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Strong convergence of the balanced Euler method for neutral stochastic differential delay equations with Markovian switching

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Abstract. In this paper, the strong convergence of the balanced Euler method for neutral stochastic differential delay equations with Markovian switching (NSDDEs-MS) without the linear growth condition is concerned. We present the balanced Euler method of NSDDEs-MS and consider its moment boundedness under polynomial growth condition plus Khasminskii-type condition. We also study its strong convergence order. A numerical example is given to support the theoretical results.

1. Introduction

As we know, stochastic differential equations (SDEs) and stochastic differential delay equations (SDDEs) are used widely and efficiently to model the phenomena in dynamical systems recently. In order to model the practical systems (see in [4, 15] and the references therein) which may be change abruptly due to environmental disturbances etc, we use the continuous-time Markovian chain $r(t)$ (see in [3]) to model such abrupt changes.

Neutral stochastic differential delay equations (NSDDEs), as a special case of SDDEs, get more and more attention. In this paper, we consider the following NSDDE with Markovian switching (MS), abbreviated as NSDDE-MS

$$d[X(t) - D(X(t-\tau), r(t))] = F(X(t), X(t-\tau), r(t))\,dt + G(X(t), X(t-\tau), r(t))\,dw(t).$$

Analytic theories of SDDEs-Ms (see [10] and references there in) are investigated extensively. Due to most SDEs-MS and SDDEs-MS have no explicit solutions generally, plenty of attention has been focused on numerical methods to approximate the analytic solutions (see e.g.,[5, 6, 8, 17, 18, 21, 22]). Stability of SDDEs-Ms and NSDDEs-MS were studied (see [7, 16, 23]). Strong convergence theory of numerical methods which is regarded as a cruel property was generally considered under the global Lipschitz condition or the local Lipschitz condition and the linear growth condition (see [19] and references there in). It is known that Euler method of SDEs (even without delay or Markovian switching) can not converge with the super-linearly growing drift coefficients (see [1]). Later, Hutzethaler et al. developed the tamed Euler method which can solve such problem (see [2]). Mao proposed the truncated Euler-Maruyama method for SDEs and

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its convergence rate in 2015 and 2016, respectively (see [12, 13]). In 2017, Zhang developed the balanced Euler method for SDEs with coefficients of super-linearly growth (see [24]). Recently, the truncated Euler-Maruyama method and other explicit numerical methods are studied by many authors (see [14],[20] and the references cited therein). However, numerical theories on NSDDEs-MS under polynomial growth condition and Khasminskii-type condition are still limited.

Due to cheap computational cost, explicit numerical methods are indeed to be considered. Inspired by the fact that the balanced methods for SDEs with coefficients of superlinearly growth satisfying a global monotone condition is of order half in the mean-square sense in [24], and in [21] I considered the truncated Euler-Maruyama method of NSDDEs which is of order closely to 1/2. In order to obtain a higher order explicit numerical method, we present the balanced Euler method of NSDDEs-MS under polynomial growth condition and Khasminskii-type condition. For the convenience, we will, in Section 3, present the balanced Euler method. We will consider the moment boundedness, one-step error and the convergence order of the balanced Euler method in Section 4. A numerical example is shown in Section 5.

2. Preliminary

Throughout this paper, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Assume $\mathbb{E}$ is the expectation corresponding to $\mathbb{P}$. Let $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Both Euclidean norm in $\mathbb{R}^n$ and the trace norm in $\mathbb{R}^{n\times m}$ are denoted by $|\cdot|$. Denote by $C([-\tau,0],\mathbb{R}^n)$ the family of continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $|\varphi| = \sup_{-\tau\leq u\leq 0} |\varphi(u)|$. For $a, b \in \mathbb{R}$, we use $a \vee b$ is used for $\max\{a, b\}$ and $a \wedge b$ is used for and $\min\{a, b\}$. If D is a set of Ω, let $\mathbf{1}_D$ be its indicator function. $\lfloor x \rfloor$ denotes the biggest integer which is not bigger than x.

A right-continuous Markovian chain on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is denoted by $r(t)$ $(t \geq 0)$. Taking values in a finite state space $S = \{1, 2, \cdots, Q\}$ with $\Gamma = (\gamma_{ij})_{Q\times Q}$, we define

$$\mathbb{P}\{(r(t+h)) = j|r(t) = i\} = \begin{cases} \gamma_{ij}h + o(h), & if\ \ i \neq j, \\ 1 + \gamma_{ij}h + o(h), & if\ \ i = j, \end{cases} \tag{1}$$

where $h > 0$. Here $\gamma_{ij} \geq 0$ is the transition from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

Assume that the Markovian chain $r(t)$ and the Brownian motion $w(t)$ are independent. Almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$. Moreover, there exists a sequence of stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ almost surely such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k)\mathbf{1}_{[\tau_k,\tau_{k+1})}(t),$$

$r(t)$ is a constant on every interval $[\tau_k, \tau_{k+1})$, for every $k \geq 0$

$$r(t) = r(\tau_k),\ \ on\ \tau_k \leq t < \tau_{k+1}.$$

Consider an n-dimensional NSDDE-MS

$$d[X(t) - D(X(t-\tau), r(t))] = F(X(t), X(t-\tau), r(t))\,dt + G(X(t), X(t-\tau), r(t))\,dw(t) \tag{2}$$

for $t \geq 0$ with initial data $\xi \in C([-\tau,0],\mathbb{R}^n)$ and $\mathbb{E}|\xi|^{2p_0} = \mathbb{E}\left(\sup_{-\tau\leq u\leq 0} |\xi(u)|^{2p_0}\right) < \infty$ which is independent of $w(t)$, $p_0 \geq 1$ and $r(0) = r_0 \in S$. Here $\tau > 0$, $D : \mathbb{R}^n \times S \longrightarrow \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^{n\times m}$.

We impose the following four assumptions as hypotheses:

(**A1**) There is a pair of constants $\gamma \geq 1$ and $K_1 > 0$ such that

$$|F(x, y, i) - F(\bar{x}, \bar{y}, i)| \vee |G(x, y, i) - G(\bar{x}, \bar{y}, i)|$$

$$\leq \quad K_1\left(1+|x|^{\gamma-1}+|\bar{x}|^{\gamma-1}+|y|^{\gamma-1}+|\bar{y}|^{\gamma-1}\right)(|x-\bar{x}|+|x-\bar{y}|),$$

where x, y $\bar{x}$, y, $\bar{y} \in \mathbb{R}^n$ and $i \in S$.

Remark 2.1. *Due to* (**A1**)*, using the Young inequality and the elementary inequality*

$$|a+b|^{\varsigma} \leq (1+c)^{\varsigma-1}(|a|^{\varsigma}+c^{1-\varsigma}|b|^{\varsigma}), \tag{3}$$

where $\varsigma \geq 1$, $c > 0$, *for* $c = 1$, *we have*

$$|F(x,y,i)| \vee |G(x,y,i)| \leq \bar{K}_1\left(1+|x|^{\gamma}+|y|^{\gamma}\right), \tag{4}$$

where x, y, $\bar{x}$, $\bar{y} \in \mathbb{R}^n$ *and* $i \in S$, *where* $\bar{K}_1 = \dfrac{2K_1(\gamma-1)}{\gamma} \vee \dfrac{K_1(\gamma+2)}{\gamma} \vee |F(0,0,i)| \vee |G(0,0,i)|$.

(**A2**) There is a pair of constants $p_0 \geq 1$ and $K_2 > 0$ such that

$$[x-D(y,i)-\bar{x}+D(\bar{y},i)]^T\left[F(x,y,i)-F(\bar{x},\bar{y},i)\right]+\frac{2p_0-1}{2}|G(x,y,i)-G(\bar{x},\bar{y},i)|^2 \leq K_2\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right),$$

where x, $\bar{x}$, y, $\bar{y} \in \mathbb{R}^n$ and $i \in S$.

(**A3**) There is a constant $K_0 > 0$ such that

$$|\xi(t)-\xi(s)| \leq K_0|t-s|^{1/2},$$

where $s, t \in [-\tau, 0]$.

(**A4**) Assume that $D(0,i) = 0$ and there exists a positive $\kappa \in (0,1)$ such that

$$|D(x,i)-D(y,i)| \leq \kappa|x-y|, \tag{5}$$

where $x, y \in \mathbb{R}^n$ and $i \in S$.

From (5), we can see that

$$|D(x,i)| \leq \kappa|x|. \tag{6}$$

Remark 2.2. *Let* (**A2**), (**A4**) *hold with sufficiently large* p_0. *Let* $p_0 > \dfrac{3}{2}$, *applying the elementary inequality* (3) *for* $c = \dfrac{2}{2p_0-3}$, *one has*

$$\begin{aligned}
&[x-D(y,i)]^T F(x,y,i)+\frac{2p_0-3}{2}|G(x,y,i)|^2\\
\leq&[x-D(y,i)-0+D(0,i)]^T\left[F(x,y,i)-F(0,0,i)\right]+[x-D(y,i)-0+D(0,i)]^T F(0,0,i)\\
&+\frac{2p_0-3}{2}\left(\frac{2p_0-1}{2p_0-3}|G(x,y,i)-G(0,0,i)|^2+\frac{2p_0-1}{2}|G(0,0,i)|^2\right)\\
\leq&K_2\left(|x|^2+|y|^2\right)+\frac{1}{2}|x-D(y,i)-0+D(0,i)|^2+\frac{1}{2}|F(0,0,i)|^2+\frac{(2p_0-3)(2p_0-1)}{4}|G(0,0,i)|^2\\
\leq&K_2\left(|x|^2+|y|^2\right)+|x|^2+|D(y,i)|^2+\frac{1}{2}|F(0,0,i)|^2+\frac{(2p_0-3)(2p_0-1)}{4}|G(0,0,i)|^2\\
\leq&K_3\left(1+|x|^2+|y|^2\right),
\end{aligned} \tag{7}$$

where $K_3 = (K_2+1) \vee \left(\dfrac{1}{2}|F(0,0,i)|^2+\dfrac{(2p_0-3)(2p_0-1)}{4}|G(0,0,i)|^2\right)$, *for all* x, $\bar{x}$, y, $\bar{y} \in \mathbb{R}^n$ *and* $i \in S$.

Using this approach to get (7), *we need* $p_0 > \dfrac{3}{2}$, *but which is not necessary.*

Let $X_{t,\phi}(t+h)$ be the solution of NSDDE-MS (2). Define

$$\begin{aligned} X_{t,\phi}(t+h) &= X_{t,\phi}(t) + D(X_{t,\phi}(t+h-\tau), r(t+h)) - D(X_{t,\phi}(t-\tau), r(t)) \\ &\quad + \int_t^{t+h} F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s))ds + \int_t^{t+h} G(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s))dw(s) \\ &= \phi(t) + D(X_{t,\phi}(t+h-\tau), r(t+h)) - D(X_{t,\phi}(t-\tau), r(t)) \\ &\quad + \int_t^{t+h} F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s))ds + \int_t^{t+h} G(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s))dw(s), \end{aligned} \tag{8}$$

where $\phi = \{\phi(s) : t-\tau \le s \le t\}$, $\mathbb{E}|\phi|^{2p_0} = \mathbb{E}\left(\sup_{-\tau\le u\le 0} |\phi(u)|^{2p_0}\right) < \infty$ and $t \in [0,T]$. It is easy to show the following lemma by the similar way of Theorem 3.1 in [9] and Theorem 2.4 in [11].

Lemma 2.3. *Assume that* (**A1**), (**A4**) *and* (7) *hold with sufficiently large p_0. For $1 \le p \le p_0 - 1$, NSDDE-MS* (2) *has a unique global solution $X_{0,\xi}(t)$ and, moreover,*

$$\mathbb{E}|X_{0,\xi}(t)|^{2p} \le K\left(1 + \mathbb{E}|\xi|^{2p}\right)$$

for all $t \in [0,T]$, where K stands for a generic positive real constant (*but independent of h later*) *and its value may change between occurrences.*

Remark 2.4. *Assume that* (**A1**), (**A4**) *and* (7) *hold with sufficiently large p_0. Due to the flow property of NSDDE-MS* (2)*, for $1 \le p \le p_0 - 1$, NSDDE-MS* (2) *has a unique global solution $X_{t,\phi}(s) = X_{0,\xi}(s)$ and, moreover,*

$$\mathbb{E}|X_{t,\phi}(s)|^{2p} \le K\left(1 + \mathbb{E}|\phi|^{2p}\right), \quad \forall s \in [t,T],$$

where $\phi = \{\phi(s) = X_{0,\xi}(s) : t-\tau \le s \le t\}$, $\mathbb{E}|\phi|^{p_0} = \mathbb{E}\left(\sup_{-\tau\le u\le 0} |\phi(u)|^{p_0}\right) < \infty$.

3. The balanced Euler method

Lemma 3.1. *For $h > 0$ and $n = 0,1,2,\cdots$, let $r_n = r(nh)$, then $\{r_n, n = 0,1,2,\cdots\}$ is discrete Markovian chain with the one-step transition probability matrix*

$$\mathbb{P}(h) = \left(\mathbb{P}_{ij}(h)\right)_{Q\times Q} = e^{h\Gamma}.$$

Due to the the independence of γ_{ij} and x, the paths of r can be generated independently of x before computing x.

Let the step size $h \in (0,1)$, $\tau/h = M$ where $M \in \mathbb{N}$. The discrete Markovian chain $\{r_n, n = 0,1,2,\cdots\}$ are simulated as follows: a random number η_1 is generated by computing the one-step transition probability matrix $\mathbb{P}(h)$ and $r_0 = i_0$, which is uniformly distributed in $[0,1]$. Define

$$r_1 = \begin{cases} i_1, \ if \ \ i_1 \in S - \{Q\} \ \ such \ \ that \ \ \sum_{j=1}^{i_1-1} \mathbb{P}_{i_0 j}(h) \le \eta_1 < \sum_{j=1}^{i_1} \mathbb{P}_{i_0 j}(h), \\ Q, \ if \ \ \sum_{j=1}^{Q-1} \mathbb{P}_{i_0 j}(h) \le \eta_1, \end{cases}$$

where $\sum_{j=1}^{0} \mathbb{P}_{i_0 j}(h) = 0$. A new random number η_2 which is uniformly distributed in $[0,1]$ is generated independently and define

$$r_2 = \begin{cases} i_2, \ if \ \ i_2 \in S - \{Q\} \ such \ \ that \ \ \sum_{j=1}^{i_2-1} \mathbb{P}_{i_1 j}(h) \le \eta_2 < \sum_{j=1}^{i_2} \mathbb{P}_{i_1 j}(h), \\ Q, \ if \ \ \sum_{j=1}^{Q-1} \mathbb{P}_{i_1 j}(h) \le \eta_2. \end{cases}$$

Repeating this procedure, the trajectory $\{r_n, n = 0, 1, 2, \cdots\}$ can be generated. The procedure can also be carried out independently to get more trajectories.

