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Tempered fractional versions of trapezoid and midpoint-type inequalities for twice-differentiable functions

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Abstract. The paper introduces a novel approach to examine Hermite–Hadamard-type inequalities taking advantage of tempered fractional operators. These inequalities are proved by means of twice-differentiable convex functions involving tempered fractional integral operators. Obtained Hermite–Hadamard-type inequalities are a generalization of some of the studies on this subject, including Riemann-Liouville fractional integrals.

1. Introduction and preliminaries

Fractional calculus is a subject that has been extensively studied in the literature over a long period of time. Several mathematicians and physicists have made important contributions in order to the development of fractional calculus over the past three centuries. Hence, books on the topic of fractional calculus have been appearing since the previous century such as Oldham and Spanier (1974), Samko, Kilbas and Marichev (1993), Podlubny (1999), and so on. Additional theories and experiments indicate that fractional calculus can be used to describe a wide range of non-classical phenomena observed in various fields of applied sciences and engineering [18, 22, 23]. In practical applications, some different kinds of fractional derivatives are introduced such as Riemann-Liouville fractional derivative, Caputo fractional derivative [22, 23], Hilfer fractional derivative [10, 27], and Riesz fractional derivative [23].

Next, we will provide some essential definitions required to establish our main results. Riemann-Liouville integral operators are defined by as follows:

Definition 1.1 (See [13]). *The Riemann-Liouville integrals of order $\alpha > 0$ are given by*

$$J_{\sigma+}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{\sigma}^{x}(x-\mu)^{\alpha-1}\mathcal{F}(\mu)d\mu, \quad x > \sigma \tag{1}$$

and

$$J_{\kappa-}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\kappa}(\mu-x)^{\alpha-1}\mathcal{F}(\mu)d\mu, \quad x < \kappa. \tag{2}$$

Here, $\mathcal{F} \in L_1[\sigma,\kappa]$ and $\Gamma(\alpha) := \int\limits_0^\infty \mu^{\alpha-1}e^{-\mu}d\mu$.

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Note that the Riemann-Liouville integrals equal to classical integrals for the condition $\alpha = 1$.

Dragomir and Agarwal first investigated trapezoid-type inequalities for the case of convex functions in [6], while Kirmacı first proved midpoint-type inequalities for the case of convex functions in [14]. Sarikaya et al. and Iqbal et al. established several fractional midpoint-type inequalities and trapezoid-type inequalities for the case of convex functions in papers [11] and [24], respectively. See [4, 21] and the references cited therein for further information about fractional integral inequalities.

Many mathematicians have directed their attention to twice differentiable functions to derive Hermite–Hadamard-type and related inequalities. For example, in paper [20], new integral inequalities of midpoint-type and trapezoid-type were obtained for twice differentiable convex functions in the form classical integral and Riemann-Liouville fractional integrals. Moreover, in paper [25], several inequalities of Simpson and Hermite–Hadamard-type were proved for functions whose absolute values of derivatives are convex. Furthermore, Budak et al. [3] established several midpoint-type and trapezoid-type inequalities for functions whose second derivatives in absolute value are convex. One can see [2, 8, 28] for results associated with these types of inequalities including twice-differentiable functions.

Now, we recall the basic definitions and new notations of tempered fractional operators.

Definition 1.2. *The incomplete gamma function and λ-incomplete gamma function are defined by*

$$\curlyvee(\alpha, x) := \int_0^x \mu^{\alpha-1} e^{-\mu} d\mu$$

and

$$\curlyvee_\lambda(\alpha, x) := \int_0^x \mu^{\alpha-1} e^{-\lambda\mu} d\mu,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Remark 1.3 (See [19]). *For the real numbers $\alpha > 0$; $x, \lambda \geq 0$ and $\sigma < \kappa$, we have*

1. $\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha, 1) = \int_0^1 \mu^{\alpha-1} e^{-\lambda\left(\frac{\kappa-\sigma}{2}\right)\mu} d\mu = \left(\frac{2}{\kappa-\sigma}\right)^\alpha \curlyvee_\lambda(\alpha, \kappa-\sigma)$,
2. $\int_0^1 \curlyvee_{\lambda(\kappa-\sigma)}(\alpha, x)\, dx = \frac{\curlyvee_\lambda(\alpha,\kappa-\sigma)}{(\kappa-\sigma)^\alpha} - \frac{\curlyvee_\lambda(\alpha+1,\kappa-\sigma)}{(\kappa-\sigma)^{\alpha+1}}$.

