. Then, [19, Lemma 3.4] implies that $D : A \oplus X \to (A \oplus X)^{(2n+1)}$ given by $D(a, x) = (-d^{(2n+1)}(x), d(a))$ is a $\mathfrak{A}$-module derivation, so it is inner. Hence, d is also inner, again by [19, Lemma 3.4]. This shows that $H^1_{\mathfrak{A}}(A, X^{(2n+1)}) = \{0\}$, as required.
Let $T : X \to A^{(2n+1)}$ and $S : X \to X^{(2n+1)}$ be $\mathfrak{A}$-A-module maps such that $S(x) \cdot y + x \cdot S(y) = 0$ in $A^{(2n+1)}$ for all $x, y \in X$. Define $D : A \oplus X \to (A \oplus X)^{(2n+1)}$ by $D(a, x) = (T(x), S(x))$. Then, Lemma 3.1 and 3.5 of [19] jointly show that D is a $\mathfrak{A}$-module derivation, so it is inner. Let $f \in A^{(2n+1)}$ and $g \in X^{(2n+1)}$ be such that $D = \mathrm{ad}_{(f,g)}$. By (1), we have

$$(T(x), S(x)) = (\mathrm{ad}_f(a) + x \cdot g - g \cdot x, \mathrm{ad}_g(a)) \qquad (a \in A, x \in X).$$

Taking $a = 0$ we obtain $S = 0$ and $T(x) = x \cdot g - g \cdot x$ for all $x \in X$. And if we take $x = 0$ we get $a \cdot g = g \cdot a$ for all $a \in A$. This proves (iii) and (iv) and completes the proof. □

Before to characterize n-weak module amenability of $A \oplus A^{(m)}$, we need the following module version of [10, Proposition 1.2]. Since the natural embedding $\iota : A^{(n)} \to A^{(n+2)}$ and the projection $P : A^{(n+2)} \to A^{(n)}$ used in the proof of [10, Proposition 1.2] are $\mathfrak{A}$-module maps, the argument of [10, Proposition 1.2] suffices to show n-weak module amenability.

Proposition 2.2. *Suppose that $n \in \mathbb{N}$ and A is $(n + 2)$-weakly module amenable. Then, A is n-weakly module amenable.*

Recall that an A-module X is called symmetric if $a \cdot x = x \cdot a$ for $a \in A$ and $x \in X$. As a consequence of Theorem 2.1, we have the next result concerning $(2n + 1)$-weak module amenability of $A \oplus A^{(2m+1)}$.

Corollary 2.3. *Suppose that A is commutative and $m \geq 0$. Then, $A\oplus A^{(2m+1)}$ is not $(2n+1)$-weakly module amenable.*

Proof. Using Proposition 2.2, we show that $A \oplus A^{(2m+1)}$ is not weakly module amenable. Set $X = A^{(2m+1)}$ in Theorem 2.1 and let $T : X = A^{(2m+1)} \to A^*$ be the adjoint map of the canonical embedding $\iota : A \to A^{(2m)}$. Then, T is a non-zero bounded $\mathfrak{A}$-A-module map. Since A is commutative, $X = A^{(2m+1)}$ is a symmetric A-module and so $x \cdot g = g \cdot x$ in $A^{(2n+1)}$ for all $x \in X$ and $g \in X^{(2n+1)}$. This follows that condition (iii) of Theorem 2.1 does not hold. Hence, $A \oplus A^{(2m+1)}$ is not weakly module amenable. □

In the next result which is a module version of [19, Theorem 2.2], we characterize $2n$-weak module amenability of $A \oplus X$. The proof is based on the argument used in Theorem 2.1 and [19, Theorem 2.2], so the details omitted.

Theorem 2.4. *Let $n \geq 0$. Then $A \oplus X$ is $2n$-weakly module amenable if and only if*

(i) If $D_A : A \to A^{(2n)}$ is a $\mathfrak{A}$-module derivation such that there is a bounded $\mathfrak{A}$-module map $S : X \to X^{(2n)}$ with $S(x \cdot a) = S(x) \cdot a + x \cdot D_A(a)$ and $S(a \cdot x) = a \cdot S(x) + D_A(a) \cdot x$ $(a \in A, x \in X)$, then D is inner.

