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New results on the Estrada index of graphs

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Abstract. Let G be a graph of order n and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G. The Estrada index of G, denoted by $EE(G)$, is the sum of the terms e^{λ_i} . In this paper, new lower and upper bounds for the Estrada index are established. Moreover, some of our bounds are extensions of the well-known bounds on the Estrada index of graphs.

1. Introduction

Throughout this paper we consider simple graphs, that is, finite and undirected graphs without loops and multiple edges. If G is a graph with vertex set $\{1, \ldots, n\}$, the adjacency matrix of G is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. Since A is a real symmetric matrix, its eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are real numbers. These are referred to as the eigenvalues of G. The multiset of eigenvalues of A is called the spectrum of G. For details of the theory of graph spectra consult [6]. We denote the complete graph on n vertices by K_n and the complete bipartite graph by $K_{m,n}$.

A graph spectrum-based invariant, put forward by Estrada [10], is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$

$EE(G)$ is usually referred to as the Estrada index. The Estrada index has a wide range of uses, including: quantify the degree of folding of long organic molecules, see [10, 11], the complex networks [12, 13], a measure of molecular branching [15], statistical thermodynamic [14] and also, for more information on the Estrada index see [4]. Some important mathematical properties of the Estrada index can be found in ([1, 8, 9, 16–22]). In this paper, we establish some bounds for the Estrada index of a graph. The bounds represent improvement of some known results from the literature.

We make use of the following results in this paper.

Theorem 1.1. *Let G be a graph of order $n \geq 1$ and size m. If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of G, then*

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(a) *[6]* $\sum_{i=1}^{n} \lambda_i = 0$.

(b) *[6]* $\sum_{i=1}^{n} \lambda_i^2 = 2m$.

(c) *[6]* $\sum_{i=1}^{n} \lambda_i^3 = 6t$ *where* t *is the number of triangle of* G.

(d) *[5]* $\sum_{i=1}^{n} \lambda_i^4 = 2Zg(G) - 2m + 8q$ *where* q *is the number of quadrangles of* G.

(e) *[5]* $\sum_{i=1}^{n} \lambda_i^5 = 30t + 10p + 10r$ *where* p *is the number of pentagons and* r *is the number of subgraphs consisting of a triangle with a pendent vertex attached in graph* G.

Using (a) we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i + \lambda_j) = n\sum_{i=1}^{n}\lambda_i + n\sum_{j=1}^{n}\lambda_j = 0, \tag{1}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i\lambda_j) = (\sum_{i=1}^{n}\lambda_i)^2 = 0, \tag{2}$$

and for any integer $k \geq 1$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^k\lambda_j + \lambda_i\lambda_j^k) = \sum_{i=1}^{n}\lambda_i^k \cdot \sum_{j=1}^{n}\lambda_j + \sum_{i=1}^{n}\lambda_i \cdot \sum_{j=1}^{n}\lambda_j^k = 0. \tag{3}$$

By (b) we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^2 + \lambda_j^2) = n\sum_{i=1}^{n}\lambda_i^2 + n\sum_{j=1}^{n}\lambda_j^2 = 4mn, \tag{4}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^2\lambda_j^2) = (\sum_{i=1}^{n}\lambda_i^2)^2 = 4m^2, \tag{5}$$

and by (c) we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^3 + \lambda_j^3) = 12nt. \tag{6}$$

Applying (b) and (c) we drive

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) = \sum_{i=1}^{n}\lambda_i^2 \cdot \sum_{j=1}^{n}\lambda_j^3 + \sum_{i=1}^{n}\lambda_i^3 \cdot \sum_{j=1}^{n}\lambda_j^2 = 24mt. \tag{7}$$

Similarly we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i^5 + \lambda_j^5) = 2n\sum_{i=1}^{n}\lambda_i^5 = 2n(30t + 10p + 10r). \tag{8}$$

We close this section by recalling some known bounds.
In [2] the following inequality was proved.

Theorem 1.2. *[2] Let G be a graph with n vertices, m edges and t triangles. Then*

$$EE(G) \geq \sqrt{n^2 + 2nm + 8nt}. \tag{9}$$

Gutman [16] derived the lower bounds on Estrada index, stated in the following theorem.

Theorem 1.3. *[16] Let G be a graph of order n and size m. Then*

$$EE(G) \geq n \cosh\left(\sqrt{\frac{2m}{n}}\right). \tag{10}$$

Rodríguez [24] obtained a lower bound for EE, stated in the next theorem.