Definition 1.4 (See [15, 17]). *The fractional tempered integral operators $\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}$ and $\mathcal{J}_{\kappa-}^{(\sigma,\lambda)}\mathcal{F}$ of order $\alpha > 0$ and $\lambda \geq 0$ are given by*

$$\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_\sigma^x (x-\mu)^{\alpha-1} e^{-\lambda(x-\mu)} \mathcal{F}(\mu) d\mu, \quad x \in [\sigma, \kappa] \tag{3}$$

and

$$\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^\kappa (\mu-x)^{\alpha-1} e^{-\lambda(\mu-x)} \mathcal{F}(\mu) d\mu, \quad x \in [\sigma, \kappa], \tag{4}$$

respectively for $\mathcal{F} \in L_1[\sigma, \kappa]$.

Note that if we choose $\lambda = 0$, then the fractional integrals in (3) and (4) become to the Riemann-Liouville fractional integral in (1) and (2), respectively.

It is well-known that tempered fractional calculus is an extension of fractional calculus. In [5], the definitions of fractional integration with exponential kernels and weak singular were firstly reported in

Buschman's earlier work. See the books [16, 23, 26] and references therein for the other different definitions of the tempered fractional integration. In paper [19], Mohammed et al. are proved several Hermite–Hadamard-type (including both trapezoidal and midpoint type) associated with tempered fractional integrals for the case of convex functions which cover the previously published results such as Riemann integrals, Riemann-Liouville fractional integrals.

Building upon the ongoing research and aforementioned papers, we will establish several Hermite–Hadamard-type inequalities via twice-differentiable convex mappings including tempered fractional integral operators. The entire form of study takes the form of four sections including the introduction. Here, the basic definitions of Riemann-Liouville integral operators and tempered fractional integrals are explained in order to build our main results. In Section 2, we prove some new version of trapezoid-type inequalities via twice-differentiable convex functions with the help of tempered fractional integrals. More precisely, Hölder and power-mean inequalities, which are well-known in the literature, will use in some of the proven inequalities. In Section 3, we present some new version of midpoint-type inequalities by twice-differentiable convex functions arising from tempered fractional integrals. Furthermore, we also present some corollaries and remarks. Finally, in Section 4, interested researchers will be informed that new versions of the inequalities we have acquired can be derived via different fractional integrals.

2. Trapezoid-type inequalities by tempered fractional integrals

In this section, we use tempered fractional operators to construct trapezoid-type inequalities for twice-differentiable convex mappings. Now, let us set up the following identity in order to prove trapezoid-type inequalities.

Lemma 2.1. *Note that $\mathcal{F}:[\sigma,\kappa]\rightarrow\mathbb{R}$ is a twice-differentiable function on (σ,κ) so that $\mathcal{F}''\in L_1[\sigma,\kappa]$. Then, it follows*

$$\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right] \tag{5}$$
$$=\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\int_0^1 E_{\alpha}(\lambda,\mu)\left[\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)+\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right]d\mu.$$

Here,

$$E_{\alpha}(\lambda,\mu)=\int_{\mu}^{1}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,u)\,du$$
$$=\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)-\mu\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)+\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,\mu).$$

Proof. By employing the integration by parts, it yields

$$I_1=\int_0^1 E_{\alpha}(\lambda,\mu)\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu \tag{6}$$
$$=\frac{2}{\kappa-\sigma}E_{\alpha}(\lambda,\mu)\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\Big|_0^1$$
$$+\frac{2}{\kappa-\sigma}\int_0^1\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu$$