(ii) $H^1_{\mathfrak{A}}(A, X^{(2n)}) = \{0\}$.

(iii) The only bounded $\mathfrak{A}$-A-module map $T : X \to A^{(2n)}$ for which $T(x) \cdot y + x \cdot T(y) = 0$ $(x, y \in X)$ in $X^{(2n)}$ is zero.

(iv) For every bounded $\mathfrak{A}$-A-module map $S : X \to X^{(2n)}$, there is $f \in A^{(2n)}$ such that $a \cdot f = f \cdot a$ and $S(x) = x \cdot f - f \cdot x$ for $a \in A$ and $x \in X$.

Proof. To prove the necessity, suppose that $A \oplus X$ is $2n$-weakly module amenable. Let $d : A \to A^{(2n)}$ be a $\mathfrak{A}$-module derivation with the property given in condition (i). Define $D : A \oplus X \to (A \oplus X)^{(2n)}$ by $D(a, x) = (d(a), S(x))$. Then, D is a $\mathfrak{A}$-module derivation, so is inner. A simple computation shows that d is also inner. This proves (i). Conditions (ii)-(iv) can be proved by analogous argument given in Theorem 2.1.

For sufficiency, let $D : A\oplus X \to (A\oplus X)^{(2n)}$ be a $\mathfrak{A}$-module derivation. Then, $D(a, x) = (D_A(a)+T(x), D_X(a)+S(x))$, where the component mappings $D_A : A \to A^{(2n)}$ and $D_X : A \to X^{(2n)}$ are $\mathfrak{A}$-module derivations, $T : X \to A^{(2n)}$ is a bounded $\mathfrak{A}$-A-module map satisfying $T(x) \cdot y + x \cdot T(y) = 0$ in $X^{(2n)}$ and $S : X \to X^{(2n)}$ is a bounded $\mathfrak{A}$-module map such that $S(x \cdot a) = S(x) \cdot a + x \cdot D_A(a)$ and $S(a \cdot x) = a \cdot S(x) + D_A(a) \cdot x$. By conditions (i) and (ii), $D_A = \mathrm{ad}_{f_0}$ and $D_X = \mathrm{ad}_g$ for some $f_0 \in A^{(2n)}$ and $g \in X^{(2n)}$ and from condition (iii), $T = 0$. Define $S_1 : X \to X^{(2n)}$ by $S_1(x) = S(x) - x \cdot f_0 + f_0 \cdot x$. It simply follows from commutativity of $\mathfrak{A}$-module A that, S_1 is a $\mathfrak{A}$-A-module map. Thus, from (iv), there exists $f_1 \in A^{(2n)}$ such that $a \cdot f_1 = f_1 \cdot a$ and $S_1(x) = x \cdot f_1 - f_1 \cdot x$. It follows that, $S(x) = x \cdot f - f \cdot x$ and $D_A = \mathrm{ad}_f$, where $f = f_0 + f_1$. Consequently, $D = \mathrm{ad}_{(f,g)}$. This complete the proof. □

As a consequence of Theorems 2.4, we have the next result.

Corollary 2.5. *If X is non-zero and symmetric, then $A \oplus X$ is not $2n$-weakly module amenable, for every $n \geq 0$. In particular, $A \oplus A^{(m)}$ is not $2n$-weakly module amenable, if $m \geq 0$ and A is commutative.*

Proof. Let $S : X \to X^{(2n)}$ be the canonical embedding. Then, it is a non-zero $\mathfrak{A}$-A-module map. Since X is symmetric, $x \cdot f = f \cdot x$ in $X^{(2n)}$, for all $x \in X$ and $f \in A^{(2n)}$. It follows that, condition (iv) of Theorem 2.4 does not hold for such X. Hence $A \oplus X$ is not $2n$-weakly module amenable, as an $\mathfrak{A}$-module. □

If we combine Corollaries 2.3 and 2.5, we get the following result.

Proposition 2.6. *Suppose that A is commutative and $m, n \geq 0$. Then, $A \oplus A^{(2m+1)}$ is not n-weakly module amenable, as an $\mathfrak{A}$-module.*

We conclude this section with the following results on direct product of two Banach algebras, that will be needed in the next section.