Now we will concern NSDDE-MS (2). The discrete-time balanced Euler numerical solutions can be formed. From now on, let the step size $h \in (0,1)$, $T = Nh$, $t_n = nh$, $n = 1, 2, \cdots, N$ and $N \in \mathbb{N}^+$. Motivated by [24], for the solution $X_{t,\phi}(t+h)$, which depends on the initial data $\phi = \{\phi(s) : t - \tau \le s \le t\}$ and $\Delta w^j := \Delta w(t) := w(t+h) - w(t)$, the one-step approximation $\bar{Y}_{t,\phi}(t+h)$ is defined as follows

$$\begin{aligned} \bar{Y}_{t,\phi}(t+h) =& X_{t,\phi}(t) + D(X_{t,\phi}(t+h-\tau), r(t+h)) - D(X_{t,\phi}(t-\tau), r(t)) \\ &+ \Upsilon(F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))h) + \Upsilon\left(G(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))\Delta w\right). \end{aligned}$$

We propose the following balanced Euler scheme

$$\begin{cases} Y(t_{n+1}) =\bar{Y}_{0,\xi}(t_{n+1}) = X_{0,\xi}(t_{n+1}) = \xi(t_{n+1}), \ \ n = -M-1, -M, \cdots, -1, \\ Y(t_{n+1}) =\bar{Y}_{t_n,\psi}(t_{n+1}) = Y(t_n) + D(Y(t_{n+1-M}), r_n) - D(Y(t_{n-M}), r_n) \\ \qquad + \Upsilon(F(Y(t_n), Y(t_{n-M}), r_n)h) + \Upsilon\left(G(Y(t_n), Y(t_{n-M}), r_n)\Delta w_n\right), \\ \qquad\qquad n = 0, 1, \cdots, N-1 \end{cases}$$

where $\psi = \{\psi(t_k) = Y(t_k) = \bar{Y}_{0,\xi}(t_k) : n - M \le k \le n\}$, $\Upsilon(\cdot)$ is either the hyperbolic tangent function or the sine function and $\Delta w_n = w(t_{n+1}) - w(t_n)$.

Remark 3.2. *Because of the similarity between the proofs of $\Upsilon(\cdot) = \sin(\cdot)$ and $\Upsilon(\cdot) = \tanh(\cdot)$, in the following section, we only consider the following balanced Euler scheme*

$$\begin{cases} Y(t_{n+1}) =\xi(t_{n+1}), \ \ n = -M-1, -M, \cdots, -1, \\ Y(t_{n+1}) =Y(t_n) + D(Y(t_{n+1-M}), r_n) - D(Y(t_{n-M}), r_n) \\ \qquad + \sin(F(Y(t_{n+1}), Y(t_{n-M}), r_n)h) + \sin\left(G(Y(t_{n+1}), Y(t_{n-M}), r_n)\Delta w_n\right), \\ \qquad\qquad n = 0, 1, \cdots, N-1. \end{cases} \tag{9}$$

4. Convergence of the balanced Euler method

4.1. Moment boundedness of the balanced Euler method

Using a stopping time technique, we show the Moment boundedness of the balanced Euler method (9).

Lemma 4.1. *Assume* (4), (6) *and* (7) *hold with sufficiently large p_0. There exists $\beta \ge 1 + \frac{(2p+1)G(\gamma)}{2p}$, we have*

$$\mathbb{E}|Y(t_u)|^{2p} \quad \le \quad K\left(1 + \mathbb{E}|\xi|^{2p\beta}\right), n = 0, 1, \cdots, N,$$

where $1 \le p \le \left(\frac{p_0 - 1}{2(2\gamma - 1)} - \frac{1}{2}\right) \wedge \left(\frac{p_0 - 1}{12(\gamma - 1)} - \frac{1}{2}\right)$ and $G(\gamma) = \max\{2\gamma - 1, \chi_{p>1} 6(\gamma - 1)\}$.

Proof. We only concern the case $\gamma > 1$ because the case of $\gamma = 1$ can be proved similarly. Define

$$\tilde{\Omega}_{R,n} := \{\omega : |Y(t_k)| \leq R(h), k = 0, 1, \cdots, n\},$$

where $R^\gamma(h) < 1/h$.

Let $\tilde{\Lambda}_{R,n}$ be the compliments of $\tilde{\Omega}_{R,n}$.

Define

$$V_{n+1} := Y(t_{n+1}) - D(Y(t_{n+1-M}), r_{n+1}). \tag{10}$$

By the elementary inequality and (6), one has

$$\begin{aligned} |V_{n+1}| \leq & |V_n| + 2 \\ \leq & |\xi(0) - D(\xi(-\tau), r(0))| + 2(n+1) \\ \leq & (1+\kappa)|\xi| + 2(n+1), \quad n = 0, 1, \cdots, N-1. \end{aligned} \tag{11}$$

Using the elementary inequality, (6), (10) and (11), we obtain

$$\begin{aligned} |Y(t_{n+1})| \leq & |V_{n+1}| + |D(Y(t_{n+1-M}), r(t_{n+1}))| \\ \leq & (1+\kappa)|\xi| + 2(n+1) + \kappa|Y(t_{n+1-M})| \\ \leq & (1+\kappa)|\xi| + 2(n+1) + \kappa \sup_{-M \leq u \leq n+1} |Y(t_u)| \\ \leq & (1+\kappa)|\xi| + 2(n+1) + \kappa \sup_{0 \leq u \leq n+1} |Y(t_u)| + \kappa|\xi|. \end{aligned}$$

Hence, one has

$$\begin{aligned} \sup_{0 \leq u \leq n+1} |Y(t_u)| \leq & (1+\kappa)|\xi| + 2(n+1) + \kappa \sup_{0 \leq u \leq n+1} |Y(t_u)| + \kappa|\xi| \\ \leq & \frac{1+2\kappa}{1-\kappa}|\xi| + \frac{2(n+1)}{1-\kappa}, \quad n = 0, 1, \cdots, N-1. \end{aligned}$$

Consequently, we can show

$$|Y(t_{n+1})| \leq \frac{1+2\kappa}{1-\kappa}|\xi| + \frac{2(n+1)}{1-\kappa}, \quad n = 0, 1, \cdots, N-1. \tag{12}$$

For any integer $p \geq 1$, we get

$$\begin{aligned} \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] & \leq \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_{n+1}|^{2p}\right] \\ & = \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|\left(V_{n+1} - V_n\right) + V_n|^{2p}\right] \\ & = \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p-2}A\right] \\ & \quad + K\sum_{l=3}^{2p} \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p-l}|V_{n+1} - V_n|^l\right], \end{aligned} \tag{13}$$

where $A = \chi_{\tilde{\Omega}_{R,n}} E[2p\langle V_n, V_{n+1} - V_n\rangle + p(2p-1)|V_{n+1} - V_n|^2|\mathcal{F}_{t_n}]$.

Since Δw_n are independent of $\mathcal{F}_{t_n}$, one obtains

$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[G(Y(t_n), Y(t_{n-M}), r_n)\Delta w_n \middle| \mathcal{F}_{t_n}\right] = 0,$$

$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\sin\left(G(Y(t_n), Y(t_{n-M}), r_n)\Delta w_n\right) \middle| \mathcal{F}_{t_n}\right] = 0$$

and

$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[|G(Y(t_n),Y(t_{n-M}),r_n)\Delta w_n|^2\middle|\mathcal{F}_{t_n}\right]=\chi_{\tilde{\Omega}_{R,n}}(\omega)\,|G(Y(t_n),Y(t_{n-M}),r_n)|^2\,h.$$

Using $|\sin x|\le|x|$, and then we can have the following estimate

$$|x-\sin x|=|(1-\cos(\theta x))x|\le 2|x|\left|\sin\left(\frac{\theta x}{2}\right)\right|^2,\ \ for\ \ some\ \ \theta\in[0,1]. \tag{14}$$

By (3), (9),(14) and the elementary inequality, one derives

$$\begin{aligned}
A &= 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\langle V_n,V_{n+1}-V_n\rangle+\frac{2p-1}{2}|V_{n+1}-V_n|^2\middle|\mathcal{F}_{t_n}\right]\\
&= 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\left(\langle V_n,F(Y(t_n),Y(t_{n-M}),r_n)h\rangle+\frac{2p_0-3}{2}\left|\sin\left(G(Y(t_n),Y(t_{n-M}),r_n)\Delta w_n\right)\right|^2\right)\right.\\
&\qquad\left.+\frac{(2p-1)(2p_0-3)}{4(p_0-p-1)}\left|\sin(F(Y(t_n),Y(t_{n-M}),r_n)h)\right|^2\middle|\mathcal{F}_{t_n}\right]\\
&\quad+2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\langle V_n,-F(Y(t_n),Y(t_{n-M}),r_n)h+\sin(F(Y(t_n),Y(t_{n-M}),r_n)h)\rangle\middle|\mathcal{F}_{t_n}\right]\\
&\le 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\langle V(t_n),F(Y(t_n),Y(t_{n-M}),r_n)\rangle h+\frac{2p_0-3}{2}\left|G(Y(t_n),Y(t_{n-M}),r_n)\Delta w_n\right|^2\middle|\mathcal{F}_{t_n}\right]\\
&\quad+\frac{p(2p-1)(2p-3)}{2(p_0-p-1)}\chi_{\tilde{\Omega}_{R,n}}(\omega)\,|F(Y(t_n),Y(t_{n-M}),r_n)|^2\,h^2\\
&\quad+4p\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|\,|F(Y(t_n),Y(t_{n-M}),r_n)|^2\,h^2.
\end{aligned}$$

Using (10), (6), (4) and the Young inequality, one obtains

$$\begin{aligned}
&\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|\,|F(Y(t_n),Y(t_{n-M}),r_n)|^2\,h^2\\
\le&\bar{K}_1^2h^2\chi_{\tilde{\Omega}_{R,n}}(\omega)\left(|Y(t_n|+\kappa|Y(t_{n-M}|\right)\left(1+|Y(t_n|^\gamma+\kappa|Y(t_{n-M}|^\gamma\right)^2\\
\le&Kh^2\chi_{\tilde{\Omega}_{R,n}}(\omega)\left(1+|Y(t_n)|^2+|Y(t_{n-M})|^2+|Y(t_n)|^{2\gamma+1}+|Y(t_{n-M})|^{2\gamma+1}\right).
\end{aligned}\tag{15}$$

Applying (4), (7) and (15), we have

$$A\le K\chi_{\tilde{\Omega}_{R,n}}(\omega)h\left(1+|Y(t_n)|^2+|Y(t_{n-M})|^2+|Y(t_n)|^{2\gamma+1}h+|Y(t_{n-M})|^{2\gamma+1}h\right). \tag{16}$$

By (9), the elementary inequality and (4), one gets

$$\begin{aligned}
&\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p-l}|V_{n+1}-V_n|^l\right]\\
\le\ &K\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p-l}\left(h^l|F(Y(t_n),Y(t_{n-M}),r_n)|^l+|G(Y(t_n),Y(t_{n-M}),r_n)|^l\,|\Delta w_n|^l\right)\right]\\
\le\ &2^{2p-2}K(1+\kappa)\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p-l}h^{l/2}\left(1+|Y(t_n)|^{l\gamma}+|Y(t_{n-M})|^{l\gamma}\right)\right].
\end{aligned}\tag{17}$$

Substituting (16) and (17) into (13), using the elementary inequality and the Young inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \le\ &\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V(t_n)|^{2p}\right]+Kh\sum_{l=3}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p+l(\gamma-1)}h^{l/2-1}\right]\\
&+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p}\right]+K\sum_{l=2}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V(t_n)|^{2p-l}h^{l/2}\right]
\end{aligned}$$

$$
\begin{aligned}
&+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p+2\gamma-1}h\right] \\
&+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V(t_n)|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p+2\gamma-1}h\right] \\
&+Kh\sum_{l=3}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p+l(\gamma-1)}h^{l/2-1}\right].
\end{aligned}
$$

Choose $R = R(h) = h^{-1/G(\gamma)}$, where $G(\gamma) = \max\{2\gamma-1, \chi_{p>1}6(\gamma-1)\}$, one has,

$$
\begin{aligned}
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p+2\gamma-1}h &\leq \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p}, \\
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p+2\gamma-1}h &\leq \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}, \\
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p+l(\gamma-1)}h^{l/2-1} &\leq \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p}, \\
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p+l(\gamma-1)}h^{l/2-1} &\leq \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}.
\end{aligned}
$$

Thus, by the Young inequality, we get

$$
\begin{aligned}
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \leq\ & \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh \\
&+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right] \\
&+K\sum_{l=1}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p-l}h^l\right] + K\sum_{l=1}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p-l}h^l\right] \\
\leq\ & \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh \\
&+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_n)|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right].
\end{aligned}
$$

Applying the elementary inequality (3), (10) and (6), we get

$$
\begin{aligned}
|Y(t_n)|^{\varsigma} &\leq (1-\kappa)^{1-\varsigma}|V_n|^{\varsigma} + \kappa^{1-\varsigma}|D(Y(t_{n-M}), r_n))|^{\varsigma} \\
&\leq (1-\kappa)^{1-\varsigma}|V_n|^{\varsigma} + \kappa|Y(t_{n-M})|^{\varsigma}.
\end{aligned} \tag{18}
$$

Using (18), we get

$$
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \leq \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right].
$$

Since $T \leq \left(\left\lfloor \frac{T}{\tau} \right\rfloor + 1\right)\tau$, we consider $n \in [kM, (k+1)M-1]$, where $k = 0, 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$.
If $n \in [0, M-1]$, one has

$$
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right] = \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|\xi(t_{n-M})|^{2p}\right] \leq \mathbb{E}|\xi|^{2p}. \tag{19}
$$

Hence, we obtain

$$
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \leq \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}|\xi|^{2p} + Kh.
$$

By the Gronwall inequality, we obtain

$$
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \leq K\left(1 + \mathbb{E}|\xi|^{2p}\right). \tag{20}
$$

Using (18), (20) and (19), one has

$$
\begin{aligned}
\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|Y_{n+1}|^{2p}\right] &\leq (1-\kappa)^{1-2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] + \kappa\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right] \\
&\leq K\left(1 + \mathbb{E}|\xi|^{2p}\right).
\end{aligned} \tag{21}
$$

We claim that for $k \geq 0$, if $n \in [kM, (k+1)M-1]$, (20) holds. By induction, we need to show that (20) and (21) still hold for $k+1$.