Theorem 1.4. *[24] Let G be a graph of order $n \geq 2$. Then*

$$EE(G) \geq \frac{\left(\sum_{i=1}^{n} e^{\frac{\lambda_i}{2}}\right)^2 - n}{n-1}. \tag{11}$$

The following result appears in [13] as well.

Theorem 1.5. *[13] Let G be a graph of order n and size m. Then*

$$EE(G) \geq e^{\frac{2m}{n}} + (n-1) - \frac{2m}{n}. \tag{12}$$

In [7], the following theorem is proved.

Theorem 1.6. *[7] Let G be a graph of order n and size m. Then*

$$\sqrt{\frac{2m}{n(n-1)}} \leq \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}$$

$$-\sqrt{\frac{2m(n-1)}{n}} \leq \lambda_n \leq -\sqrt{\frac{2m}{n(n-1)}}.$$

2. Main result

In this section, we establish some sharp bounds for the Estrada index and show that some of our bounds improve the well-known results. First, we recall an inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers for which there exist real constants a, b, A and B, such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for each $i \in \{1, 2, \ldots, n\}$. It is proved in [3] that

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b), \tag{13}$$

where, $\alpha(n) = n[\frac{n}{2}]\left(1 - \frac{1}{n}[\frac{n}{2}]\right)$.
Equality in (13) holds if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

In the next theorem we establish an upper bound and a lower bound for the $EE(G)$ in terms of parameters n and $\alpha(n)$.

Theorem 2.1. *Let G be a graph of order n and let $\alpha(n) = n\left\lfloor \frac{n}{2} \right\rfloor\left(1 - \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor\right)$. Then*

$$\sqrt{n\sum_{i=1}^{n} e^{2\lambda_i} - \alpha(n)(e^{\lambda_1} - e^{\lambda_n})^2} \leq EE(G) \leq \sqrt{n\sum_{i=1}^{n} e^{2\lambda_i} + \alpha(n)(e^{\lambda_1} - e^{\lambda_n})^2}.$$

Proof. Replacing $a = b = e^{\lambda_n}$, $A = B = e^{\lambda_1}$ and $a_i = b_i = e^{\lambda_i}$ for each $i \in \{1, 2, \ldots, n\}$ in (13), we obtain

$$\left| n\sum_{i=1}^{n} e^{2\lambda_i} - \sum_{i=1}^{n} e^{\lambda_i} \sum_{i=1}^{n} e^{\lambda_i} \right| \leq \alpha(n)(e^{\lambda_1} - e^{\lambda_n})(e^{\lambda_1} - e^{\lambda_n}).$$

This implies that

$$n\sum_{i=1}^{n} e^{2\lambda_i} - \alpha(n)(e^{\lambda_1} - e^{\lambda_n})^2 \leq \left(\sum_{i=1}^{n} e^{\lambda_i}\right)^2 \leq n\sum_{i=1}^{n} e^{2\lambda_i} + \alpha(n)(e^{\lambda_1} - e^{\lambda_n})^2$$

which leads to the desired bounds. □

Since $\alpha(n) = n[\frac{n}{2}]\left(1 - \frac{1}{n}[\frac{n}{2}]\right) \leq \frac{n^2}{4}$, the next result is an immediate consequence of Theorem 2.1. It shows that the lower bound of Theorem 2.1 is stronger than the lower bound presented by Rodríguez in 2019 [24].

Corollary 2.2. *For any graph G of order n,*

$$\sqrt{n\sum_{i=1}^{n} e^{2\lambda_i} - \frac{n^2}{4}(e^{\lambda_1} - e^{\lambda_n})^2} \leq EE(G) \leq \sqrt{n\sum_{i=1}^{n} e^{2\lambda_i} + \frac{n^2}{4}(e^{\lambda_1} - e^{\lambda_n})^2}.$$

To prove another lower bound, we recall a numerical inequality presented in [23].

Lemma 2.3. *[23] Let $n \geq 1$ be an integer and $a_1, a_2, \ldots, a_n$ be some non-negative real numbers such that $a_1 \geq a_2 \geq \ldots \geq a_n$. Then*

$$(a_1 + \ldots + a_n)(a_1 + a_n) \geq a_1^2 + \ldots + a_n^2 + na_1a_n.$$

The next theorem reveals a connection between the Estrada index, $e^{\lambda_1+\lambda_n}$ and $e^{\lambda_1} + e^{\lambda_n}$.