Theorem 2.7. *For $n \geq 0$, the direct product $A \times B$ is n-weakly module amenable, as an $\mathfrak{A}$-module, if and only if*

(i) both A and B are n-weakly module amenable.

(ii) The only bounded $\mathfrak{A}$-module map $S : A \to B^{(n)}$ for which $S(ac) = 0$ and $S(a) \cdot b = b \cdot S(a) = 0$ for all $a, c \in A$ and $b \in B$ is $S = 0$.

(iii) If $T : B \to A^{(n)}$ is a bounded $\mathfrak{A}$-module map such that $T(bd) = 0$ and $a \cdot T(b) = T(b) \cdot a = 0$ for all $a \in A$ and $b, d \in B$, then $T = 0$.

Proof. To prove the necessity, let $d_A : A \to A^{(n)}$ and $d_B : B \to B^{(n)}$ be $\mathfrak{A}$-module derivations. Then, $D : A \times B \to (A \times B)^{(n)}$ defined by $D(a, b) = (d_A(a), d_B(b))$ is a $\mathfrak{A}$-module derivation and so it is inner. Thus, $D = \mathrm{ad}_{(f,g)}$, for some $(f, g) \in A^{(n)} \times B^{(n)} \simeq (A \times B)^{(n)}$. From the equality $\mathrm{ad}_{(f,g)}(a, b) = (\mathrm{ad}_f(a), \mathrm{ad}_g(b))$, it follows that d_A and d_B are inner, so (i) holds.

Let $S : A \to B^{(n)}$ be a bounded $\mathfrak{A}$-module map satisfying the hypotheses in (ii). Then, $D : A\times B \to (A\times B)^{(n)}$ given by $D(a, b) = (0, S(a))$, is a bounded $\mathfrak{A}$-module derivation, and so $D = \mathrm{ad}_{(f,g)}$, for some $(f, g) \in (A \times B)^{(n)}$. Applying the equality, $(0, S(a)) = (\mathrm{ad}_f(a), \mathrm{ad}_g(b))$, for $b = 0$, we get $S = 0$. This proves (ii). Similarly we can prove (iii).

For sufficiency, suppose that $D : A \times B \to (A \times B)^{(n)}$ is a $\mathfrak{A}$-module derivation. A direct verification shows that D enjoys the presentation

$$D(a, b) = (D_A(a) + T(b), S(a) + D_B(b)) \quad ((a, b) \in A \times B),$$

where $D_A : A \to A^{(n)}$ and $D_B : B \to B^{(n)}$ are $\mathfrak{A}$-module derivations and $T : B \to A^{(n)}$ and $S : A \to B^{(n)}$ are bounded $\mathfrak{A}$-module map satisfying $T(bd) = 0$, $a \cdot T(b) = T(b) \cdot a = 0$, $S(ac) = 0$ and $b \cdot S(a) = S(a) \cdot b = 0$, for every $a, c \in A$ and $b, d \in B$. By condition (ii) and (iii), $S = 0$ and $T = 0$. From conditions (i), it follows that $D_A = \mathrm{ad}_f$ and $D_B = \mathrm{ad}_g$, for some $f \in A^{(n)}$ and $g \in B^{(n)}$. Consequently, $D(a, b) = (\mathrm{ad}_f(a), \mathrm{ad}_g(b)) = \mathrm{ad}_{(f,g)}(a, b)$, for all $(a, b) \in A \times B$. Thus, D is inner, as claimed. □

Let A be a Banach algebra and X be a Banach left A-module. By $\langle AX \rangle$, we denote the linear span of $AX = \{a \cdot x \mid a \in A, x \in X\}$, in X. We also recall that, X is non-degenerate if

$$\mathrm{Ann}_A(X) = \{x \in X; a \cdot x = 0 \quad \forall a \in A\} = \{0\}.$$

Non-degenerate right A-module are defined similarly.

Corollary 2.8. *Let $n \geq 0$. If the direct product $A \times B$ is n-weakly module amenable then both A and B are also n-weakly module amenable. The converse holds if any of the following statements holds.*

(1) $\langle A^2 \rangle$ is dense in A and $\langle B^2 \rangle$ is dense in B.