$$
\begin{aligned}
&= -\frac{2}{\kappa-\sigma}\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\right]\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad+\frac{2}{\kappa-\sigma}\left[\frac{2}{\kappa-\sigma}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\Bigg|_0^1\right.\\
&\quad\left.-\frac{2}{\kappa-\sigma}\int_0^1\mu^{\alpha-1}e^{-\lambda\frac{\kappa-\sigma}{2}\mu}\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu\right]\\
&= -\frac{2}{\kappa-\sigma}\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\right]\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad+\frac{4}{(\kappa-\sigma)^2}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\mathcal{F}(\kappa)-\left(\frac{2}{\kappa-\sigma}\right)^{\alpha+2}\Gamma(\alpha)J_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right).
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
I_2 &= \int_0^1 E_\alpha(\lambda,\mu)\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)d\mu \qquad (7)\\
&= \frac{2}{\kappa-\sigma}\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\right]\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad+\frac{4}{(\kappa-\sigma)^2}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\mathcal{F}(\sigma)-\left(\frac{2}{\kappa-\sigma}\right)^{\alpha+2}\Gamma(\alpha)J_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right).
\end{aligned}
$$

From (6) and (7), we have

$$
\begin{aligned}
&\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}[I_1+I_2]\\
&=\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right],
\end{aligned}
$$

which concluded the proof of Lemma 2.1. □

Theorem 2.2. *Suppose that $\mathcal{F}:[\sigma,\kappa]\to\mathbb{R}$ is a twice-differentiable functions on (σ,κ) and $|\mathcal{F}''|$ is convex on $[\sigma,\kappa]$. Under these conditions, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|\\
&\le\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\varphi_1(\alpha,\lambda)\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].
\end{aligned}
$$

Here,

$$
\begin{aligned}
\varphi_1(\alpha,\lambda) &= \int_0^1\left|E_\alpha(\lambda,\mu)\right|d\mu \qquad (8)\\
&= \int_0^1\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)-\mu\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)+\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,\mu)\right]d\mu.
\end{aligned}
$$

Proof. Taking the absolute value of both sides of (5), it yields

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right| \qquad (9)\\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\int_0^1\left|E_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\right|d\mu\\
&\quad+\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\int_0^1\left|E_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right|d\mu.
\end{aligned}$$

The fact that $|\mathcal{F}''|$ is convex on $[\sigma,\kappa]$ yields

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|\\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\int_0^1 E_\alpha(\lambda,\mu)\left[\frac{1-\mu}{2}|\mathcal{F}''(\kappa)|+\frac{1+\mu}{2}|\mathcal{F}''(\sigma)|+\frac{1-\mu}{2}|\mathcal{F}''(\sigma)|+\frac{1+\mu}{2}|\mathcal{F}''(\kappa)|\right]d\mu\\
&=\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1 E_\alpha(\lambda,\mu)\,d\mu\right)\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].
\end{aligned}$$

This ends the proof of Theorem 2.2. □

Remark 2.3. *If we assign $\lambda=0$ in Theorem 2.2, then we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^{\alpha}}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|\\
&\leq\frac{(\kappa-\sigma)^2}{8(\alpha+2)}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right],
\end{aligned}$$

which is established by Budak et al. in [3, Corollary 3.6].

Theorem 2.4. *Let $\mathcal{F}:[\sigma,\kappa]\to\mathbb{R}$ be a twice-differentiable function on (σ,κ) such that $\mathcal{F}''\in L_1[\sigma,\kappa]$ and let $|\mathcal{F}''|^q$ be convex on $[\sigma,\kappa]$ with $q>1$. Then, the following double inequality holds:*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|\\
&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\psi_1^{\alpha}(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(4\psi_1^{\alpha}(\lambda,p)\right)^{\frac{1}{p}}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi_1^{\alpha}(\lambda,p)=\int_0^1\left(E_\alpha(\lambda,\mu)\right)^p d\mu.$$

Proof. Let us consider Hölder's inequality in (9). Then, we have

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right| \\
&\le \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1\left|E_\alpha(\lambda,\mu)\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\quad+\frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1\left|E_\alpha(\lambda,\mu)\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

If we apply the convexity of $|\mathcal{F}''|^q$ on $[\sigma,\kappa]$, then we have the following inequality

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right| \\
&\le \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1\left(E_\alpha(\lambda,\mu)\right)^p d\mu\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&= \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\psi_1^{\alpha}(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The second inequality of Theorem 2.4 can be acquired immediately by letting $\vartheta_1 = |\mathcal{F}''(\sigma)|^q$, $\varrho_1 = 3|\mathcal{F}''(\kappa)|^q$, $\vartheta_2 = 3|\mathcal{F}''(\sigma)|^q$ and $\varrho_2 = |\mathcal{F}''(\kappa)|^q$ and applying the inequality:

$$\sum_{k=1}^{n}(\vartheta_k+\varrho_k)^s \le \sum_{k=1}^{n}\vartheta_k^s+\sum_{k=1}^{n}\varrho_k^s, \quad 0\le s<1.$$