Theorem 2.4. *Let G be a graph of order n. Then*

$$EE(G) \geq \frac{\sum_{i=1}^{n} e^{2\lambda_i} + ne^{(\lambda_1+\lambda_n)}}{e^{\lambda_1} + e^{\lambda_n}}. \tag{14}$$

Proof. Replacing $a_1 = e^{\lambda_1}$, $a_n = e^{\lambda_n}$ and $a_i = e^{\lambda_i}$ for $i = 1, 2, \ldots, n$ in the inequality of Lemma 2.3, we have

$$\sum_{i=1}^{n} e^{\lambda_i}\left(e^{\lambda_1} + e^{\lambda_n}\right) \geq \sum_{i=1}^{n}\left(e^{\lambda_i}\right)^2 + ne^{\lambda_1}e^{\lambda_n}.$$

It follows that

$$EE(G) \geq \frac{\sum_{i=1}^{n} e^{2\lambda_i} + ne^{\lambda_1+\lambda_n}}{e^{\lambda_1} + e^{\lambda_n}}.$$

□

Next result shows that the bound in Theorem 2.5 improves the lower bound $EE(G) \geq \sqrt{n^2 + 4m + 8t}$ presented in [8].

Theorem 2.5. Let G be a graph with n vertices, m edges and t triangles. Then

$$EE(G) \geq \sqrt{n^2 + 4m + 8t + \frac{4}{15}(30t + 10p + 10r)}\ .$$

Proof. By the definition of the Estrada index and Equationes (1)-(8), we have

$$\begin{aligned}
EE(G)^2 &= (\textstyle\sum_{i=1}^n e^{\lambda_i})^2\\
&= (\textstyle\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_1}+e^{\lambda_n}) + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&\geq \textstyle\sum_{i=1}^n e^{2\lambda_i} + ne^{\lambda_1+\lambda_n} + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&= \textstyle\sum_{i=1}^n (1+2\lambda_i+2\lambda_i^2+\frac{8}{6}\lambda_i^3+\frac{16}{24}\lambda_i^4+\frac{32}{120}\lambda_i^5)\\
&+ ne^{\lambda_1+\lambda_n} + (\textstyle\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&= n + 2\textstyle\sum_{i=1}^n \lambda_i^2 + \frac{8}{6}\sum_{i=1}^n \lambda_i^3 + \frac{16}{24}\sum_{i=1}^n \lambda_i^4 + \frac{32}{120}\sum_{i=1}^n \lambda_i^5\\
&+ ne^{\lambda_1+\lambda_n} + (\textstyle\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&\geq n+4m+8t+\tfrac{2}{3}\textstyle\sum_{i=1}^n \lambda_i^4 + \frac{4}{15}\sum_{i=1}^n \lambda_i^5 + n(1+\lambda_1+\lambda_n)\\
&\quad +n((1+\lambda_2)+\ldots+(1+\lambda_{n-1}))\\
&\geq n^2+4m+8t+\tfrac{4}{15}(30t+10p+10r),
\end{aligned}$$

and this leads to the desired bound. □

Applying Theorem 1.6 and a similar argument described in the proof of Theorem 2.5, we drive the next result.

Theorem 2.6. Let G be a graph with n vertices, m edges and t triangles. Then

$$EE(G) \geq \sqrt{n^2+(n+4)m+(n+8)t+\frac{2m(n-1)\sqrt{\frac{2m}{n(n-1)}}}{3n}-\frac{2m}{n}-\frac{2m\sqrt{\frac{2m(n-1)}{n}}}{3n(n-1)}}.$$