(2) $\langle B^2 \rangle$ is dense in B and $B^{(n)}$ is a non-degenerate left or right B-module.

(3) $\langle A^2 \rangle$ is dense in A and $A^{(n)}$ is a non-degenerate left or right A-module.

Proof. For n-weak module amenability of $A \times B$, we need to prove conditions (ii) and (iii) of Theorem 2.7. The other side is clear. Suppose that S and T are $\mathfrak{A}$-module maps satisfying conditions (ii) and (iii) of Theorem 2.7, respectively. Then, $S(a) \in \mathrm{Ann}_B(B^{(n)})$ and $T(b) \in \mathrm{Ann}_A(A^{(n)})$, for $a \in A$ and $b \in B$. Since S is a $\mathfrak{A}$-module map and $S = 0$ on A^2, we have $S = 0$ on $\langle A^2 \rangle$. Indeed, if $z \in \langle A^2 \rangle$ then $z = \sum_{i=1}^{m} \lambda_i a_i c_i$, for some $\lambda_i \in \mathbb{C}$ and $a_i, c_i \in A$. Thus, $S(z) = \sum_{i=1}^{m} S((\lambda_i a_i) c_i) = 0$. As the same way, $T = 0$ on $\langle B^2 \rangle$. Now conditions (ii) and (iii) of Theorem 2.7, will be simply concluded from each of the assumptions (1) to (3). □

3. Application to triangular Banach algebras

In this section we apply the results of the previous section, to give necessary and sufficient conditions for n-weak module amenability of triangular Banach algebras. Our approach not only provides a direct proof for some known results in the literature, but also it improves and extends the main results of [6, 18] and [16].

Let A and B be Banach algebras and let X be a Banach (A, B)-module. Then,

$$\mathrm{Tri}(A, X, B) = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; a \in A, x \in X, b \in B \right\},$$

under matrix-like operations and equipped with the ℓ^1-norm $\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|b\| + \|x\|$, becomes a Banach algebra, which is called a triangular Banach algebra. This Banach algebra was first introduced and studied in [12]. Some aspects of triangular Banach algebras have been discussed in [6, 13] and [18].

The triangular Banach algebra $\mathrm{Tri}(A, X, B)$, can be viewed as a module extension Banach algebra $(A \times B) \oplus X$, where $A \times B$ is the direct product of A and B and X as an $(A \times B)$-module is equipped with the module operations

$$(a, b) \cdot x = a \cdot x \quad \text{and} \quad x \cdot (a, b) = x \cdot b \qquad (a \in A, b \in B, x \in X).$$

In the whole of this section, we shall assume that $\mathrm{Tri}(A, X, B)$ is a commutative $\mathfrak{A}$-module with compatible actions. The first result, gives a necessary and sufficient conditions for $(2n + 1)$-weak module amenability of $\mathrm{Tri}(A, X, B)$, for the case where $\langle AX + XB \rangle$, the linear span of $AX + XB = \{a \cdot x + y \cdot b\,; a \in A, b \in B, x, y \in X\}$, is dense in X.

Theorem 3.1. *Suppose that $\langle AX + XB \rangle$ is dense in X and $n \geq 0$. Then, $\mathrm{Tri}(A, X, B)$ is $(2n + 1)$-weakly module amenable if and only if*

(1) $A \times B$ is $(2n + 1)$-weakly module amenable.

(2) $H^1_{\mathfrak{A}}(A \times B, X^{(2n+1)}) = \{0\}$.

Proof. Let conditions (1) and (2) hold, it is enough to prove conditions (iii) and (iv) of Theorem 2.1. Suppose that $T : X \to (A\times B)^{(2n+1)}$ be a $\mathfrak{A}$-$(A\times B)$-module map. Then, for all $(a,x) \in A\times X$ and $(a^{(2n)}, b^{(2n)}) \in A^{(2n)}\times B^{(2n)}$ we have

$$\begin{aligned}\langle T(a\cdot x), (a^{(2n)}, b^{(2n)})\rangle &= \langle (a,0)\cdot T(x), (a^{(2n)}, b^{(2n)})\rangle\\ &= \langle T(x), (a^{(2n)}, b^{(2n)})\cdot (a,0)\rangle\\ &= \langle T(x)\cdot (a^{(2n)},0), (a,0)\rangle\\ &= \langle T^{(2n)}(x\cdot (a^{(2n)},0)), (a,0)\rangle = 0.\end{aligned}$$