If $n \in [(k+1)M, \cdots, (k+2)M-1]$, one gets

$$\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y(t_{n-M})|^{2p}\right] \leq K\left(1+\mathbb{E}|\xi|^{2p}\right), \tag{22}$$

and

$$\begin{aligned}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|V_{n+1}|^{2p}\right] \quad \leq \quad & \mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] + Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|V_n|^{2p}\right] \\ & +Kh\left(1+\mathbb{E}|\xi|^{2p}\right) + Kh.\end{aligned} \tag{23}$$

The well-known Gronwall inequality yield that (17) holds for any $k \in \left[0, \left\lfloor \frac{T}{\tau} \right\rfloor\right]$.

By (18), (22) and (23), we obtain (21).

Since

$$\begin{aligned}\chi_{\tilde{\Lambda}_{R,n}} &= 1 - \chi_{\tilde{\Omega}_{R,n}} \\ &= 1 - \chi_{\tilde{\Omega}_{R,n-1}}\chi_{|Y(t_n)|\leq R} \\ &= \chi_{\tilde{\Lambda}_{R,n-1}} + \chi_{\tilde{\Omega}_{R,n-1}}\chi_{|Y(t_n)|>R} \\ &= \cdots \\ &= \sum_{k=0}^{n} \chi_{\tilde{\Omega}_{R,k-1}}\chi_{|Y(t_k)|>R},\end{aligned}$$

where we put $\chi_{\tilde{\Omega}_{R,-1}} = 1$, using (12) and (21), Hölder's inequality, Markov's inequality and Jensen's inequality, one obtains

$$\begin{aligned}\mathbb{E}[\chi_{\tilde{\Lambda}_{R,n}}(\omega)|Y(t_n)|^{2p}] \quad &\leq \quad \left[\mathbb{E}\left(\frac{1+2\kappa}{1-\kappa}|\xi| + \frac{2(n+1)}{1-\kappa}\right)^{4p}\right]^{1/2} \sum_{k=0}^{n} \frac{\left(\mathbb{E}\left[\chi_{\tilde{\Omega}_{R(h),k-1}}(\omega)|Y(t_k)|^{2(2p+1)G(\gamma)}\right]\right)^{1/2}}{R(h)^{(2p+1)G(\gamma)}} \\ &\leq \quad K\left[\mathbb{E}\left(\frac{1+2\kappa}{1-\kappa}|\xi| + \frac{2(n+1)}{1-\kappa}\right)^{4p}\right]^{1/2} \left(1+\mathbb{E}|\xi|^{2(2p+1)G(\gamma)}\right)^{1/2} nh^{2p+1} \\ &\leq \quad K\left(1+\mathbb{E}|\xi|^{2p\beta}\right),\end{aligned}$$

where $\beta \geq 1 + \frac{(2p+1)G(\gamma)}{2p}$. The proof is complete. □

Remark 4.2. *Due to the flow property of the NSDDE-MS* (2), *let* (4), (6) *and* (7) *hold with sufficiently large* p_0. *There exists* $\beta \geq 1 + \frac{(2p+1)G(\gamma)}{2p}$, *we have*

$$\mathbb{E}|Y(t_u)|^{2p} = E|\bar{Y}_{t_n,\psi}(t_u)|^{2p} \leq K\left(1+\mathbb{E}|\psi|^{2p\beta}\right), n \leq u \leq N, \tag{24}$$

where $\psi = \{\psi(t_k) = Y(t_k) = \bar{Y}_{0,\xi}(t_k) : n-1-M \leq k \leq n-1\}$ *with* $\mathbb{E}|\psi|^{p_0} < \infty$ *and* $1 \leq p \leq \left(\frac{p_0-1}{2(2\gamma-1)} - \frac{1}{2}\right) \wedge \left(\frac{p_0-1}{12(\gamma-1)} - \frac{1}{2}\right)$ *and* $G(\gamma) = \max\{2\gamma-1, \chi_{p>1}6(\gamma-1)\}$.

4.2. One-step error of the balanced Euler method

We present the following four lemmas to show the estimate for the one-step error of the balanced Euler method (9).

Lemma 4.3. *Under* (4), (6) *and* (7) *with sufficiently large* p_0. *Let the function* $f(x,y,i)$ *satisfy*

$$|f(x,y,i)| \le \bar{C}_1\left(1+|x|^\gamma+|y|^\gamma\right), \tag{25}$$

where $x, y \in \mathbb{R}^n$ *and* $i \in S$ *and* $\bar{C}_1 > 0$. *Then for all* $1 \le l \le \frac{p_0-1}{\gamma}$ *and* $0 \le t \le s \le T$, *we have*

$$\mathbb{E}|f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s)) - f(X_{t,\phi}(t), X(_{t,\phi}t-\tau), r(t))|^l \quad \le \quad K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)[(s-t)+o((s-t))],$$

where $\phi = \{\phi(s) = X_{0,\xi}(s) : t-\tau \le s \le t\}$ *with* $\mathbb{E}|\phi|^{p_0} < \infty$.

Proof. Using the elementary inequality, (1), (4), Lemma 2.3 and Remark 2.4, we get

$$\begin{aligned}
& \mathbb{E}|f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \\
= \;& \mathbb{E}\left[|f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& 2^{l-1}\mathbb{E}\left[\left(|f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s))|^l + |f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l\right)\mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& \bar{C}_1 2^l \mathbb{E}\left[\left(1+|X_{t,\phi}(t)|^{l\gamma}+|X_{t,\phi}(t-\tau)|^{l\gamma}\right)\mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)E\left(\mathbf{1}_{\{r(s)\neq r(t)\}}\right) \\
= \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\sum_{i\in S}\mathbf{1}_{\{r(t)\neq i\}}P(r(s)\neq i|r(t)=i) \\
= \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\sum_{i\in S}\mathbf{1}_{\{r(t)\neq i\}}\sum_{j\neq i}\left[\gamma_{ij}(s-t)+o(s-t)\right] \\
\le \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\max_{0\le i\le Q}\left[-\gamma_{ii}(s-t)+o((s-t))\right]\sum_{i\in S}\mathbf{1}_{\{r(t)\neq i\}} \\
\le \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\max_{0\le i\le Q}\left[-\gamma_{ii}(s-t)+o((s-t))\right] \\
\le \;& K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)[(s-t)+o((s-t))].
\end{aligned}$$

□

Lemma 4.4. *Under* (**A1**), (**A3**), (6) *and* (7) *with sufficiently large* p_0. *Then for all* $1 \le l \le \dfrac{p_0-1}{\gamma}$ *and* $0 \le t \le s \le T$, *we have*

$$\mathbb{E}|D(X_{t,\phi}(t-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \quad \le \quad K|\phi(t-\tau)|^l\left[(s-t)+o((s-t))\right],$$

where $\phi = \{\phi(s) = X_{0,\xi}(s) : t-\tau \le s \le t\}$ *with* $\mathbb{E}|\phi|^{p_0} < \infty$.

Proof. Using the elementary inequality, (1), (6), Lemma 2.3 and Remark 2.4, we get

$$\begin{aligned}
& \mathbb{E}|D(X_{t,\phi}(t-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \\
= \;& \mathbb{E}\left[|D(X_{t,\phi}(t-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& 2^{l-1}\mathbb{E}\left[\left(|D(X_{t,\phi}(t-\tau), r(s))|^l + |D(X_{t,\phi}(t-\tau), r(t))|^l\right)\mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& 2^l\mathbb{E}\left[\left(\kappa^l|X_{t,\phi}(t-\tau)|^l\right)\mathbf{1}_{\{r(s)\neq r(t)\}}\right] \\
\le \;& K|\phi(t-\tau)|^l\sum_{i\in S}\mathbf{1}_{\{r(t)\neq i\}}P(r(s)\neq i|r(t)=i)
\end{aligned}$$

$$
\begin{aligned}
&= K|\phi(t-\tau)|^l \sum_{i\in S} \mathbf{1}_{\{r(t)\neq i\}} \sum_{j\neq i} \left[\gamma_{ij}(s-t) + o(s-t)\right] \\
&\leq K|\phi(t-\tau)|^l \max_{0\leq i\leq Q} \left[-\gamma_{ii}(s-t) + o((s-t))\right] \sum_{i\in S} \mathbf{1}_{\{r(t)\neq i\}} \\
&\leq K|\phi(t-\tau)|^l \max_{0\leq i\leq Q} \left[-\gamma_{ii}(s-t) + o((s-t))\right] \\
&\leq K|\phi(t-\tau)|^l \left[(s-t) + o((s-t))\right].
\end{aligned}
$$

□

Lemma 4.5. *Under* (**A1**), (**A3**), (**A4**) *and* (7) *with sufficiently large* p_0. *Let the function* $f(x, y, i)$ *satisfy*

$$
|f(x,y,i) - f(\hat{x},\hat{y},i)| \leq K_1 \left(1 + |x|^{\gamma-1} + |\hat{x}|^{\gamma-1} + |y|^{\gamma-1} + |\hat{y}|^{\gamma-1}\right)(|x-\hat{x}| + |x-\hat{y}|), \tag{26}
$$

where x, y $\bar{x}$, y, $\bar{y} \in \mathbb{R}^n$ *and* $i \in S$. *Then for all* $1 \leq l \leq \dfrac{p_0 - 1}{\gamma}$ *and* $0 \leq t \leq s \leq T$, *we have*

$$
\mathbb{E}|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \leq K\left(1 + |\phi(t-\tau)|^l\right)\left[(s-t) + o((s-t))\right], \tag{27}
$$

where $\phi = \{\phi(s) = X_{0,\xi}(s) : t - \tau \leq s \leq t\}$ *with* $\mathbb{E}|\phi|^{p_0} < \infty$.

Proof. Using the elementary inequality and Lemma 4.4, we get

$$
\begin{aligned}
&\mathbb{E}|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \\
\leq\ & 2^{l-1}\mathbb{E}\left[|D(X_{t,\phi}(t-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l + |D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(s))|^l\right] \\
\leq\ & K|\phi(t-\tau)|^l\left[(s-t) + o((s-t))\right] + 2^{l-1}B_1,
\end{aligned} \tag{28}
$$

where

$$
B_1 := \mathbb{E}|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(s))|^l.
$$

Applying (**A4**), one has

$$
B_1 \leq \kappa^l \mathbb{E}|X_{t,\phi}(s-\tau) - X_{t,\phi}(t-\tau)|^l.
$$

Now we consider $s \in [k\tau, (k+1)\tau]$, where $k = 0, 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$.
If $s \in [0, \tau]$, that is $-\tau \leq t - \tau \leq s - \tau \leq 0$, we obtain

$$
\left|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(s))\right| \leq \kappa|\xi(s-\tau) - \xi(t-\tau)|.
$$

Therefore, applying (**A3**), we have $B_1 \leq \kappa^l \mathbb{E}|\xi(s-\tau) - \xi(t-\tau)|^l \leq \kappa^l K_0^l (s-t)^{l/2} \leq K(s-t)^{l/2}$.
One claims for $k > 0$, $s \in [k\tau, (k+1)\tau]$,

$$
B_1 \leq K(s-t)^{l/2} \tag{29}
$$

and

$$
\mathbb{E}|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))|^l \leq K\left(1 + |\phi(t-\tau)|^l\right)\left[(s-t) + o((s-t))\right]. \tag{30}
$$

Next we only should show that (29) holds for $k+1$.
If $s \in [(k+1)\tau, (k+2)\tau]$, we divided the following into two cases.
Case (I) If $k\tau \leq t \leq (k+1)\tau \leq s \leq (k+2)\tau$, we have $s - \tau \in [k\tau, (k+1)\tau]$ and $t - \tau \in [(k-1)\tau, k\tau]$.
Case (II) If $(k+1)\tau \leq t \leq s \leq (k+2)\tau$, we have $s - \tau \in [k\tau, (k+1)\tau]$ and $t - \tau \in [k\tau, (k+1)\tau]$.

Applying the elementary inequality, by (8) and (**A4**), one gets

$$\begin{aligned} B_1 \le& \kappa^l \left|X_{t,\phi}(s-\tau) - X_{t,\phi}(t-\tau)\right|^l \\ \le& 3^{l-1}\kappa^l \Bigg[\mathbb{E}\left|D(X_{t,\phi}(s-2\tau), r(s-\tau)) - D(X_{t,\phi}(t-2\tau), r(t-\tau))\right|^l \\ &+ \mathbb{E}\left|\int_{t-\tau}^{s-\tau} F(X_{t,\phi}(s_1), X_{t,\phi}(s_1-\tau), r(s_1))ds_1\right|^l \\ &+ \mathbb{E}\left|\int_{t-\tau}^{s-\tau} G(X_{t,\phi}(s_1), X_{t,\phi}(s_1-\tau), r(s_1))dw(s_1)\right|^l\Bigg]. \end{aligned} \tag{31}$$

By hypothesis, we have

$$\mathbb{E}\left|D(X_{t,\phi}(s-2\tau), r(s-\tau)) - D(X_{t,\phi}(t-2\tau), r(t-\tau))\right|^l \le K\left(1+|\phi(t-\tau)|^l\right)[(s-t)+o((s-t))]. \tag{32}$$

Due to the properties of Itô integral, (4), Remark 2.1 and Remark 2.4, we obtian

$$\begin{aligned} &\mathbb{E}\left|\int_t^s F(X_{t,\phi}(s_1), X_{t,\phi}(s_1-\tau), r(s_1))ds_1\right|^l \\ \le\ & K(s-t)^{l-1}\int_t^s \left(1+\mathbb{E}|X_{t,\phi}(s_1)|^{l\gamma} + \mathbb{E}|X_{t,\phi}(s_1-\tau)|^{l\gamma}\right)ds_1 \\ \le\ & K(s-t)^l \end{aligned} \tag{33}$$

and

$$\begin{aligned} &\mathbb{E}\left|\int_t^s G(X_{t,\phi}(s_1), X_{t,\phi}(s_1-\tau), r(s_1))dw(s_1)\right|^l \\ \le\ & K(s-t)^{(2l-2)/2}\int_t^s \left(1+\mathbb{E}|X_{t,\phi}(s_1)|^{l\gamma} + \mathbb{E}|X_{t,\phi}(s_1-\tau)|^{l\gamma}\right)ds_1 \\ \le\ & K(s-t)^{l/2}. \end{aligned} \tag{34}$$

Substituting (32) - (34) into (31), one gets (29).