Thus, the proof of Theorem 2.4 is completed. □

Corollary 2.5. *If Theorem 2.4 is evaluated as $\lambda = 0$, then the following result is obtained:*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^{\alpha}}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right| \\
&\le \frac{(\kappa-\sigma)^2}{8}\left(\frac{1}{\alpha+1}\right)^{1+\frac{1}{p}}\left[\mathscr{B}\left(p+1,\frac{1}{\alpha+1}\right)\right]^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{|\mathcal{F}''(\kappa)|^q+3|\mathcal{F}''(\sigma)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{(\kappa-\sigma)^2}{8}\left(\frac{1}{\alpha+1}\right)^{1+\frac{1}{p}}\left[4\mathscr{B}\left(p+1,\frac{1}{\alpha+1}\right)\right]^{\frac{1}{p}}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].
\end{aligned}$$

Here, $\mathscr{B}(\cdot,\cdot)$ is a beta function defined as

$$\mathscr{B}(x,y) := \int_0^1 \mu^{x-1}(1-\mu)^{y-1}d\mu, \quad x,y\in\mathbb{R}^+.$$

Theorem 2.6. *Assume that $\mathcal{F} : [\sigma, \kappa] \to \mathbb{R}$ is a twice-differentiable function on (σ, κ) so that $\mathcal{F}'' \in L_1[\sigma, \kappa]$ and assume also that $|\mathcal{F}''|^q$ is convex on $[\sigma, \kappa]$ with $q \geq 1$. Then, it follows*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha} \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right) + \mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|$$
$$\leq \frac{(\kappa-\sigma)^2}{8 \curlyvee_{\lambda}(\alpha,1)} (\varphi_1(\alpha,\lambda))^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}''(\kappa)|^q + \frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}''(\sigma)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}''(\kappa)|^q + \frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}''(\sigma)|^q\right)^{\frac{1}{q}}\right].$$

Here, $\varphi_1(\alpha, \lambda)$ is described as in (8) *and*

$$\varphi_2(\alpha,\lambda) = \int_0^1 \mu\left|E_\alpha(\lambda,\mu)\right| d\mu$$
$$= \int_0^1 \mu\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1) - \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1) - \mu \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu) + \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,\mu)\right] d\mu.$$

Proof. By using the power-mean inequality, we get

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha} \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right) + \mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|$$
$$\leq \frac{(\kappa-\sigma)^2}{8 \curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1 \left|E_\alpha(\lambda,\mu)\right| d\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1 \left|E_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \left|E_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

Since $|\mathcal{F}''|^q$ is convex on $[\sigma, \kappa]$, we obtain

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha} \curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right) + \mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|$$
$$\leq \frac{(\kappa-\sigma)^2}{8 \curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1 E_\alpha(\lambda,\mu) d\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1 E_\alpha(\lambda,\mu)\left(\frac{1+\mu}{2}|\mathcal{F}''(\kappa)|^q + \frac{1-\mu}{2}|\mathcal{F}''(\sigma)|^q\right) d\mu\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 E_\alpha(\lambda,\mu)\left(\frac{1+\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathcal{F}''(\kappa)\right|^q\right)d\mu\right)^{\frac{1}{q}}\Bigg].$$

□

Remark 2.7. *Let us consider $\lambda = 0$ in Theorem 2.6. Then, the following inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^\alpha}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]\right|$$
$$\leq\frac{(\kappa-\sigma)^2}{8(\alpha+2)}\left[\left(\frac{(3\alpha+8)\left|\mathcal{F}''(\kappa)\right|^q+(\alpha+4)\left|\mathcal{F}''(\sigma)\right|^q}{4(\alpha+3)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\alpha+4)\left|\mathcal{F}''(\kappa)\right|^q+(3\alpha+8)\left|\mathcal{F}''(\sigma)\right|^q}{4(\alpha+3)}\right)^{\frac{1}{q}}\right],$$

which is established by Hezenci et al. in paper [9].