Similarly, $T(y\cdot b) = 0$, for all $(y,b) \in X\times B$, and so $T(a\cdot x + y\cdot b) = 0$. Let $z \in \langle AX + XB\rangle$. Then, $z = \sum_{i=1}^{k}(\lambda_i(a_i\cdot x_i) + \gamma_i(y_i\cdot b_i))$ for some $\lambda_i, \gamma_i \in \mathbb{C}$, $a_i \in A$, $b_i \in B$ and $x_i, y_i \in X$. Since T is a $\mathfrak{A}$-module map, we have $T(z) = \sum_{i=1}^{k} T((\lambda_i a_i)\cdot x_i + (\gamma_i y_i)\cdot b_i) = 0$. From continuity of T and density of $\langle AX + XB\rangle$ in X, we get $T = 0$, so condition (iii) of Theorem 2.1 holds.

For (iv), let $S : X \to X^{(2n+1)}$ be a $\mathfrak{A}$-$(A\times B)$-module map with $S(x)\cdot y + x\cdot S(y) = 0$, for all $x, y \in X$. Then, $S(a\cdot x + y\cdot b) = (a,0)\cdot S(x) + S(y)\cdot (0,b) = 0$. Continuity of S and density of $\langle AX + XB\rangle$ in X imply that $S = 0$, as required. □

It is proved in [6, Theorem 4.3] that $(2n+1)$-weak module amenability of $\mathrm{Tri}(A,X,B)$ is equivalent to $(2n+1)$-weak module amenability of A and B, if A and B both have bounded approximate identity and X is a non-degenerate (A,B)-module. They use [6, Proposition 4.2] in their proof, but the assumptions of this proposition do not appear in [6, Theorem 4.3]. Thus, the result will be valid, if $A^{(2n-1)}$, $B^{(2n-1)}$ and $X^{(2n-1)}$ are also non-degenerate. In the next, we improve [6, Theorem 4.3] and extend the main result of [18] and give a simple proof for them. In fact we obtain the same result with different conditions.

Theorem 3.2. *Let B (resp. A) has a bounded right (resp. left) approximate identity, and let $X^{(2n+1)}$ be a non-degenerate left B-module (resp. right A-module). Then, $\mathrm{Tri}(A,X,B)$ is $(2n+1)$-weakly module amenable if and only if A and B are $(2n+1)$-weakly module amenable.*

Proof. Using Corollary 2.8 and Theorem 3.1, it is enough to show that $H^1_{\mathfrak{A}}(A\times B, X^{(2n+1)}) = \{0\}$. For this, let $D : A\times B \to X^{(2n+1)}$ be a $\mathfrak{A}$-module derivation. Then, $D(a,b) = D_A(a) + D_B(b)$ for some right A-module map $D_A : A \to X^{(2n+1)}$ and left B-module map $D_B : B \to X^{(2n+1)}$. Moreover, $b\cdot D_A(a) = -D_B(b)\cdot a$ for all $a\in A, b\in B$. Since B has a bounded right approximate identity, there is $g \in X^{(2n+1)}$ such that $D_B(b) = b\cdot g$. Thus, $b\cdot D_A(a) = -D_B(b)\cdot a = -b\cdot g\cdot a$. Since $X^{(2n+1)}$ is non-degenerate, we get $D_A(a) = -g\cdot a$. Therefore,

$$D(a,b) = D_A(a) + D_B(b) = -g\cdot a + b\cdot g = \mathrm{ad}_g(a,b).$$

□

If we apply Theorem 3.2 for $\mathrm{Tri}(A,X,A)$, we get the following result.

Corollary 3.3. *Let A has a bounded right (resp. left) approximate identity, and $X^{(2n+1)}$ be a non-degenerate left (resp. right) A-module. Then, $\mathrm{Tri}(A,X,A)$ is $(2n+1)$-weakly module amenable if and only if A is $(2n+1)$-weakly module amenable.*

To give our results on $2n$-weak module amenability of $\mathrm{Tri}(A,X,B)$, we need the following lemma, which can be proved by a similar argument used in Theorem 3.2.