Proof. By the definition of the Estrada index and Theorems 2.4 and 1.6, we have

$$\begin{aligned}
EE(G)^2 &= (\textstyle\sum_{i=1}^n e^{\lambda_i})^2\\
&= (\textstyle\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_1}+e^{\lambda_n}) + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&\geq \textstyle\sum_{i=1}^n e^{2\lambda_i} + ne^{\lambda_1+\lambda_n} + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&= \textstyle\sum_{i=1}^n (1+2\lambda_i+2\lambda_i^2+\frac{8}{6}\lambda_i^3) + ne^{\lambda_1+\lambda_n} + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&= n+2\textstyle\sum_{i=1}^n \lambda_i^2 + \frac{8}{6}\sum_{i=1}^n \lambda_i^3 + ne^{\lambda_1+\lambda_n} + (\sum_{i=1}^n e^{\lambda_i})(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&\geq n+4m+8t+ne^{\lambda_1+\lambda_n} + n(e^{\lambda_2}+\ldots+e^{\lambda_{n-1}})\\
&\geq n+4m+8t+n(1+(\lambda_1+\lambda_n)+\tfrac{1}{2}(\lambda_1+\lambda_n)^2+\tfrac{1}{6}(\lambda_1+\lambda_n)^3)\\
&\quad +n(\textstyle\sum_{i=2}^{n-1}(1+\lambda_i+\frac{1}{2}\lambda_i^2+\frac{1}{6}\lambda_i^3))\\
&= n^2+(n+4)m+(n+8)t+\lambda_1\lambda_n+\tfrac{1}{3}(\lambda_1^2\lambda_n+\lambda_1\lambda_n^2)\\
&= n^2+(n+4)m+(n+8)t+\frac{2m(n-1)\sqrt{\frac{2m}{n(n-1)}}}{3n}-\frac{2m}{n}-\frac{2m\sqrt{\frac{2m(n-1)}{n}}}{3n(n-1)}
\end{aligned}$$

as desired. □

Using the bound of Theorem 2.4 we obtain a lower bound on the Estrada index of bipartite graphs.

Theorem 2.7. Let G be a bipartite connected graph with n vertices and m edges. Then

$$EE(G) \geq 2\cosh(\frac{2m}{n}) + \frac{4m+2(n-2)-\frac{8m(n-1)}{n}}{2\cosh(\sqrt{\frac{2m(n-1)}{n}})}.$$

Proof. Using the bound of Theorem 2.4 and the fact $\lambda_n = -\lambda_1$ we obtain

$$\begin{aligned}
EE(G) &\geq \frac{\sum_{i=1}^{n} e^{2\lambda_i} + ne^{(\lambda_1+\lambda_n)}}{e^{\lambda_1}+e^{\lambda_n}} \\
&= \frac{\sum_{i=1}^{n} e^{2\lambda_i} + n}{2\cosh(\lambda_1)} \\
&= \frac{(e^{2\lambda_1}+e^{2\lambda_n}) + \sum_{i=2}^{n-1} e^{2\lambda_i} + n}{2\cosh(\lambda_1)} \\
&= \frac{2\cosh(2\lambda_1) + \sum_{i=2}^{n-1} e^{2\lambda_i} + n}{2\cosh(\lambda_1)} \\
&= \frac{2(2\cosh^2(\lambda_1)-1) + \sum_{i=2}^{n-1} e^{2\lambda_i} + n}{2\cosh(\lambda_1)} \\
&= 2\cosh(\lambda_1) + \frac{\sum_{i=2}^{n-1} e^{2\lambda_i} + n - 2}{2\cosh(\lambda_1)} \\
&\geq 2\cosh(\lambda_1) + \frac{\sum_{i=2}^{n-1}(1+2\lambda_i+2\lambda_i^2+8\lambda_i^3) + n - 2}{2\cosh(\lambda_1)} \\
&= 2\cosh(\lambda_1) + \frac{\sum_{i=2}^{n-1}(2\lambda_i^2) + 2(n-2)}{2\cosh(\lambda_1)} \\
&= 2\cosh(\lambda_1) + \frac{4m + 2(n-2) - 4\lambda_1^2}{2\cosh(\lambda_1)}.
\end{aligned}$$

Since, $\cosh x$, is an increasing function on $[0, +\infty)$ and $\sqrt{\frac{2m(n-1)}{n}} \geq \lambda_1 \geq \frac{2m}{n} \geq 0$, we obtain

$$EE(G) \geq 2\cosh(\frac{2m}{n}) + \frac{4m + 2(n-2) - \frac{8m(n-1)}{n}}{2\cosh(\sqrt{\frac{2m(n-1)}{n}})}.$$

□

Remark 2.8. *Let's see some examples of families of graphs where Theorem 2.4 is verified, more than one, it is shown that the lower bound in Theorem 2.4 is better than the Estrada index existing in the literature. For our analysis we will use the following notation, from (14)* $EE(G) \geq \frac{\sum_{i=1}^{n} e^{2\lambda_i} + ne^{(\lambda_1+\lambda_n)}}{e^{\lambda_1}+e^{\lambda_n}} = A_1(G)$*, from (9)* $EE(G) \geq \sqrt{n^2 + 2nm + 8nt} = A_2(G)$*, from (10)* $EE(G) \geq n\cosh\left(\sqrt{\frac{2m}{n}}\right) = A_3(G)$*, and from (11)* $EE(G) \geq \frac{\left(\sum_{i=1}^{n} e^{\frac{\lambda_i}{2}}\right)^2 - n}{n-1} = A_4(G)$*. We show that for some kind of graphs G,* $A_1(G) > \max\{A_2(G), A_3(G), A_4(G)\}$*.*