Remark 2.8. *If we select $\alpha = 1$ and $\lambda = 0$ in Theorem 2.6, then the following inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\kappa)}{2}-\frac{1}{\kappa-\sigma}\int_\sigma^\kappa \mathcal{F}(x)\,dx\right|$$
$$\leq\frac{(\kappa-\sigma)^2}{24}\left[\left(\frac{11\left|\mathcal{F}''(\kappa)\right|^q+5\left|\mathcal{F}''(\sigma)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{5\left|\mathcal{F}''(\kappa)\right|^q+11\left|\mathcal{F}''(\sigma)\right|^q}{16}\right)^{\frac{1}{q}}\right],$$

which is given in paper [25, Proposition 6].

3. Midpoint-type inequalities by tempered fractional integrals

In this section, we consider tempered fractional integrals to construct midpoint-type inequalities with the help of the twice-differentiable convex functions. First, let's set up the following equality to get midpoint-type inequalities.

Lemma 3.1. *Under the assumptions of Lemma 2.1, we have the following equality*

$$\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\,\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\tag{10}$$
$$=\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\int_0^1 F_\alpha(\lambda,\mu)\left[\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)+\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right]d\mu.$$

Here,

$$F_\alpha(\lambda,\mu)=\int_\mu^1\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,u)\right]du=(1-\mu)\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-E_\alpha(\lambda,\mu)$$
$$=\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,\mu)-\mu\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)+\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)\right].$$

Proof. With the help of the integration by parts, we obtain

$$
\begin{aligned}
I_3 &= \int_0^1 F_\alpha(\lambda,\mu)\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu \qquad (11)\\
&= \frac{2}{\kappa-\sigma}F_\alpha(\lambda,\mu)\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\Bigg|_0^1\\
&\quad -\frac{2}{\kappa-\sigma}\int_0^1\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\right]\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu\\
&= -\frac{2}{\kappa-\sigma}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad -\frac{2}{\kappa-\sigma}\Bigg[\frac{2}{\kappa-\sigma}\left[\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,\mu)-\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\right]\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\Bigg|_0^1\\
&\quad -\frac{2}{\kappa-\sigma}\int_0^1\mu^{\alpha-1}e^{-\lambda\frac{\kappa-\sigma}{2}\mu}\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)d\mu\Bigg]\\
&= -\frac{2}{\kappa-\sigma}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad -\frac{4}{(\kappa-\sigma)^2}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\left(\frac{2}{\kappa-\sigma}\right)^{\alpha+2}\Gamma(\alpha)J_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right).
\end{aligned}
$$

In a similar manner, we have

$$
\begin{aligned}
I_4 &= \int_0^1 F_\alpha(\lambda,\mu)\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)d\mu \qquad (12)\\
&= \frac{2}{\kappa-\sigma}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha+1,1)\mathcal{F}'\left(\frac{\sigma+\kappa}{2}\right)\\
&\quad -\frac{4}{(\kappa-\sigma)^2}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\left(\frac{2}{\kappa-\sigma}\right)^{\alpha+2}\Gamma(\alpha)J_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right).
\end{aligned}
$$

From (11) and (12), we get the following equality

$$
\begin{aligned}
&\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}[I_3+I_4]\\
&= \frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right).
\end{aligned}
$$

Finally, the proof of Lemma 3.1 is completed. □

Theorem 3.2. *Under the assumptions of Theorem 2.4, we have the following midpoint-type inequality*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|\\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\varphi_3(\alpha,\lambda)\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].
\end{aligned}
$$

Here,

$$\varphi_3(\alpha,\lambda)=\int_0^1\left((1-\mu)\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-E_\alpha(\lambda,\mu)\right)d\mu.$$

Proof. Let us take the absolute value of both sides of (10). Then, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right| \qquad (13)\\&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\int_0^1\left|F_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\right|d\mu\\&\quad+\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\int_0^1\left|F_\alpha(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right|d\mu.\end{aligned}$$