Lemma 3.4. *Let $n \in \mathbb{N}$ and B (resp. A) has a bounded left (resp. right) approximate identity. If $X^{(2n)}$ is a non-degenerate right B-module (resp. left A-module), then $H^1_{\mathfrak{A}}(A\times B, X^{(2n)}) = \{0\}$.*

If we use Theorem 2.4 for $\mathrm{Tri}(A,X,B)$, we arrive at the following result, which is a generalization of [6, Theorem 5.1(iii) and 5.3]. By $\langle AXB\rangle$, we denote the linear span of $AXB = \{a\cdot x\cdot b\,; a\in A, b\in B, x\in X\}$ in X.

Theorem 3.5. *Let $n \in \mathbb{N}$ and B (resp. A) has a bounded left (resp. right) approximate identity, and $X^{(2n)}$ be a non-degenerate right B-module (resp. left A-module). If $\langle AXB\rangle$ is dense in X, then $\mathrm{Tri}(A, X, B)$ is $2n$-weakly module amenable if and only if*

(1) The only $\mathfrak{A}$-module derivations $D : A \times B \to (A \times B)^{(2n)}$ for which there is a bounded $\mathfrak{A}$-module map $S : X \to X^{(2n)}$ such that $S(x \cdot b) = S(x) \cdot b + x \cdot D(a, b)$ and $S(a \cdot x) = a \cdot S(x) + D(a, b) \cdot x$ $(a \in A, x \in X)$ are inner $\mathfrak{A}$-module derivations.

(2) For every bounded $\mathfrak{A}$-$(A\times B)$-module map $S : X \to X^{(2n)}$, there is $(f, g) \in (A\times B)^{(2n)}$ such that $(a, b) \cdot (f, g) = (f, g) \cdot (a, b)$ for $(a, b) \in A \times B$ and $S(x) = x \cdot (f, g) - (f, g) \cdot x$ for $x \in X$.

Proof. Using Lemma 3.4, it is enough to prove condition (*iii*) of Theorem 2.4. Let $T : X \to (A \times B)^{(2n)}$ be $\mathfrak{A}$-$(A \times B)$-module map. Then, for every $f \in A^{(2n-1)}$ and $g \in B^{(2n-1)}$, we have

$$\begin{aligned}\langle T(a \cdot y \cdot b), (f, g)\rangle &= \langle (a, 0) \cdot T(y) \cdot (0, b), (f, g)\rangle \\ &= \langle T(y), (0, b) \cdot (f, g) \cdot (a, 0)\rangle = 0.\end{aligned}$$

Since $\langle AXB\rangle$ is dense in X, we obtain $T(x) = 0$, for all $x \in X$. So $T = 0$. □

Remark 3.6. *It is worthwhile mentioning that, the condition $\overline{\langle AXB\rangle} = X$, in Theorem 3.5, can be replaced by any of the following statements:*

(a) $\overline{\langle AX\rangle} = X$ and $A^{(2n)}$ is a non-degenerate right A-module.

(b) $\overline{\langle XB\rangle} = X$ and $B^{(2n)}$ is a non-degenerate left B-module.

Indeed, if (a) holds and $T : X \to (A \times B)^{(2n)}$ is a $\mathfrak{A}$-$(A \times B)$-module map. Then, for $f \in A^{(2n-1)}$,

$$\langle T(x), (a \cdot f, 0)\rangle = \langle T(x), (a, 0) \cdot (f, 0)\rangle = \langle T(x \cdot (a, 0)), (f, 0)\rangle = 0.$$

And for all $g \in B^{(2n-1)}$,

$$\langle T(a \cdot y), (0, g)\rangle = \langle (a, 0) \cdot T(y), (0, g)\rangle = \langle T(y), (0, g) \cdot (a, 0)\rangle = 0.$$

So, by assumption we get $\langle T(x), (f, g)\rangle = \langle T(x), (f, 0)\rangle + \langle T(x), (0, g)\rangle = 0$, for $x \in X$, $(f, g) \in A^{(2n-1)} \times B^{(2n-1)}$. Therefore, $T = 0$. A similar argument can be used for (b).