Hence, for any k, we have (29). Consequently, substituting (29) into (28), we have (27).

□

Lemma 4.6. *Under* (**A1**), (**A3**), (**A4**) *and* (7) *with sufficiently large* p_0. *Let the function* $f(x, y, i)$ *satisfy* (26). *Then for all* $1 \le l \le \frac{p_0-1}{\gamma}$ *and* $0 \le t \le s \le T$, *we have*

$$\begin{aligned} &\mathbb{E}|f(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \\ \le\ & K\left(1+|\phi(t)|^{l\gamma} + |\phi(t-\tau)|^{l\gamma}\right)\left[(s-t)^l \vee (s-t)^{l/2} \vee (s-t)\right], \end{aligned} \tag{35}$$

where $\phi = \{\phi(s) = X_{0,\xi}(s) : t-\tau \le s \le t\}$ *with* $\mathbb{E}|\phi|^{p_0} < \infty$.

Proof. For all $1 \le l \le \frac{p_0-1}{\gamma}$ and $0 \le t \le s \le T$, applying Lemma 4.3, we have

$$\begin{aligned} &\mathbb{E}|f(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \\ \le\ & 2^{l-1}\mathbb{E}|f(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s))|^l \\ &+2^{l-1}\mathbb{E}|f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \\ \le\ & 2^{l-1}B_2 + K\left(1+|\phi(t)|^{l\gamma} + |\phi(t-\tau)|^{l\gamma}\right)[(s-t)+o((s-t))], \end{aligned}$$

where

$$B_2 := \mathbb{E}|f(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(s))|^l.$$

By (26), one gets

$$\begin{aligned} B_2 \quad \leq \quad & K_1^l \mathbb{E}\Big[\Big(1 + |X_{t,\phi}(s)|^{\gamma-1} + |X_{t,\phi}(s-\tau)|^{\gamma-1} + |X_{t,\phi}(t)|^{\gamma-1} + |X_{t,\phi}(t-\tau)|^{\gamma-1}\Big) \\ & \cdot \Big(\Big|X_{t,\phi}(s) - X_{t,\phi}(t)\Big| + \Big|X_{t,\phi}(s-\tau) - X_{t,\phi}(t-\tau)\Big|\Big)\Big]^l. \end{aligned} \tag{36}$$

We consider $s \in [k\tau, (k+1)\tau]$, where $k = 0, 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$.

If $s \in [0, \tau]$, let $0 \leq t \leq s \leq \tau$, applying (**A3**), we have

$$\Big|X_{t,\phi}(s-\tau) - X_{t,\phi}(t-\tau)\Big| = |\xi(s-\tau) - \xi(t-\tau)| \leq K_0(s-t)^{1/2}.$$

Hence, by Lemma 2.3 and Lemma 4.5, substituting the estimate above into (36), using Cauchy-Schwarz inequality, one obtain

$$\begin{aligned} B_2 \quad \leq \quad & K\Bigg[\Big(1 + |\phi(t)|^{2l(\gamma-1)} + |\phi(t-\tau)|^{2l(\gamma-1)}\Big)^{1/2}\Bigg(\Big|D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\phi}(t-\tau), r(t))\Big|^{2l} \\ & + \mathbb{E}\left|\int_t^s F(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))ds_1\right|^{2l} \\ & + \mathbb{E}\left|\int_t^s G(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))dw(s_1)\right|^{2l} + K(s-t)^l\Bigg)^{1/2}\Bigg] \\ \leq \quad & K\Bigg[\Big(1 + |\phi(t)|^{2l(\gamma-1)} + |\phi(t-\tau)|^{2l(\gamma-1)}\Big)^{1/2}\Bigg(K(s-t)\Big(1 + |\phi(t-\tau)|^{2l}\Big) + K(s-t)^l \\ & + \mathbb{E}\left|\int_t^s F(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))ds_1\right|^{2l} \\ & + \mathbb{E}\left|\int_t^s G(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))dw(s_1)\right|^{2l}\Bigg)^{1/2}\Bigg]. \end{aligned} \tag{37}$$

Due to the properties of Itô integral, (4), Remark 2.1 and Remark 2.4, we get

$$\begin{aligned} & \left(\mathbb{E}\left|\int_t^s F(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))ds_1\right|^{2l}\right)^{1/2} \\ \leq \quad & \left((s-t)^{2l-1}\mathbb{E}\int_t^s \Big|F(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))\Big|^{2l} ds_1\right)^{1/2} \\ \leq \quad & K\left((s-t)^{2l-1}\int_t^s (1 + \mathbb{E}|X_{0,\phi}(s_1)|^{2l\gamma} + \mathbb{E}|X_{0,\phi}(s_1-\tau)|^{2l\gamma})ds_1\right)^{1/2} \\ \leq \quad & K(s-t)^l \end{aligned} \tag{38}$$

and

$$\begin{aligned} & \left(\mathbb{E}\left|\int_t^s G(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))dw(s_1)\right|^{2l}\right)^{1/2} \\ \leq \quad & \left((s-t)^{(2l-2)/2}\mathbb{E}\int_t^s \Big|G(X_{0,\phi}(s_1), X_{0,\phi}(s_1-\tau), r(s_1))\Big|^{2l} ds_1\right)^{1/2} \end{aligned}$$

$$\leq K\left((s-t)^{(2l-2)/2}\int_t^s(1+\mathbb{E}|X_{0,\phi}(s_1)|^{2l\gamma}+\mathbb{E}|X_{0,\phi}(s_1-\tau)|^{2l\gamma})ds_1\right)^{1/2}$$
$$\leq K(s-t)^{l/2}. \tag{39}$$

Applying Remark 2.4 and substituting (38) and (39) to (37), we get

$$B_2 \leq K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\left[(s-t)^l\vee(s-t)^{l/2}\vee(s-t)\right]. \tag{40}$$

We claim that for $n\geq 0$, if $s\in[n\tau,(n+1)\tau]$, (40) hold. Using induction, we should prove that (40) still holds for $n+1$.

If $s\in[(n+1)\tau,(n+2)\tau]$, we divide the following proof into two cases.

Case (I) If $n\tau\leq t\leq(n+1)\tau\leq s\leq(n+2)\tau$, we have $s-\tau\in[n\tau,(n+1)\tau]$ and $t-\tau\in[(n-1)\tau,n\tau]$.

Case (II) If $(n+1)\tau\leq t\leq s\leq(n+2)\tau$, we have $s-\tau\in[n\tau,(n+1)\tau]$ and $t-\tau\in[n\tau,(n+1)\tau]$.

Hence, using the Hölder inequality, Lemma 4.5 and Remark 2.4, we have

$$\begin{aligned}B_2 \leq\ & K\Bigg[\left(1+|\phi(t)|^{2l(\gamma-1)}+|\phi(t-\tau)|^{2l(\gamma-1)}\right)^{1/2}\Bigg(\left|D(X_{t,\phi}(s-\tau),r(s))-D(X_{t,\phi}(t-\tau),r(t))\right|^{2l}\\&+\mathbb{E}\left|\int_t^s F(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))ds_1\right|^{2l}+\mathbb{E}\left|\int_t^s G(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))dw(s_1)\right|^{2l}\\&+\mathbb{E}\left|\int_{t-\tau}^{s-\tau} F(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))ds_1\right|^{2l}\\&+\mathbb{E}\left|\int_{t-\tau}^{s-\tau} G(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))dw(s_1)\right|^{2l}\Bigg)^{1/2}\Bigg].\end{aligned} \tag{41}$$

By the properties of Itô integral, (4), Remark 2.1 and Remark 2.4, we have

$$\begin{aligned}&\left(\mathbb{E}\left|\int_{t-\tau}^{s-\tau}F(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))ds_1\right|^{2l}\right)^{1/2}\\ \leq\ &\left((s-t)^{2l-1}\mathbb{E}\int_{t-\tau}^{s-\tau}\left|F(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))\right|^{2l}ds_1\right)^{1/2}\\ \leq\ & K\left((s-t)^{2l-1}\int_{t-\tau}^{s-\tau}\left(1+\mathbb{E}|X_{t,\phi}(s_1)|^{2l\gamma}+\mathbb{E}|X_{t,\phi}(s_1-\tau)|^{2l\gamma}\right)ds_1\right)^{1/2}\\ \leq\ & K(s-t)^l\end{aligned} \tag{42}$$

and

$$\begin{aligned}&\left(\mathbb{E}\left|\int_{t-\tau}^{s-\tau}G(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))dw(s_1)\right|^{2l}\right)^{1/2}\\ \leq\ &\left((s-t)^{(2l-2)/2}\mathbb{E}\int_{t-\tau}^{s-\tau}\left|G(X_{t,\phi}(s_1),X_{t,\phi}(s_1-\tau),r(s_1))\right|^{2l}ds_1\right)^{1/2}\\ \leq\ & K\left((s-t)^{(2l-2)/2}\int_{t-\tau}^{s-\tau}\left(1+\mathbb{E}|X_{t,\phi}(s_1)|^{2l\gamma}+\mathbb{E}|X_{t,\phi}(s_1-\tau)|^{2l\gamma}\right)ds_1\right)^{1/2}\\ \leq\ & K(s-t)^{l/2}.\end{aligned} \tag{43}$$

Using Lemma 4.5 and substituting (38), (42), (39), (43) to (41), we get (40).

Consequently, one shows that (40) holds for any n.

Therefore, one derives

$$\begin{aligned} & \mathbb{E}|f(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - f(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^l \\ \leq\ & K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)[(s-t)+o((s-t))] + K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\left((s-t)^{l/2}\vee(s-t)\right) \\ \leq\ & K\left(1+|\phi(t)|^{l\gamma}+|\phi(t-\tau)|^{l\gamma}\right)\left[(s-t)^l\vee(s-t)^{l/2}\vee(s-t)\right]. \end{aligned} \tag{44}$$

Hence, we obtain (35). □

In the following theorem, we can obtain the local truncation error of the one-step method.

Theorem 4.7. *Suppose* (**A1**), (**A3**), (**A4**) *and* (7) *hold with sufficiently large* p_0. *For* $1 \leq p \leq \left(\frac{p_0-1}{2\gamma}\right) \wedge \left(\frac{p_0-1}{2(2\gamma-1)}-\frac{1}{2}\right) \wedge \left(\frac{p_0-1}{12(\gamma-1)}-\frac{1}{2}\right)$ *and* $t+h \in [0,T]$, *we have*

$$\left|\mathbb{E}\left[X_{t,\phi}(t+h)-\bar{Y}_{t,\phi}(t+h)\right]\right| \leq Kh^{3/2}\left(1+|\phi(t)|^{3\gamma}+|\phi(t-\tau)|^{3\gamma}\right) \tag{45}$$

and

$$\mathbb{E}\left|X_{t,\phi}(t+h)-\bar{Y}_{t,\phi}(t+h)\right|^{2p} \leq Kh^{p+1}\left(1+|\phi(t)|^{6p\gamma}+|\phi(t-\tau)|^{6p\gamma}\right). \tag{46}$$

where $\mathbb{E}|\phi|^{p_0} < \infty$.

Proof. We divided the proof into to three steps.

Step 1: The one-step approximation to the explicit Euler scheme is given as the following:

$$\begin{aligned} \tilde{X}_{t,\phi}(t+h) =& X_{t,\phi}(t) + D(X_{t,\phi}(t+h-\tau), r(t+h)) - D(X_{t,\phi}(t-\tau), r(t)) \\ & + F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))h + G(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))\Delta w. \end{aligned}$$

Define

$$\tilde{\rho}(t+h) = X_{t,\phi}(t+h) - \tilde{X}_{t,\phi}(t+h). \tag{47}$$

By (47) and Lemma 4.6, we have

$$\begin{aligned} |\mathbb{E}\tilde{\rho}(t+h)| \leq\ & \left|\mathbb{E}\int_t^{t+h}\left[F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))\right]ds\right| \\ \leq\ & \mathbb{E}\int_t^{t+h}|F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|ds \\ \leq\ & Kh^{3/2}\left(1+|\phi(t)|^{\gamma}+|\phi(t-\tau)|^{\gamma}\right) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \mathbb{E}|\tilde{\rho}(t+h)|^{2p} \leq\ & \mathbb{E}\left|\int_t^{t+h}\left[F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))\right]ds\right|^{2p} \\ & + K\mathbb{E}\left|\int_t^{t+h}\left[G(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - G(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))\right]dw(s)\right|^{2p} \\ \leq\ & Kh^{2p-1}\int_t^{t+h}\mathbb{E}|F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - F(X_{t,\phi}(t), X_{t,\phi}(t-\tau), r(t))|^{2p}ds \end{aligned}$$

$$
\begin{aligned}
&+Kh^{p-1}\int_t^{t+h}\mathbb{E}|G(X_{t,\phi}(s),X_{t,\phi}(s-\tau),r(s))-F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))|^{2p}ds\\
\leq\ & Kh^{2p+1}\left(1+|\phi(t)|^{2p\gamma}+|\phi(t-\tau)|^{2p\gamma}\right)+Kh^{p+1}\left(1+|\phi(t)|^{2p\gamma}+|\phi(t-\tau)|^{2p\gamma}\right)\\
\leq\ & Kh^{p+1}\left(1+|\phi(t)|^{2p\gamma}+|\phi(t-\tau)|^{2p\gamma}\right). \qquad (49)
\end{aligned}
$$