- *Complete graphs* K_n $(n \geq 3)$.
  We have $A_1(K_n) = \frac{e^{2(n-1)} + \frac{n-1}{e^2} + ne^{(n-2)}}{e^{(n-1)} + e^{-1}}$*. It is easy to see that* $A_1(K_n) > \sqrt{\frac{n^4+2n^2}{2}} = A_2(K_n)$*,* $A_1(K_n) > n\left(\frac{e^{\sqrt{n-1}} + e^{-\sqrt{n-1}}}{2}\right) = A_3(K_n)$ *and* $A_1(K_n) > \frac{\left(e^{\frac{n-1}{2}} + (n-1)e^{\frac{-1}{2}}\right)^2 - n}{n-1} = A_4(K_n)$*.*
- *Complete bipartite graph* $K_{m,n}$*.*
  We have $A_1(K_{m,n}) = \frac{e^{2\sqrt{nm}} + e^{-2\sqrt{nm}} + 2n + 2m - 2}{e^{\sqrt{nm}} + e^{-\sqrt{nm}}}$*. It is not hard to verify that* $A_1(K_{m,n}) > \sqrt{(m+n)^2 + 2(m+n)(mn)} = A_2(K_{m,n})$*,*
  $A_1(K_{m,n}) > (n+m)\left(\frac{e^{\sqrt{\frac{2(mn)}{m+n}}} + e^{-\sqrt{\frac{2(mn)}{m+n}}}}{2}\right) = A_3(K_{m,n})$ *and*
  $A_1(K_{m,n}) > \frac{\left(e^{\frac{\sqrt{nm}}{2}} + e^{\frac{-\sqrt{nm}}{2}} + (m+n)\right)^2 - (n+m)}{n-1} = A_4(K_{m,n})$*.*

3. Comparing bounds and final remarks

In this section, we present some computational experiments to compare our new bound (Theorem 2.4) to previously published lower bounds for some of the graphs in Figure 1 and some of the well-known graphs. We compare the estimates obtained by Theorem 1.3 (Th. 1.3), Theorem 1.4 (Th. 1.4), Theorem 1.5 (Th. 1.5), and Theorem 2.4 (Th. 2.4).

In all of our test cases, the lower bound in Theorem 2.4 were better than existing bounds, see Table 1.