Using Theorem 3.5, we obtain the next result, which improves [6, Corollary 5.3.1].

Corollary 3.7. *Let $n \in \mathbb{N}$ and A has a bounded left (resp. right) approximate identity, and $A^{(2n)}$ be a non-degenerate right A-module (resp. left A-module). Then, $\mathrm{Tri}(A, A, A)$ is $2n$-weakly module amenable if and only if A is $2n$-weakly module amenable.*

Proof. To prove the necessity, suppose that $d : A \to A^{(2n)}$ is a $\mathfrak{A}$-module derivation. Define $D : A \times A \to (A \times A)^{(2n)}$ by $D(a, c) = (d(a), d(c))$. Then, D is a $\mathfrak{A}$-module derivation. Using Part (1) of Theorem 3.5 with D and $S = d$, we conclude that D is inner. A simple calculation shows that d is also inner.

For sufficiency, it is enough to prove conditions (1) and (2) of Theorem 3.5, by Cohen's factorization property. From Corollary 2.8, it follows that $A \times A$ is $2n$-weakly module amenable. So, condition (1) of Theorem 3.5 holds.

For condition (2), let $S : A \to A^{(2n)}$ be a bounded $\mathfrak{A}$-$(A \times A)$-module map. Then, S is an A-module map. Since A has a bounded left approximate identity, there is $f \in A^{(2n)}$ such that $S(a) = f \cdot a$, for all $a \in A$. But,

$$a \cdot f \cdot x = a \cdot S(x) = S(a \cdot x) = f \cdot a \cdot x \quad (x \in A).$$

This implies that $a \cdot f = f \cdot a$, since $A^{(2n)}$ is a non-degenerate right A-module. Now $(-f, 0) \cdot (a, c) = (a, c) \cdot (-f, 0)$ for all $a, c \in A$ and $S(x) = x \cdot (-f, 0) - (-f, 0) \cdot x$ for all $x \in A$. □

We close this paper, with some examples.

Example 3.8. *1. Let G be an abelian locally compact Hausdorff group. Since $L^1(G)$ has a bounded approximate identity and $L^p(G)$, $1 \leq p \leq \infty$, is a commutative $L^1(G)$-module, it follows from Corollary 3.3 that, $\mathrm{Tri}(L^1(G), L^p(G), L^1(G))$ is weakly module amenable, as an $L^1(G)$-module. Moreover, Proposition 2.6 shows that $L^1(G) \oplus L^\infty(G)$ is not n-weakly module amenable, as an $L^1(G)$-module, for each $n \in \mathbb{N}$.*

2. Let S be an inverse semigroup with the set of idempotents E. Then, E is a commutative sub-semigroup of S and $l^1(E)$ could be regarded as a commutative sub-algebra of $l^1(S)$. It is well known that $l^1(S)$ has a bounded approximate identity if and only if E satisfies condition D_k for some $k \in \mathbb{N}$, [6].

Let $l^1(E)$ act trivially on $l^1(S)$ from left and by multiplication from right. Then, $l^1(S)$ is a Banach $l^1(E)$-module with compatible actions. Although $l^1(S)$ is n-weakly module amenable (as an $l^1(E)$-module) [4, 11], Proposition 2.6 shows that $l^1(S) \oplus l^\infty(S)$ is not n-weakly module amenable, as an $l^1(E)$-module, for each $n \in \mathbb{N}$. Furthermore, it follows from Corollaries 3.3 and 3.7 that $\mathcal{T}_2 \otimes l^1(S) = \mathrm{Tri}(l^1(S), l^1(S), l^1(S))$ is n-weakly module amenable (as an $l^1(E)$-module) if E satisfies condition D_k for some $k \in \mathbb{N}$. Theorem 2.7 in [5] shows that, the same conclusion is also true when S is commutative and $l^1(E)$ acts on $l^1(S)$ by usual multiplication from both sides.

4. Conclusions

We study and characterize the n-weak module amenability of module extension and triangular Banach algebras. We also address a gap in the proof of [6, Theorem 4.3] and extend and improve it by discussing general necessary and sufficient conditions for $\mathrm{Tri}(A, X, B)$ to be n-weakly module amenable, for an integer $n \geq 0$.

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