From the fact that $|\mathcal{F}''|$ is convex on $[\sigma,\kappa]$, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|\\&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\int_0^1F_\alpha(\lambda,\mu)\left(\frac{1-\mu}{2}|\mathcal{F}''(\kappa)|+\frac{1+\mu}{2}|\mathcal{F}''(\sigma)|+\frac{1-\mu}{2}|\mathcal{F}''(\sigma)|+\frac{1+\mu}{2}|\mathcal{F}''(\kappa)|\right)d\mu\\&=\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\left(\int_0^1F_\alpha(\lambda,\mu)\,d\mu\right)\left[|\mathcal{F}''(\kappa)|+|\mathcal{F}''(\sigma)|\right].\end{aligned}$$

Hence, the proof of Theorem 3.2 is completed. □

Remark 3.3. *If we assign $\lambda=0$ in Theorem 3.2, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^\alpha}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|\leq\frac{(\kappa-\sigma)^2\alpha}{16(\alpha+2)}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right],$$

which is given by Budak et al. in [3, Corollary 4.6].

Theorem 3.4. *Under the assumptions of Theorem 2.4, we obtain the following midpoint-type inequalities*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^\alpha\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|\\&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\left(\psi_2^\alpha(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\kappa)|^q+3|\mathcal{F}''(\sigma)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\kappa)|^q}{4}\right)^{\frac{1}{q}}\right]\\&\leq\frac{(\kappa-\sigma)^2}{8\curlyvee_\lambda(\alpha,1)}\left(4\psi_2^\alpha(\lambda,p)\right)^{\frac{1}{p}}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi_2^\alpha(\lambda,p)=\int_0^1\left((1-\mu)\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-E_\alpha(\lambda,\mu)\right)^p d\mu.$$

Proof. Let us consider Hölder's inequality in (13). Then, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq\frac{(\kappa-\sigma)^{2}}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_{0}^{1}\left|F_{\alpha}(\lambda,\mu)\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\right|^{q}d\mu\right)^{\frac{1}{q}}$$
$$+\frac{(\kappa-\sigma)^{2}}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_{0}^{1}\left|F_{\alpha}(\lambda,\mu)\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right|^{q}d\mu\right)^{\frac{1}{q}}.$$

By using the convexity of $|\mathcal{F}''|^{q}$ on $[\sigma,\kappa]$, we have the following

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq\frac{(\kappa-\sigma)^{2}}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_{0}^{1}(F_{\alpha}(\lambda,\mu))^{p}d\mu\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^{q}+3|\mathcal{F}''(\kappa)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^{q}+|\mathcal{F}''(\kappa)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\kappa-\sigma)^{2}}{8\curlyvee_{\lambda}(\alpha,1)}\left(\psi_{2}^{\alpha}(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\kappa)|^{q}+3|\mathcal{F}''(\sigma)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\sigma)|^{q}+3|\mathcal{F}''(\kappa)|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

The second inequality of Theorem 3.4 can be acquired immediately by letting $\vartheta_1=3|\mathcal{F}''(\sigma)|^{q}$, $\varrho_1=|\mathcal{F}''(\kappa)|^{q}$, $\vartheta_2=|\mathcal{F}''(\sigma)|^{q}$ and $\varrho_2=3|\mathcal{F}''(\kappa)|^{q}$ and applying the inequality:

$$\sum_{k=1}^{n}(\vartheta_k+\varrho_k)^{s}\leq\sum_{k=1}^{n}\vartheta_k^{s}+\sum_{k=1}^{n}\varrho_k^{s},\qquad 0\leq s<1.$$

□

Corollary 3.5. *If it is chosen $\lambda=0$ in Theorem 3.4, then the following result is obtained*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^{\alpha}}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq\frac{(\kappa-\sigma)^{2}\alpha}{8}\left(\psi_{2}^{p}(\alpha,0)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\kappa)|^{q}+3|\mathcal{F}''(\sigma)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\sigma)|^{q}+3|\mathcal{F}''(\kappa)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{(\kappa-\sigma)^{2}\alpha}{8}\left(4\psi_{2}^{p}(\alpha,0)\right)^{\frac{1}{p}}\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\kappa)|\right].$$

Remark 3.6. *If we assign $\alpha=1$ and $\lambda=0$ in Theorem 3.4, then the following inequalities hold:*

$$\left|\frac{1}{\kappa-\sigma}\int_{\sigma}^{\kappa}\mathcal{F}(x)\,dx-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq\frac{(\kappa-\sigma)^{2}}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\kappa)|^{q}+3|\mathcal{F}''(\sigma)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\sigma)|^{q}+3|\mathcal{F}''(\kappa)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\kappa-\sigma)^2}{16}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\kappa)\right|\right],$$

which is given in paper [3, Corollary 4.8].