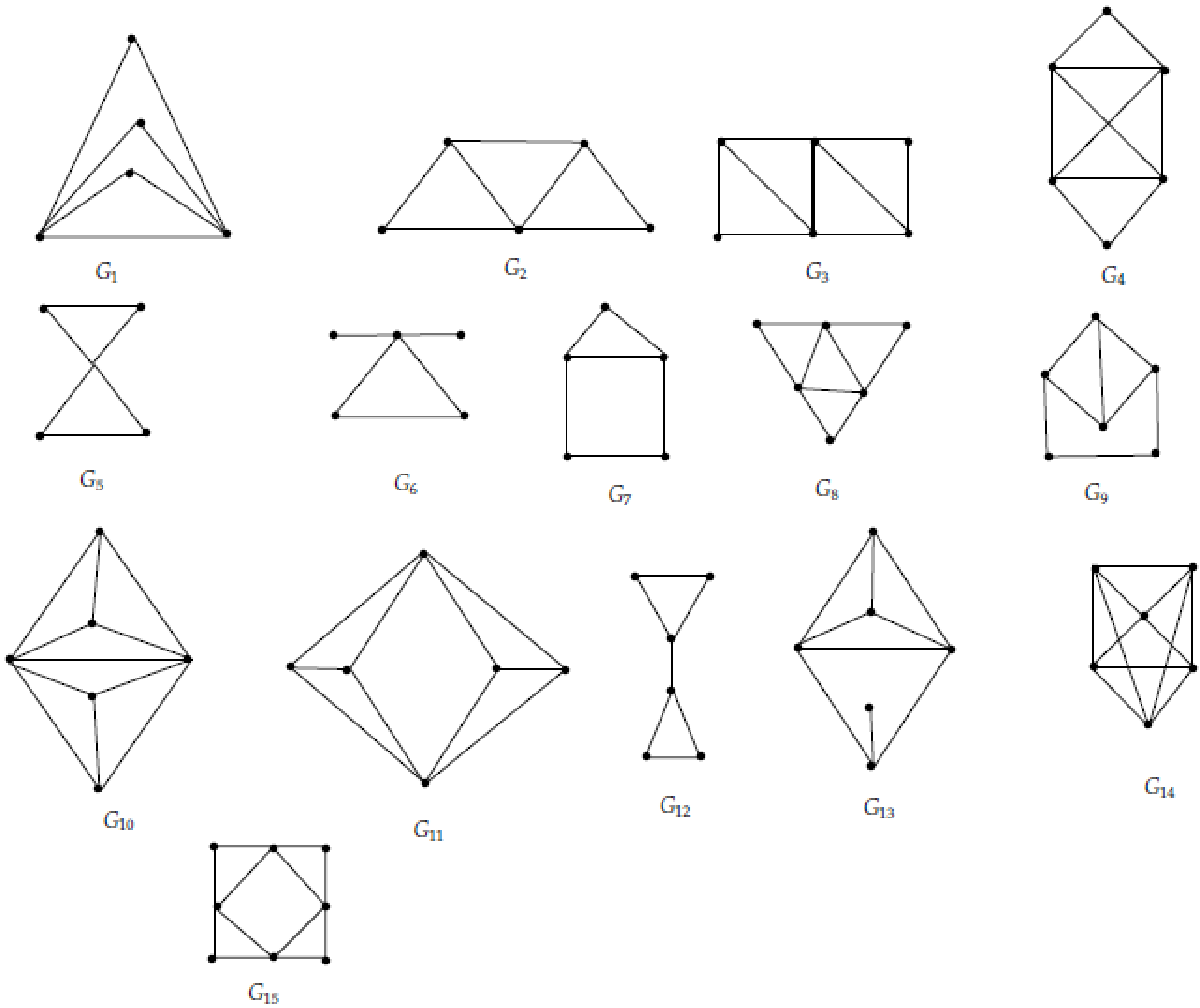

Figure 1: The graphs G_i for $i = 1, 2, \ldots, 15$

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest

We declare that there is no conflict of interest.

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Table 1: Compare bounds

G_i	λ_i	Th.2.4	Th.1.3	Th.1.4	Th.1.5
G_1	3, 0, 0, -1, -2	**20.7297612723145**	13.793642244254	11.7728887374146	17.644646771097
G_2	2.9354, 0.6180, -0.4626, -1.4728, -1.6180	**19.8205001720493**	13.793642244254	12.5101859069609	17.644646771097
G_3	3.182, 1.247, -0.445, -0.594,-1.588, -1.802	**25.4456273771905**	17.4874646410556	15.6392673037286	22.0855369231877
G_4	3.562, 1, -0.562, -1, -1, -2	**46.9409143878271**	20.7999382590903	18.5346150059134	29.6982915611928
G_5	2.6518, 1, -1, -1, -1.6518	**14.3674686507902**	12.3002260213754	10.7193172418443	12.6231763806416
G_6	2.3429, 0.4707, 0, -1, -1.8136	**12.911099129215**	12.3002260213754	10.2891281328215	12.6231763806416
G_7	2.4812, 0.6889, 0 , -1.1701, -2	**11.4059024130837**	10.8909177830429	9.3201860399	9.3890560989306
G_8	3.236, 0.618, 0.618,-1.236, -1.618, -1.618	**26.6905328816204**	17.4874646410556	15.7160298736336	22.0855369231877
G_9	2.791,1, 0.618, -1,-1.618, -1.791	**19.8205001720493**	15.9435549879962	13.2772582599538	16.7252494284832
G_{10}	3.828, 1 , -1, -1, -1, -1.828	**46.9409143878271**	20.7999382590903	21.4818837684105	40.4546173314865
G_{11}	3.372, 1, 0, -1, -1, -2.372	**29.8981294023778**	19.1054892285039	17.0981099769904	29.6982915611928
G_{12}	2.414, 1.732, -0.414, -1, -1, -1.732	**14.9276489437469**	14.4715027240215	12.1417198292866	12.9789251679924
G_{13}	3.354, 1, -0.476, -1, -1, -1.877	**29.6586145257653**	17.4874646410556	16.4369851486481	22.0855369231877
G_{14}	4, 0, 0, 0, -2,-2	**55.3286336035092**	19.1054892285039	23.5523016735136	29.6982915611928
G_{15}	3.236, 1.414, 1.414, 0, -1.236, -1.414, -1.414, -2	**27.7447974525628**	23.3166195214074	19.4009280117335	24.0855369231877

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