Step 2: The one-step approximation to the balanced Euler scheme is defined as the following

$$
\begin{aligned}
\bar{Y}_{t,\phi}(t+h) =& X_{t,\phi}(t)+D(X_{t,\phi}(t+h-\tau),r(t+h))-D(X_{t,\phi}(t-\tau),r(t))\\
&+\sin(F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h)+\sin\left(G(X_{t,\phi}(t),X_{t,\phi}(t-\tau,r(t)))\Delta w\right).
\end{aligned}
$$

Define

$$
\begin{aligned}
\bar{\rho}(t+h) \ :=\ & \tilde{X}_{t,\phi}(t+h)-\bar{Y}_{t,\phi}(t+h) \qquad (50)\\
=\ & F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h-\sin(F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h)\\
&+G(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))\Delta w-\sin\left(G(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))\Delta w\right). \qquad (51)
\end{aligned}
$$

Using (14) and (50), we get

$$
\begin{aligned}
|\mathbb{E}\bar{\rho}(t+h)| \ =\ & \left|\mathbb{E}\left[F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h-\sin(F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h)\right]\right|\\
\leq\ & K\mathbb{E}\left|F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h\right|^3\\
\leq\ & Kh^3\left(1+|\phi(t)|^{3\gamma}+|\phi(t-\tau)|^{3\gamma}\right) \qquad (52)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}|\bar{\rho}(t+h)|^{2p} \ \leq\ & K\mathbb{E}\left|F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h-\sin(F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h)\right|^{2p}\\
&+K\mathbb{E}\left|G(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))\Delta w-\sin\left(G(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))\Delta w\right)\right|^{2p}\\
\leq\ & K\mathbb{E}\left|F(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))h\right|^{6p}+K\mathbb{E}|G(X_{t,\phi}(t),X_{t,\phi}(t-\tau),r(t))\Delta w|^{6p}\\
\leq\ & Kh^{3p}\left(1+|\phi(t)|^{6p\gamma}+|\phi(t-\tau)|^{6p\gamma}\right). \qquad (53)
\end{aligned}
$$

Step 3. Define

$$
\rho(t+h) \ :=\ X_{t,\phi}(t+h)-\bar{Y}_{t,\phi}(t+h)=\tilde{\rho}(t+h)-\bar{\rho}(t+h). \qquad (54)
$$

Using (48), (52) and (54), it is easy to see that

$$
\begin{aligned}
|\mathbb{E}\rho(t+h)| \ \leq\ & |\mathbb{E}\tilde{\rho}(t+h)|+|\mathbb{E}\bar{\rho}(t+h)|\\
\leq\ & Kh^{3/2}\left(1+|\phi(t)|^{3\gamma}+|\phi(t-\tau)|^{3\gamma}\right).
\end{aligned}
$$

Using (49), (53) and (54), one has

$$
\begin{aligned}
\mathbb{E}|\rho(t+h)|^{2p} \ \leq\ & K\mathbb{E}|\tilde{\rho}(t+h)|^{2p}+K\mathbb{E}|\bar{\rho}(t+h)|^{2p}\\
\leq\ & Kh^{p+1}\left(1+|\phi(t)|^{6p\gamma}+|\phi(t-\tau)|^{6p\gamma}\right).
\end{aligned}
$$

□

4.3. Convergence order of the balanced Euler method

To show the convergence theorem of the balanced Euler method (9), we uses the following lemma.

Lemma 4.8. *Suppose* (**A1**) – (**A4**) *hold with sufficiently large* p_0. *Define*

$$V_{t,\phi}(s) := X_{t,\phi}(s) - D(X_{t,\phi}(s-\tau), r(s)), \tag{55}$$

$$V_{t,\psi}(s) := X_{t,\psi}(s) - D(X_{t,\psi}(s-\tau), r(s)), \tag{56}$$

$$\Delta D_{t,\phi,\psi}(s) := D(X_{t,\phi}(s-\tau), r(s)) - D(X_{t,\psi}(s-\tau), r(s)), \tag{57}$$

$$S_{t,\phi,\psi}(s) := S(s) := X_{t,\phi}(s) - X_{t,\psi}(s), \;\; S(t) := \phi(t) - \psi(t), \tag{58}$$

$$V_{t,\phi,\psi}(s) := V(s) := V_{t,\phi}(s) - V_{t,\psi}(s) = S(s) - \Delta D_{t,\phi,\psi}(s), \;\; V(t) := V_{t,\phi,\psi}(t) = S(t) - \Delta D_{t,\phi,\psi}(t), \tag{59}$$

and

$$Z_{t,\phi,\psi}(s) := Z(s) := V(s) - V(t) = S(s) - S(t) - \Delta D_{t,\phi,\psi}(s) + \Delta D_{t,\phi,\psi}(t), \tag{60}$$

for $\theta \in [0,h]$, *where* $\phi = \{\phi(s) : t-\tau \le s \le t, \phi(s) \in \mathbb{R}^n\}$, $\psi = \{\psi(s) : t-\tau \le s \le t, \psi(s) \in \mathbb{R}^n\}$ *and* $|\phi-\psi|^{2p} = \sup_{t-\tau\le s\le t} |\phi(s)-\psi(s)|^{2p} < \infty$. *For* $1 \le p \le \left(\frac{p_0-1}{2\gamma}\right) \wedge \left(\frac{p_0-1}{2(2\gamma-1)} - \frac{1}{2}\right) \wedge \left(\frac{p_0-1}{12(\gamma-1)} - \frac{1}{2}\right)$ *and* $t+h \in [0,T]$, *we have*

$$\mathbb{E}|V_{t,\phi,\psi}(t+h|^{2p} \le \left(|\phi(t)-\psi(t)|^{2p} + |\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)(1+Kh), \tag{61}$$

$$\mathbb{E}|S_{t,\phi,\psi}(t+h)|^{2p} \le \left(|\phi(t)-\psi(t)|^{2p} + |\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)(1+Kh), \tag{62}$$

$$\begin{aligned}\mathbb{E}|Z_{t,\phi,\psi}(t+h)|^{2p} \le \quad & K\left(1+|\phi(t)|^{\gamma-1}+|\psi(t)|^{\gamma-1}+|\phi(t-\tau)|^{\gamma-1}+|\psi(t-\tau)|^{\gamma-1}\right)^p \\ & \cdot\left(|\phi(t)-\psi(t)|^{2p} + |\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)h^p.\end{aligned} \tag{63}$$

Proof. Define

$$\Delta F_{t,\phi,\psi}(s) := F(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - F(X_{t,\psi}(s), X_{t,\psi}(s-\tau), r(s)) \tag{64}$$

and

$$\Delta G_{t,\phi,\psi}(s) := G(X_{t,\phi}(s), X_{t,\phi}(s-\tau), r(s)) - G(X_{t,\psi}(s), X_{t,\psi}(s-\tau), r(s)) \tag{65}$$

We consider $t+h \in [k\tau, (k+1)\tau]$, where $k = 0, 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$.

If $t+h \in [0,\tau]$, namely $0 \le t \le s \le t+h \le \tau$, we have $-\tau \le t-\tau \le s-\tau \le t+h-\tau \le 0$.

By (58), (3) and the elementary inequality, one has

$$\begin{aligned}|S(s-\tau)|^{2p} =& |X_{t,\phi}(s-\tau) - X_{t,\psi}(s-\tau)|^{2p} \\ =& |\phi(s-\tau) - \psi(s-\tau)|^{2p} \\ \le& \sup_{t-\tau\le s\le t+h-\tau} |\phi(s)-\psi(s)|^{2p} \\ \le& |\phi-\psi|^{2p},\end{aligned} \tag{66}$$

$$|S(t)|^{2p} = |\phi(t)-\psi(t)|^{2p} \tag{67}$$

and

$$|S(t-\tau)|^{2p} = |\phi(t-\tau) - \psi(t-\tau)|^{2p}. \tag{68}$$

Applying (57), (6), one can see

$$\begin{aligned}|\Delta D_{t,\phi,\psi}(s))|^{2p} =&|D(X_{t,\phi}(s-\tau), r(s) - D(X_{t,\varphi}(s-\tau), r(s)|^{2p}\\ \le&\kappa^{2p}|S(s-\tau)|^{2p}\end{aligned} \tag{69}$$

and

$$\begin{aligned}|\Delta D_{t,\phi,\psi}(t))|^{2p} =&|D(X_{t,\phi}(t-\tau), r(t) - D(X_{t,\psi}(t-\tau), r(t)|^{2p}\\ \le&\kappa^{2p}|S(t-\tau)|^{2p}\\ =&\kappa^{2p}|\phi(t-\tau) - \psi(t-\tau)|^{2p}.\end{aligned} \tag{70}$$

Using (3), (59) and (70), we get

$$\begin{aligned}|V(t)|^{2p} \le&(1-\kappa)^{1-2p}|S(t)|^{2p} + \kappa^{1-2p}|\Delta D_{t,\phi,\psi}(t)|^{2p}\\ \le&(1-\kappa)^{1-2p}|S(t)|^{2p} + \kappa|\phi(t-\tau) - \psi(t-\tau)|^{2p},\end{aligned} \tag{71}$$

$$|S(s)|^{2p} \le(1-\kappa)^{1-2p}|V(s)|^{2p} + \kappa^{1-2p}|\Delta D_{t,\phi,\psi}(s)|^{2p}$$
$$\le(1-\kappa)^{1-2p}|V(s)|^{2p} + \kappa|\phi(s-\tau) - \psi(s-\tau)|^{2p} \tag{72}$$
$$=(1-\kappa)^{1-2p}|V(s)|^{2p} + \kappa|S(s-\tau)|^{2p} \tag{73}$$

and

$$\begin{aligned}|V(s)|^{2p} \le&(1-\kappa)^{1-2p}|S(s)|^{2p} + \kappa^{1-2p}|\Delta D_{t,\phi,\psi}(s)|^{2p}\\ \le&(1-\kappa)^{1-2p}|S(s)|^{2p} + \kappa|\phi(s-\tau) - \psi(s-\tau)|^{2p}\\ =&(1-\kappa)^{1-2p}|S(s)|^{2p} + \kappa|S(s-\tau)|^{2p}.\end{aligned} \tag{74}$$

By (63) and (3), one has

$$\begin{aligned}|Z(s)|^{2p} =&|V(s) - V(t)|^{2p}\\ \le&(1-\kappa)^{1-2p}|V(t)|^{2p} + \kappa^{1-2p}|V(s)|^{2p},\end{aligned} \tag{75}$$

and

$$\begin{aligned}|V(s)|^{2p} =&|Z(s) + V(t)|^{2p}\\ \le&(1-\kappa)^{1-2p}|V(t)|^{2p} + \kappa^{1-2p}|Z(s)|^{2p}.\end{aligned} \tag{76}$$

Using the Itô formula, (**A2**), (66), (68), (71), (72), Remark 3.2 and the Young inequality, for $\theta \ge 0$, one obtains

$$\begin{aligned}\mathbb{E}|V(t+\theta)|^{2p} \quad \le& \quad |V(t)|^{2p} + 2p\mathbb{E}\int_t^{t+\theta} |V(s)|^{2p-2}\left(V^T(s)\Delta F_{t,\phi,\psi}(s) + \frac{2p-1}{2}|\Delta G_{t,\phi,\psi}(s)|^2\right)ds\\ \le& \quad |V(t)|^{2p} + 2pK_2\mathbb{E}\int_t^{t+\theta} |V(s)|^{2p-2}\left(|S(s)|^2 + |S(s-\tau)|^2\right)ds\\ \le& \quad |V(t)|^{2p} + 2(2p-2)K_2\mathbb{E}\int_t^{t+\theta} |V(s)|^{2p}ds\end{aligned}$$

$$
\begin{aligned}
&+2K_2E\int_t^{t+\theta}|S(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s-\tau)|^{2p}ds\\
\leq\ & (1-\kappa)^{1-2p}|\phi(t)-\psi(t)|^{2p}+\kappa|\phi(t-\tau)-\psi(t-\tau)|^{2p}+K\mathbb{E}\int_t^{t+\theta}|V(s)|^{2p}ds\\
&+K\theta|\phi-\psi|^{2p}.
\end{aligned}
$$

Applying the Gronwall inequality, we get

$$
\mathbb{E}|V(t+h)|^{2p} \leq (1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right).
$$

By (72) and (3), we has

$$
\begin{aligned}
\mathbb{E}|S(t+h)|^{2p} &\leq (1-\kappa)^{1-2p}\mathbb{E}|V(t+h)|^{2p}+\kappa\mathbb{E}|S(t+h-\tau)|^{2p}\\
&\leq (1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right).
\end{aligned}
$$

Noting (58), (60), (64), (65), by the Itô formula and (**A2**), for $\theta\geq 0$, one has

$$
\begin{aligned}
\mathbb{E}|Z(t+\theta)|^{2p} &\leq 2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(Z^T(s)\Delta F_{t,\phi,\psi}(s)ds+\frac{2p-1}{2}|\Delta G_{t,\phi,\psi}(s)|^2\right)ds\\
&\leq 2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(V^T(s)\Delta F_{t,\phi,\psi}(s)ds+\frac{2p-1}{2}|\Delta G_{t,\phi,\psi}(s)|^2\right)ds\\
&\quad -2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}V^T(t)\Delta F_{t,\phi,\psi}(s)ds\\
&\leq 2pK_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(|S(s)|^2+|S(s-\tau)|^2\right)ds\\
&\quad -2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}V^T(t)\Delta F_{\phi,\psi}(s)ds.
\end{aligned}
$$

Using the Young inequality, (66), (72), (76) and (71), we derive

$$
\begin{aligned}
&2pK_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(|S(s)|^2+|S(s-\tau)|^2\right)ds\\
\leq\ & 2(2p-2)K_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s-\tau)|^{2p}ds\\
\leq\ & 2(2p-2)K_2\int_t^{t+\theta}\mathbb{E}|Z(s)|^{2p}ds+K\theta(1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right).
\end{aligned}
$$

Noting (58), (60), (64), using the Hölder inequality, (**A1**) and Lemma 4.6, one obtains

$$
\begin{aligned}
&-2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}V^T(t)\Delta F_{t,\phi,\psi}(s)ds\\
\leq\ & 2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}|\phi(t)-\psi(t)||\Delta F_{t,\phi,\psi}(s)|ds\\
\leq\ & K|\phi(t)-\psi(t)|\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}\left(\mathbb{E}|\Delta F_{t,\phi,\psi}(s)|^p\right)^{1/p}ds\\
\leq\ & K|\phi(t)-\psi(t)|\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}\Big(\mathbb{E}\Big[(1+|X_{t,\phi}(s)|^{\gamma-1}+|X_{t,\phi}(s-\tau)|^{\gamma-1}\\
&\quad +|X_{t,\psi}(s)|^{\gamma-1}+|X_{t,\psi}(s-\tau))|^{\gamma-1})^p\,(|S(s)|+|S(s-\tau)|)^p\Big]\Big)^{1/p}ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq K|\phi(t)-\psi(t)|\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}\left(\mathbb{E}|S(s)|^p+\mathbb{E}|S(s-\tau)|^p\right)^{1/p} \\
&\qquad\times\left(\mathbb{E}\left[1+|X_{t,\phi}(s)|^{p(\gamma-1)}+|X_{t,\phi}(s-\tau)|^{p(\gamma-1)}+|X_{t,\psi}(s)|^{p(\gamma-1)}+|X_{t,\psi}(s-\tau))|^{p(\gamma-1)}\right]\right)^{1/p}ds \\
&\leq K\left(|\phi(t)-\psi(t)|^2+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)\left(1+|\phi(t)|^{\gamma-1}+|\psi(t)|^{\gamma-1}+|\phi(t-\tau)|^{\gamma-1}+|\psi(t-\tau)|^{\gamma-1}\right) \\
&\qquad\cdot\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}ds.
\end{aligned}
$$

Hence, we derive

$$
\begin{aligned}
\mathbb{E}|Z(t+\theta)|^{2p} \leq\ & K\int_t^{t+\theta}\mathbb{E}|Z(s)|^{2p}ds+K\theta(1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right) \\
&+K\left(|\phi(t)-\psi(t)|^2+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right) \\
&\cdot\left(1+|\phi(t)|^{\gamma-1}+|\psi(t)|^{\gamma-1}+|\phi(t-\tau)|^{\gamma-1}+|\psi(t-\tau)|^{\gamma-1}\right)\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}ds.
\end{aligned}
$$

Using the Young inequality and the Gronwall inequality, we get (63).