Theorem 3.7. *Under the conditions of Theorem 2.6, the following midpoint-type inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right| \\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\varphi_3(\alpha,\lambda)\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\frac{(\varphi_3(\alpha,\lambda)+\varphi_4(\alpha,\lambda))}{2}\left|\mathcal{F}''(\kappa)\right|^q+\frac{(\varphi_3(\alpha,\lambda)-\varphi_4(\alpha,\lambda))}{2}\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{(\varphi_3(\alpha,\lambda)-\varphi_4(\alpha,\lambda))}{2}\left|\mathcal{F}''(\kappa)\right|^q+\frac{(\varphi_3(\alpha,\lambda)+\varphi_4(\alpha,\lambda))}{2}\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here, $\varphi_3(\alpha,\lambda)$ is defined as in (8) and

$$\varphi_4(\alpha,\lambda)=\int_0^1 \mu\left((1-\mu)\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)-E_{\alpha}(\lambda,\mu)\right)d\mu.$$

Proof. By using the power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right| \\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1\left|F_{\alpha}(\lambda,\mu)\right|d\mu\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|F_{\alpha}(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left|F_{\alpha}(\lambda,\mu)\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\kappa\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $\left|\mathcal{F}''\right|^q$ is convex on $[\sigma,\kappa]$, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(\kappa-\sigma)^{\alpha}\curlyvee_{\lambda\left(\frac{\kappa-\sigma}{2}\right)}(\alpha,1)}\left[\mathcal{J}_{\kappa-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right| \\
&\leq \frac{(\kappa-\sigma)^2}{8\curlyvee_{\lambda}(\alpha,1)}\left(\int_0^1 F_{\alpha}(\lambda,\mu)d\mu\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1 F_{\alpha}(\lambda,\mu)\left(\frac{1+\mu}{2}\left|\mathcal{F}''(\kappa)\right|^q+\frac{1-\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1 F_{\alpha}(\lambda,\mu)\left(\frac{1+\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathcal{F}''(\kappa)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Remark 3.8. *If $\lambda = 0$ in Theorem 3.7, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa-\sigma)^{\alpha}}\left[J_{\kappa-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq \frac{(\kappa-\sigma)^{2}\alpha}{16(\alpha+2)}\left[\left(\frac{(4\alpha+11)\left|\mathcal{F}''(\kappa)\right|^{q}+(2\alpha+7)\left|\mathcal{F}''(\sigma)\right|^{q}}{6(\alpha+3)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2\alpha+7)\left|\mathcal{F}''(\kappa)\right|^{q}+(4\alpha+11)\left|\mathcal{F}''(\sigma)\right|^{q}}{6(\alpha+3)}\right)^{\frac{1}{q}}\right],$$

which is given in paper [9].

Remark 3.9. *If we select $\alpha = 1$ and $\lambda = 0$ in Theorem 3.7, then the following inequality holds:*

$$\left|\frac{1}{\kappa-\sigma}\int_{\sigma}^{\kappa}\mathcal{F}(x)\,dx-\mathcal{F}\left(\frac{\sigma+\kappa}{2}\right)\right|$$
$$\leq \frac{(\kappa-\sigma)^{2}}{48}\left[\left(\frac{5\left|\mathcal{F}''(\kappa)\right|^{q}+3\left|\mathcal{F}''(\sigma)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{F}''(\kappa)\right|^{q}+5\left|\mathcal{F}''(\sigma)\right|^{q}}{8}\right)^{\frac{1}{q}}\right],$$

which is given in paper [25, Proposition 5].

4. Conclusions

In this paper, we prove trapezoid-type, and midpoint-type inequalities by making use of tempered fractional integrals. Convexity of the twice-differentiable functions, Hölder and power-mean inequalities are used in these inequalities. Moreover, special choices of the variables in the theorems, generalizations of several papers, and new results were found. In the future, the mathematicians may provide new inequalities of different fractional types related to these Hermite–Hadamard-type inequalities. Interested readers can also prove new inequalities by using different kinds of convexities. These inequalities created are new as far as we know and according to the literature review. These inequalities will inspire new studies in various fields of mathematics.

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