We claim that for $n \geq 0$, if $t+h \in [n\tau,(n+1)\tau]$, (62) and (63) hold. Applying induction, we need to show that (62) and (63) still hold for $n+1$.

If $t+h \in [(n+1)\tau,(n+2)\tau]$, one divides the following proof into three cases.

Case (I) If $(n+1)\tau \leq t \leq s \leq t+h \leq (n+2)\tau$, we have $s-\tau \in [n\tau,(n+1)\tau]$.

Case (II) If $n\tau \leq t \leq (n+1)\tau \leq s \leq t+h \leq (n+2)\tau$, we have $s-\tau \in [n\tau,(n+1)\tau]$.

Case (III) If $n\tau \leq t \leq s \leq (n+1)\tau \leq t+h \leq (n+2)\tau$, we have $s-\tau \in [(n-1)\tau, n\tau]$.

Using the Itô formula, (**A2**), (58), (66), (72), Remark 3.2 and the Young inequality, for $\theta \geq 0$, one obtains

$$
\begin{aligned}
\mathbb{E}|V(t+\theta)|^{2p} \leq\ & \mathbb{E}|V(t)|^{2p}+(2p-2)K_2\mathbb{E}\int_t^{t+\theta}|V(s)|^{2p}ds \\
&+2K_2\mathbb{E}\int_t^{t+\theta}|S(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s-\tau)|^{2p}ds \\
\leq\ & (1-\kappa)^{1-2p}|\phi(t)-\psi(t)|^{2p}+\kappa|\phi(t-\tau)-\psi(t-\tau)|^{2p}+K\mathbb{E}\int_t^{t+\theta}|V(s)|^{2p}ds \\
&+K\theta|\phi-\psi|^{2p}.
\end{aligned}
$$

Applying the Gronwall inequality, we has

$$
\mathbb{E}|V(t+h)|^{2p} \leq (1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right).
$$

Hence, using (72), (74), one gets (62).

By the Itô formula, (58), (60), (64), (65) and (**A2**), for $\theta \geq 0$, one has

$$
\begin{aligned}
\mathbb{E}|Z(t+\theta)|^{2p} \leq\ & 2pK_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(|S(s)|^2+|S(s-\tau)|^2\right)ds \\
&-2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}[\phi(t)-\psi(t)]^T\Delta F_{\phi,\psi}(s)ds.
\end{aligned}
$$

Using (66), (72), (76) and the Young inequality, we derive

$$
\begin{aligned}
&2pK_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}\left(|S(s)|^2+|S(s-\tau)|^2\right)ds \\
\leq\ & 2(2p-2)K_2\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s)|^{2p}ds+2K_2\mathbb{E}\int_t^{t+\theta}|S(s-\tau)|^{2p}ds
\end{aligned}
$$

$$\leq \quad 2(2p-2)K_2\int_t^{t+\theta}\mathbb{E}|Z(s)|^{2p}ds + K\theta(1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right).$$

Applying (64), (**A2**), the Hölder inequality and Lemma 4.6, one obtains

$$\begin{aligned}&-2p\mathbb{E}\int_t^{t+\theta}|Z(s)|^{2p-2}(V(t))^T\Delta F_{t,\phi,\psi}(s)ds\\ \leq\ & K\left(|\phi(t)-\psi(t)|^2+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)\\ &\quad\cdot\left(1+|\phi(t)|^{\gamma-1}+|\psi(t)|^{\gamma-1}+|\phi(t-\tau)|^{\gamma-1}+|\psi(t-\tau)|^{\gamma-1}\right)\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}ds.\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathbb{E}|Z(t+\theta)|^{2p} \leq\ & K\int_t^{t+\theta}\mathbb{E}|Z(s)|^{2p}ds + K\theta(1+Kh)\left(|\phi(t)-\psi(t)|^{2p}+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)\\ &+K\left(|\phi(t)-\psi(t)|^2+|\phi(t-\tau)-\psi(t-\tau)|^{2p}\right)\\ &\cdot\left(1+|\phi(t)|^{\gamma-1}+|\psi(t)|^{\gamma-1}+|\phi(t-\tau)|^{\gamma-1}+|\psi(t-\tau)|^{\gamma-1}\right)\int_t^{t+\theta}\left(\mathbb{E}|Z(s)|^{2p}\right)^{(p-1)/p}ds.\end{aligned}$$

Using the Young inequality and the Gronwall inequality, we get (63).

That is, (61), (62) and (63) hold for any n. The proof is complete. □

The following theorem follows from Lemma 4.1, Theorem 4.7 and Lemma 4.8, and we can see the balanced Euler method (9) has a mean-square convergence order half.

Theorem 4.9. *Suppose* (**A1**) – (**A4**) *hold with sufficiently large* p_0. *For* $1 \leq p \leq \left(\frac{p_0-1}{2\gamma}\right)\wedge\left(\frac{p_0-1}{2(2\gamma-1)}-\frac{1}{2}\right)\wedge\left(\frac{p_0-1}{12(\gamma-1)}-\frac{1}{2}\right)$ *and* $n=0,1,\cdots,N$, *we have*

$$\mathbb{E}|X_{0,\xi}(t_n)-\bar{Y}_{0,\xi}(t_n)|^{2p}\leq K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^p. \tag{77}$$

Proof. For $n=0,1,\cdots,N-1$, define

$$r_{n+1} := X_{t_n,\psi}(t_{n+1})-\bar{Y}_{t_n,\psi}(t_{n+1}), \tag{78}$$

$$r'_{n+1} := D(X_{t_n,\psi}(t_{n+1-M}),r(t_{n+1}))-D(\bar{Y}_{t_n,\psi}(t_{n+1-M}),r(t_{n+1})), \tag{79}$$

$$\begin{aligned}R_{n+1} :=& \left[X_{t_n,\psi}(t_{n+1})-D(X_{t_n,\psi}(t_{n+1-M}),r(t_{n+1}))\right]-\left[\bar{Y}_{t_n,\psi}(t_{n+1})-D(\bar{Y}_{t_n,\psi}(t_{n+1-M}),r(t_{n+1}))\right]\\ :=& r_{n+1}-r'_{n+1},\end{aligned} \tag{80}$$

Noting (58), we define

$$S_{n+1} \quad := \quad S_{t_n,\phi,\psi}(t_{n+1}) := X_{t_n,\phi}(t_{n+1})-X_{t_n,\psi}(t_{n+1}), \tag{81}$$

By (54), define

$$\begin{aligned}\rho_{n+1} &:= X_{0,\xi}(t_{n+1})-\bar{Y}_{0,\xi}(t_{n+1})\\ &= X_{t_n,\phi}(t_{n+1})-\bar{Y}_{t_n,\psi}(t_{n+1})\\ &= \left[X_{t_n,\phi}(t_{n+1})-X_{t_n,\psi}(t_{n+1})\right]+\left[X_{t_n,\psi}(t_{n+1})-\bar{Y}_{t_n,\psi}(t_{n+1})\right]\\ &:= S_{n+1}+r_{n+1}.\end{aligned} \tag{82}$$

Using (57), we obtain

$$\Delta D_{n+1} := \Delta D_{t_n,\phi,\psi}(t_{n+1}) = D(X_{t_n,\phi}(t_{n+1-M}), r(t_{n+1})) - D(X_{t_n,\psi}(t_{n+1-M}), r(t_{n+1})). \tag{83}$$

Recalling (60), one has

$$Z_{n+1} := Z_{t_n,\phi,\psi}(t_{n+1}) = S_{n+1} - \rho_n - \Delta D_{n+1} + \Delta D_n.$$

Consequently, one has

$$S_{n+1} = \rho_n + Z_{n+1} + \Delta D_{n+1} - \Delta D_n. \tag{84}$$

Applying (59), we define

$$V_n := \rho_n - \Delta D_n \tag{85}$$

and

$$\begin{aligned} V_{n+1} :=& V_{t_n,\phi,\psi}(t_{n+1}) \\ :=& X_{t_n,\phi}(t_{n+1}) - D(X_{t_n,\phi}(t_{n+1-M}), r(t_{n+1})) - X_{t_n,\psi}(t_{n+1}) + D(X_{t_n,\psi}(t_{n+1-M}), r(t_{n+1})) \\ =& S_{n+1} - \Delta D_{n+1} \\ =& V_n + Z_{n+1}. \end{aligned} \tag{86}$$

Hence, one gets

$$V_{n+1} := \rho_n - \Delta D_n + Z_{n+1}. \tag{87}$$

Then, we define

$$\begin{aligned} Q_{n+1} :=& [X_{0,\xi}(t_{n+1}) - D(X_{0,\xi}(t_{n+1-M}), r(t_{n+1}))] - [\bar{Y}_{0,\xi}(t_{n+1}) - D(\bar{Y}_{0,\xi}(t_{n+1-M}), r(t_{n+1}))] \\ =& \left[X_{t_n,\phi}(t_{n+1}) - D(X_{t_n,\phi}(t_{n+1-M}), r(t_{n+1}))\right] - \left[\bar{Y}_{t_n,\psi}(t_{n+1}) - D(\bar{Y}_{t_n,\psi}(t_{n+1-M}), r(t_{n+1}))\right] \\ :=& \rho_{n+1} - r'_{n+1} \\ =& \left[X_{t_n,\phi}(t_{n+1}) - D(X_{t_n,\phi}(t_{n+1-M}), r(t_{n+1}))\right] - \left[X_{t_n,\psi}(t_{n+1}) - D(X_{t_n,\psi}(t_{n+1-M}), r(t_{n+1}))\right] \\ &+ \left[X_{t_n,\psi}(t_{n+1}) - D(X_{t_n,\psi}(t_{n+1-M}), r(t_{n+1}))\right] - \left[\bar{Y}_{t_n,\psi}(t_{n+1}) - D(\bar{Y}_{t_n,\psi}(t_{n+1-M}), r(t_{n+1}))\right] \\ :=& V_{n+1} + R_{n+1}. \end{aligned} \tag{88}$$

If $n \in [0, M-1]$, we have $\rho_{n-M} = 0$, $\Delta D_n = 0$ and $\Delta D_{n+1} = 0$. Hence, by (86) and (87), one gets

$$V_{n+1} = S_{n+1} = \rho_n + Z_{n+1}. \tag{89}$$

By (89) and (82), we get

$$\begin{aligned} \mathbb{E}|\rho_{n+1}|^{2p} &= \mathbb{E}|S_{n+1} + r_{n+1}|^{2p} \\ &= \mathbb{E}\left[\langle S_{n+1}, S_{n+1}\rangle + 2\langle S_{n+1}, r_{n+1}\rangle + \langle r_{n+1}, r_{n+1}\rangle\right]^p \\ &\le \mathbb{E}|S_{n+1}|^{2p} + 2p\mathbb{E}\left[|S_{n+1}|^{2p-2}\langle \rho_n + Z_{n+1}, r_{n+1}\rangle\right] + K\sum_{l=2}^{2p} \mathbb{E}\left[|S_{n+1}|^{2p-l}|r_{n+1}|^l\right]. \end{aligned} \tag{90}$$

Using Lemma 4.8, we have

$$\begin{aligned} \mathbb{E}|S_{n+1}|^{2p} &\le \left(\mathbb{E}|\rho_n|^{2p} + \mathbb{E}|\rho_{n-M}|^{2p}\right)(1 + Kh) \\ &= \mathbb{E}|\rho_n|^{2p}(1 + Kh). \end{aligned} \tag{91}$$

By the elementary inequality, one gets

$$\mathbb{E}\left[|S_{n+1}|^{2p-2}\langle \rho_n + Z_{n+1}, r_{n+1}\rangle\right]$$

$$= \mathbb{E}\left[|\rho_n|^{2p-2}\langle\rho_n, r_{n+1}\rangle\right] + \mathbb{E}\left[|S_{n+1}|^{2p-2}\langle Z_{n+1}, r_{n+1}\rangle\right] + \mathbb{E}\left[[|S_{n+1}|^{2p-2} - |\rho_n|^{2p-2}]\langle\rho_n, r_{n+1}\rangle\right]. \quad (92)$$

See from (78) and Theorem 4.7, for $l \geq 2$, we have

$$\mathbb{E}|r_{n+1}|^l \leq Kh^{l/2+1}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3l\gamma}\right) \quad (93)$$

and

$$\mathbb{E}|r_{n+1}| \leq Kh^{3/2}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma}\right). \quad (94)$$

Due to $\mathcal{F}_{t_n}$-measurability of ρ_n, by the elementary inequality, the Young inequality and (94), one gets

$$\begin{aligned}\mathbb{E}\left[|\rho_n|^{2p-2}\langle\rho_n, r_{n+1}\rangle\right] &\leq \mathbb{E}\left[|\rho_n|^{2p-1}\mathbb{E}\left(|r_{n+1}||\mathcal{F}_{t_n}\right)\right]\\ &\leq K\mathbb{E}\Big[|\rho_n|^{2p-1}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma}\right)\Big]h^{3/2}\\ &\leq Kh\mathbb{E}|\rho_n|^{2p} + K(1 + \mathbb{E}|\xi|^{6p\beta\gamma})h^{p+1}, \end{aligned} \quad (95)$$

By Lemma 2.3, Lemma 4.1, Lemma 4.8, (89), (94) and the Young inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\left(|S_{n+1}|^{2p-2} - |\rho_n|^{2p-2}\right)\langle\rho_n, r_{n+1}\rangle\right]\\ \leq&\mathbb{E}\left[|Z_{n+1}||\rho_n||r_{n+1}|\sum_{l=0}^{2p-3}|S_{n+1}|^{2p-3-l}|\rho_n|^l\right]\\ \leq&K\sum_{l=0}^{2p-3}\mathbb{E}\Big[|\rho_n|^{2p-1}\left(1 + |X_{0,\xi}(t_{n+1})|^{\gamma-1} + |\bar{Y}_{0,\xi}(t_n)|^{\gamma-1} + |X_{0,\xi}(t_{n-M})|^{\gamma-1} + |\bar{Y}_{0,\xi}(t_{n-M})|^{\gamma-1}\right)^{(2p-2-l)/2}\\ &\qquad\cdot(1 + |\bar{Y}_{0,\xi}(t_n)|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma})h^{3/2}\Big]\\ \leq&Kh\mathbb{E}|\rho_n|^{2p} + K\left(1 + \mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1}, \end{aligned} \quad (96)$$

$$\begin{aligned}&\mathbb{E}|S_{n+1}|^{2p-2}\langle Z_{n+1}, r_{n+1}\rangle\\ \leq&\mathbb{E}\left[\mathbb{E}\left(|S_{n+1}|^{4p-4}|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|Z_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}\mathbb{E}\left(|r_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}\right]\\ \leq&K\mathbb{E}\Big[|\rho_n|^{2p-1}\left(1 + |X_{0,\xi}(t_n)|^{2(\gamma-1)} + |\bar{Y}_{0,\xi}(t_n)|^{2(\gamma-1)} + |X_{0,\xi}(t_{n-M})|^{2(\gamma-1)} + |\bar{Y}_{0,\xi}(t_{n-M})|^{2(\gamma-1)}\right)^{1/4}\\ &\qquad\cdot(1 + |\bar{Y}_{0,\xi}(t_n)|^{12\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{12\gamma})^{1/4}h^{3/2}\Big]\\ \leq&Kh\mathbb{E}|\rho_n|^{2p} + K\left(1 + \mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1} \end{aligned} \quad (97)$$

and

$$\begin{aligned}K\sum_{l=2}^{2p}\mathbb{E}|S_{n+1}|^{2p-l}|r_{n+1}|^l &\leq K\sum_{l=2}^{2p}\mathbb{E}\left[\mathbb{E}\left(|S_{n+1}|^{4p-2l}|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|r_{n+1}|^{2l}|\mathcal{F}_{t_n}\right)^{1/2}\right]\\ &\leq K\sum_{l=2}^{2p}\mathbb{E}\left[|\rho_n|^{2p-l}(1 + Kh)(1 + |\bar{Y}_{0,\xi}(t_n)|^{6l\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{6l\gamma})^{1/2}h^{l/2+1}\right]\\ &\leq K\mathbb{E}h|\rho_n|^{2p} + K\left(1 + \mathbb{E}|\xi|^{6p\beta\gamma}\right)h^{p+1}. \end{aligned} \quad (98)$$

Substituting (91), (95)-(98) into (90), we have

$$\mathbb{E}|\rho_{n+1}|^{2p} \leq \mathbb{E}|\rho_n|^{2p} + Kh\mathbb{E}|\rho_n|^{2p} + K\left(1 + \mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1}.$$

Applying the Gronwall inequality, we get (77).

We claim that for $k \geq 0$, if $n \in [kM, (k+1)M - 1]$, (77) holds. By induction, we need to show that (77) still holds for $k+1$.

If $n \in [(k+1)M, (k+2)M - 1]$, for $l \geq 1$, applying (**A4**), (79) and (93), one gets

$$\begin{aligned}\mathbb{E}|r'_{n+1}|^l &\leq \kappa^l \mathbb{E}|X_{t_n,\psi}(t_{n+1-M}) - \bar{Y}_{t_n,\psi}(t_{n+1-M})|^l \\ &= \kappa^l \mathbb{E}|r_{n+1-M}|^l \\ &\leq Kh^{l/2+1}\left(1 + |\bar{Y}_{0,\xi}(t_{n-M})|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-2M})|^{3l\gamma}\right). \end{aligned} \tag{99}$$

By (85), (99), and (3), one gets

$$\begin{aligned}\mathbb{E}|V_n|^l \leq& (1-\kappa)^{1-l}\mathbb{E}|\rho_n|^l + \kappa^{1-l}E|\Delta D_n|^l \\ \leq& (1-\kappa)^{1-l}\mathbb{E}|\rho_n|^l + \kappa\mathbb{E}|S_{n-M}|^l. \end{aligned} \tag{100}$$

Using (3), (80), (93) and (99), we have

$$\begin{aligned}\mathbb{E}|R_{n+1}|^l \leq& (1-\kappa)^{1-l}\mathbb{E}|r_{n+1}|^l + \kappa^{1-l}\mathbb{E}|r'_{n+1}|^l \\ \leq& Kh^{l/2+1}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3l\gamma}\right) + Kh^{l/2+1}\left(1 + |\bar{Y}_{0,\xi}(t_{n-M})|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-2M})|^{3l\gamma}\right) \\ \leq& Kh^{l/2+1}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3l\gamma} + |\bar{Y}_{0,\xi}(t_{n-2M})|^{3l\gamma}\right). \end{aligned} \tag{101}$$

By (80), (94), and (99), one gets

$$\begin{aligned}\mathbb{E}|R_{n+1}| \leq& \mathbb{E}|r_{n+1}| + \mathbb{E}|r'_{n+1}| \\ \leq& Kh^{3/2}\left(1 + |\bar{Y}_{0,\xi}(t_n)|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-2M})|^{3\gamma}\right). \end{aligned} \tag{102}$$

By (81), (80), and (88), we get

$$\begin{aligned}\mathbb{E}|Q_{n+1}|^{2p} &= \mathbb{E}|V_{n+1} + R_{n+1}|^{2p} \\ &= \mathbb{E}\left[\langle V_{n+1}, V_{n+1}\rangle + 2\langle V_{n+1}, R_{n+1}\rangle + \langle R_{n+1}, R_{n+1}\rangle\right]^p \\ &\leq \mathbb{E}|V_{n+1}|^{2p} + 2p\mathbb{E}\left[|V_{n+1}|^{2p-2}\langle V_n + Z_{n+1}, R_{n+1}\rangle\right] + K\sum_{l=2}^{2p}\mathbb{E}\left[|V_{n+1}|^{2p-l}|R_{n+1}|^l\right]. \end{aligned} \tag{103}$$

Applying Lemma 4.8 and the elementary inequality, we have

$$\mathbb{E}|V_{n+1}|^{2p} \leq \left(E|\rho_n|^{2p} + E|\rho_{n-M}|^{2p}\right)(1 + Kh). \tag{104}$$

$$\begin{aligned}\mathbb{E}\left[|V_{n+1}|^{2p-2}\langle V_n + Z_{n+1}, R_{n+1}\rangle\right] =& \mathbb{E}\left[|V_n|^{2p-2}\langle V_n, R_{n+1}\rangle\right] + \mathbb{E}\left[|V_{n+1}|^{2p-2}\langle Z_{n+1}, R_{n+1}\rangle\right] \\ &+\mathbb{E}\left[[|V_{n+1}|^{2p-2} - |V_n|^{2p-2}]\langle V_n, R_{n+1}\rangle\right]. \end{aligned} \tag{105}$$

Due to $\mathcal{F}_{t_n}$-measurability of ρ_n, by the elementary inequality, the Young inequality, (101) and (99), one gets

$$\begin{aligned}\mathbb{E}\left[|V_n|^{2p-2}\langle V_n, R_{n+1}\rangle\right] &\leq \mathbb{E}\left[|V_n|^{2p-1}|\mathbb{E}|R_{n+1}||\mathcal{F}_{t_n}\right] \\ &\leq K\mathbb{E}\left(|V_n|^{2p-1}\left[1 + |\bar{Y}_{0,\xi}(t_n)|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma} + |\bar{Y}_{0,\xi}(t_{n-2M})|^{3\gamma}\right]h^{3/2}\right) \\ &\leq K\mathbb{E}|V_n|^{2p} + Kh\mathbb{E}|V_n|^{2p} + K\left(1 + \mathbb{E}|\xi|^{6p\beta\gamma}\right)h^{p+1}, \end{aligned} \tag{106}$$

$$\begin{aligned}&\mathbb{E}\left[\left(|V_{n+1}|^{2p-2} - |V_n|^{2p-2}\right)\langle V_n, R_{n+1}\rangle\right] \\ \leq& \mathbb{E}\left[|Z_{n+1}||V_n||R_{n+1}|\sum_{l=0}^{2p-3}|V_{n+1}|^{2p-3-l}|V_n|^l\right]\end{aligned}$$

$$\leq K\sum_{l=0}^{2p-3}\mathbb{E}\Big[|V_n|^{2p-1}\left(1+|X_{0,\xi}(t_n)|^{\gamma-1}+|\bar{Y}_{0,\xi}(t_n)|^{\gamma-1}+|X_{0,\xi}(t_{n-M})|^{\gamma-1}+|\bar{Y}_{0,\xi}(t_{n-M})|^{\gamma-1}\right)$$
$$\cdot\left(1+|\bar{Y}_{0,\xi}(t_n)|^{3\gamma}+|\bar{Y}_{0,\xi}(t_{n-M})|^{3\gamma}+|\bar{Y}_{0,\xi}(t_{n-2M})|^{3\gamma}\right)h^{3/2}\Big]$$
$$\leq K\mathbb{E}|V_n|^{2p}+Kh\mathbb{E}|V_n|^{2p}+K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1}, \tag{107}$$

$$\mathbb{E}|V_{n+1}|^{2p-2}\langle Z_{n+1},R_{n+1}\rangle$$
$$\leq\mathbb{E}\left[\mathbb{E}\left(|V_{n+1}|^{4p-4}|\mathcal{F}_{t_n}\right)^{1/2}E\left(|Z_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}E\left(|R_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}\right]$$
$$\leq K\mathbb{E}\Big[|V_n|^{2p-1}\left(1+|X_{0,\xi}(t_{n+1})|^{2(\gamma-1)}+|\bar{Y}_{0,\xi}(t_n)|^{2(\gamma-1)}+|X_{0,\xi}(t_{n-M})|^{2(\gamma-1)}+|\bar{Y}_{0,\xi}(t_{n-M})|^{2(\gamma-1)}\right)^{1/4}$$
$$\cdot\left(1+|\bar{Y}_{0,\xi}(t_n)|^{12\gamma}+|\bar{Y}_{0,\xi}(t_{n-M})|^{12\gamma}+|\bar{Y}_{0,\xi}(t_{n-2M})|^{12\gamma}\right)^{1/4}h^{3/2}\Big]$$
$$\leq K\mathbb{E}|V_n|^{2p}+Kh\mathbb{E}|V_n|^{2p}+K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1} \tag{108}$$

and

$$K\sum_{l=2}^{2p}\mathbb{E}|V_{n+1}|^{2p-l}|R_{n+1}|^l\leq K\sum_{l=2}^{2p}\mathbb{E}\left[\mathbb{E}\left(|V_{n+1}|^{4p-2l}|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|R_{n+1}|^{2l}|\mathcal{F}_{t_n}\right)^{1/2}\right]$$
$$\leq K\sum_{l=2}^{2p}\mathbb{E}\Big[|V_n|^{2p-l}(1+Kh)$$
$$\cdot\left(1+|\bar{Y}_{0,\xi}(t_n)|^{6l\gamma}+|\bar{Y}_{0,\xi}(t_{n-M})|^{6l\gamma}+|\bar{Y}_{0,\xi}(t_{n-2M})|^{6l\gamma}\right)^{1/2}h^{(l+1)/2}\Big]$$
$$\leq K\mathbb{E}|V_n|^{2p}+K\mathbb{E}h|V_n|^{2p}+K\left(1+\mathbb{E}|\xi|^{6p\beta\gamma}\right)h^{p+1}. \tag{109}$$

Substituting (104)-(109) into (103), we have

$$\mathbb{E}|Q_{n+1}|^{2p}\leq K\left(\mathbb{E}|\rho_n|^{2p}+\mathbb{E}|\rho_{n+1-M}|^{2p}\right)(1+Kh)+K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1}. \tag{110}$$

By (104) and (110), one gets

$$\begin{aligned}\mathbb{E}|Q_{n+1}|^{2p}&\leq K\left(\mathbb{E}|\rho_n|^{2p}+\mathbb{E}|\rho_{n+1-M}|^{2p}\right)(1+Kh)+K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^{p+1}\\&\leq K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^p.\end{aligned} \tag{111}$$

Applying (88), (101) and (111), one obtains

$$\begin{aligned}\mathbb{E}|V_{n+1}|^{2p}&\leq(1-\kappa)^{1-2p}\mathbb{E}|Q_{n+1}|^{2p}+\kappa^{1-2p}\mathbb{E}|R_{n+1}|^{2p}\\&\leq K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^p.\end{aligned} \tag{112}$$

Using (3) , (88), (99) and (111), we has

$$\begin{aligned}\mathbb{E}|\rho_{n+1}|^{2p}&\leq(1-\kappa)^{1-2p}\mathbb{E}|Q_{n+1}|^{2p}+\kappa^{1-2p}\mathbb{E}|r'_{n+1}|^{2p}\\&\leq(1-\kappa)^{1-2p}\mathbb{E}|V_{n+1}|^{2p}+\kappa\mathbb{E}|S_{n+1-M}|^{2p}\\&\leq K\left(1+\mathbb{E}|\xi|^{p\beta(7\gamma-1)}\right)h^p.\end{aligned} \tag{113}$$

Hence, one proves that (77) holds for any k. □

5. Numerical experiments

In this section, to illustrate the theoretical results, we present two numerical examples.

We use discrete Brownian paths over $[0,1]$ with $\Delta = 2^{-12}$. We take the numerical solution with $h = \Delta$ to be a approximation X_Δ of the exact solution and compare this with the numerical approximation using $h = 2^6\Delta$, $h = 2^7\Delta$, $h = 2^8\Delta$ and $h = 2^9\Delta$ over $M = 500$ sample paths. Here the mean-square error is denoted as follows:

$$Error_h := \left(\frac{1}{M} \sum_{i=1}^{M} \left| Y_h^i(T) - X_\Delta^i(T) \right|^2 \right)^{1/2} \tag{114}$$

where $Y_h^i(T)$ denotes the numerical solution along the ith sample path at $t = T$ with step size h, and the strong convergence order is defined numerically by

$$Order = \log \frac{Error_h}{Error_{h/2}} / \log(2).$$

Example 5.1. *Consider the NSDDE-MS*

$$d[x(t) - 0.1\sin(x(t-1))] \quad = \quad -a(r(t))x^3(t)dt + [x(t) - 0.1x(t-1)]dw(t), \tag{115}$$

on $t \ge 0$ with initial data $\{y(t) = 1 : -\tau \le t \le 0\}$. where $w(t)$ is a 1-dimension Brownian motion, $r(t)$ is a Markovian chain on the state space $S = \{1,2\}$ and they are independent. Let the generator of Markovain chain that

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \tag{116}$$

Moreover, $a(1) = 2$, $a(2) = 1$.

Here $D(y,i) = 0.1\sin y$, $F(x,y,i) = -a(r(t))x^3$ and $G(x,y,i) = x - 0.1y$. It is obvious that (A1), (A3) and (A4) are satisfied.

If $a(1) = 2$, by the elementary inequality (3), for $c = 1$, and the Young inequality, we have

$$\begin{aligned}
& [x - D(y,i) - \bar{x} - D(\bar{y},i)]^T[F(x,y,i) - F(\bar{x},\bar{y},i)] + \frac{2p_0 - 1}{2}|G(x,y,i) - G(\bar{x},\bar{y},i)|^2 \\
= \quad & [x - 0.1\sin y - (\bar{x} - 0.1\sin\bar{y})]^T 2\left[-x^3 + \bar{x}^3\right] + \frac{2p_0 - 1}{2}|x - 0.1y - \bar{x} + 0.1\bar{y}|^2 \\
\le \quad & 2|x-\bar{x}|^2\left[-x^2 - \bar{x}^2 - x\bar{x}\right] + 2|-0.1\sin y + 0.1\sin\bar{y}||-x^3 + \bar{x}^3| \\
& +(2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2\left[-x^2 - \bar{x}^2 - x\bar{x}\right] + 0.4|x-\bar{x}||x^2 + \bar{x}^2 + x\bar{x}| \\
& +(2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2\left[-x^2 - \bar{x}^2 - x\bar{x}\right] + 0.4|x-\bar{x}||(x-\bar{x})^2 + 3x\bar{x}| \\
& +(2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2\left[-x^2 - \bar{x}^2 - x\bar{x}\right] + 0.4|x-\bar{x}||(x-\bar{x})^2 + \frac{3}{2}(x-\bar{x})^2| \\
& +(2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2\left[-x^2 - \bar{x}^2 - x\bar{x} + |x-\bar{x}|\right] + (2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2\left[-0.5x^2 - 0.5\bar{x}^2 + |x| + |\bar{x}|\right] + (2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right) \\
\le \quad & 2|x-\bar{x}|^2 + (2p_0 - 1)\left(|x-\bar{x}|^2 + 0.01|y-\bar{y}|^2\right)
\end{aligned}$$

$$\leq \quad (2p_0 + 1)\left(|x - \bar{x}|^2 + |y - \bar{y}|^2\right).$$

Similarly, we can prove that (**A2**) is also fulfilled if $a(2) = 1$.

Next we define the balanced Euler method

$$\begin{cases} Y(t_{n+1}) = 1, \quad n = -M-1, -M, \cdots, -1, \\ Y(t_{n+1}) = Y(t_n) + 0.1\sin(Y(t_{n+1-M})) - 0.1\sin(Y(t_{n-M})) \\ \qquad + \sin\left(-a(r_n)Y^3(t_n)h\right) + \sin([Y(t_n) - 0.1Y(t_{n-M})]\Delta w_n), \quad n = 0, 1, \cdots, N-1. \end{cases}$$

Table 1: Mean-square convergence order of balanced Euler method for NSDDE-MS (115)

balanced Euler		
stepsize	Error	order
$2^5\Delta t$	0.1267	-
$2^6\Delta t$	0.1579	0.3174
$2^7\Delta t$	0.2064	0.3862
$2^8\Delta t$	0.2751	0.4146

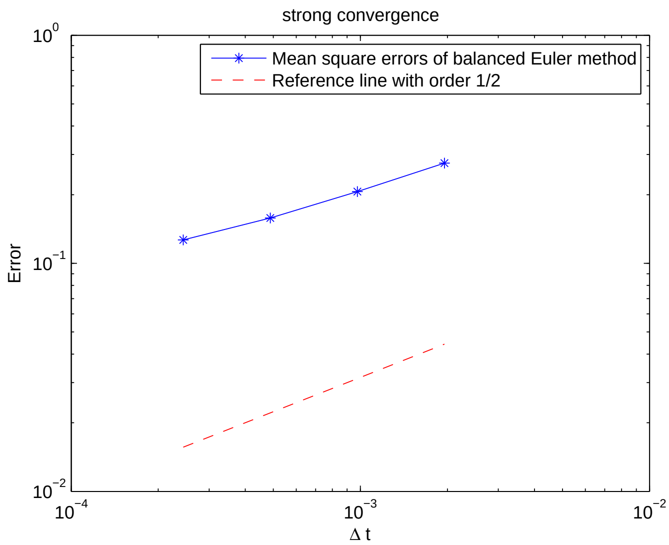

Figure 1: Strong convergence of the balanced Euler method for NSDDE-MS (115).

The strong convergence results of the balanced Euler method for Example 5.1 are shown in Table 1 and Figure 1. From this table, we can see that the smaller the step size, the smaller the error, we also found that the balanced Euler method of NSDDE-MS is convergent of order 1/2.

Example 5.2. *Consider the NSDDE-MS*

$$\begin{aligned} d(x(t) - 0.1x(t-1)) &= a(r(t))\left[x(t) - 0.1x(t-1) - (x(t) - 0.1x(t-1))^3\right]dt \\ &\quad + b(r(t))|x(t) - 0.1x(t-1)|^{3/2}dw(t), \end{aligned} \tag{117}$$

on $t \geq 0$ with initial data $\{y(t) = 1 : -\tau \leq t \leq 0\}$, where $w(t)$ is a 1-dimension Brownian motion, $r(t)$ is a Markovian chain on the state space $S = \{1, 2\}$ and they are independent. Let the generator of Markovain chain that

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \tag{118}$$

Moreover, $a(1) = 2$, $b(1) = 1$, $a(2) = 1$ *and* $b(2) = 2$.

Here $D(y,i) = 0.1y$, $F(x,y,i) = a(r(t))[x - 0.1y - (x - 0.1y)^3]$ and $G(x,y,i) = b(r(t))|x - 0.1y|^{3/2}$.

If $a(1) = 2$ and $b(1) = 1$, using the elementary inequality and the Young inequality, it is easy to check that

$$\begin{aligned}
& |F(x,y,i) - F(\bar{x},\bar{y},i)| \\
= \ & 2\left|x - 0.1y - (x - 0.1y)^3 - (\bar{x} - 0.1\bar{y}) + (\bar{x} - 0.1\bar{y})^3\right| \\
\le \ & 2\left|1 - (x - 0.1y)^2 - (\bar{x} - 0.1\bar{y})^2 - (x - 0.1y)(\bar{x} - 0.1\bar{y})\right| (|x - \bar{x}| + 0.1|y - \bar{y}|) \\
\le \ & 2\left|1 - \frac{1}{2}(x - 0.1y)^2 - \frac{1}{2}(\bar{x} - 0.1\bar{y})^2\right| (|x - \bar{x}| + 0.1|y - \bar{y}|) \\
\le \ & 2\left(1 + |x|^2 + 0.01|y|^2 + |\bar{x}|^2 + 0.01|\bar{y}|^2\right)(|x - \bar{x}| + 0.1|y - \bar{y}|) \\
\le \ & 2\left(1 + |x|^2 + |y|^2 + |\bar{x}|^2 + |\bar{y}|^2\right)(|x - \bar{x}| + |y - \bar{y}|)
\end{aligned}$$

and

$$\begin{aligned}
& |G(x,y,i) - G(\bar{x},\bar{y},i)| = \left||x - 0.1y|^{3/2} - |\bar{x} - 0.1\bar{y}|^{3/2}\right| \\
\le \ & ||x - 0.1y|^{1/2} - |\bar{x} - 0.1\bar{y}|^{1/2}|\left(|x - 0.1y| + |\bar{x} - 0.1\bar{y}| + |x - 0.1y|^{1/2}|\bar{x} - 0.1\bar{y}|^{1/2}\right) \\
\le \ & ||x - 0.1y|^{1/2} - |\bar{x} - 0.1\bar{y}|^{1/2}|\left(|x - 0.1y|^{1/2} + |\bar{x} - 0.1\bar{y}|^{1/2}\right)^2 \\
\le \ & ||x - 0.1y| - |\bar{x} - 0.1\bar{y}||\left(|x - 0.1y|^{1/2} + |\bar{x} - 0.1\bar{y}|^{1/2}\right) \\
\le \ & \left(|x|^{1/2} + |y|^{1/2} + |\bar{x}|^{1/2} + |\bar{y}|^{1/2}\right)|x - 0.1y - \bar{x} + 0.1\bar{y}| \\
\le \ & \left(|x|^{1/2} + |y|^{1/2} + |\bar{x}|^{1/2} + |\bar{y}|^{1/2}\right)(|x - \bar{x}| + |y - \bar{y}|) \\
\le \ & 3\left(1 + |x|^2 + |y|^2 + |\bar{x}|^2 + |\bar{y}|^2\right)(|x - \bar{x}| + |y - \bar{y}|).
\end{aligned}$$

Therefore, (**A1**) is satisfied with $\gamma = 3$.

Applying the elementary inequality and the Young inequality, one has

$$\begin{aligned}
& [x - D(y,i) - \bar{x} - D(\bar{y},i)]^T[F(x,y,i) - F(\bar{x},\bar{y},i)] + \frac{3(p_0 - 1)}{2}|G(x,y,i) - G(\bar{x},\bar{y},i)|^2 \\
= \ & [x - 0.1y - (\bar{x} - 0.1\bar{y})]^T 2\left[x - 0.1y - (x - 0.1y)^3 - (\bar{x} - 0.1\bar{y}) + (\bar{x} - 0.1\bar{y})^3\right] \\
& \qquad + \frac{3(p_0 - 1)}{2}\left||x - 0.1y|^{3/2} - |\bar{x} - 0.1\bar{y}|^{3/2}\right|^2 \\
\le \ & [x - 0.1y - (\bar{x} - 0.1\bar{y})]^2\left[2 - ((x - 0.1y)^2 + (\bar{x} - 0.1\bar{y})^2)\right] \\
& \qquad + \frac{3(p_0 - 1)}{2}\left||x - 0.1y| - |\bar{x} - 0.1\bar{y}|\right|^2\left(|x - 0.1y|^{1/2} + |\bar{x} - 0.1\bar{y}|^{1/2}\right)^2 \\
\le \ & [x - 0.1y - (\bar{x} - 0.1\bar{y})]^2\left(2 - \left[(x - 0.1y)^2 + (\bar{x} - 0.1\bar{y})^2\right] + 3(p_0 - 1)\left[|x - 0.1y| + |\bar{x} - 0.1\bar{y}|\right]\right) \\
\le \ & 2\left[2 + \frac{9}{8}(p_0 - 1)^2\right]\left(|x - \bar{x}|^2 + |y - \bar{y}|^2\right).
\end{aligned}$$

So (**A2**) is also fulfilled.

Similarly, we can prove that (**A1**) and (**A2**) are fulfilled if $a(2) = 1$ and $b(2) = 2$.

Now we define the balanced Euler method

$$\begin{cases}
Y(t_{n+1}) = 1, \quad n = -M - 1, -M, \cdots, -1, \\
Y(t_{n+1}) = Y(t_n) + 0.1Y(t_{n+1-M}) - 0.1Y(t_{n-M}) \\
\qquad + \sin\left(a(r_n)\left[Y(t_n) - 0.1Y(t_{n-M}) - (Y(t_n) - 0.1Y(t_{n-M}))^3\right]h\right) \\
\qquad + \sin\left(b(r_n)|Y(t_n) - 0.1Y(t_{n-M})|^{3/2}\Delta w_n\right), \quad n = 0, 1, \cdots, N - 1.
\end{cases}$$

Then we define the truncated EM method which was developed in [21]. Let $\phi(s) = 2s^3$ such that

$$\sup_{|x|\vee|y|\leq s} \left(|F(x,y,i)| \vee |F(x,y,i)|\right) = \sup_{|x|\vee|y|\leq s} \left(2|x|^3 \vee 2|x|^{3/2}\right) \leq 2s^3, \quad \forall s \geq 1.$$

Let $\psi(\Delta) = \Delta^{-1/10}$ for any $\Delta^* \in (0,1]$. Now we define the truncated functions as follows:

$$\begin{aligned} F_\Delta(x,y,i) =& F\left(\left(|x| \wedge \Delta^{-1/30}\right)\frac{x}{|x|}, \left(|y| \wedge \Delta^{-1/30}\right)\frac{y}{|y|}, i\right), \\ G_\Delta(x,y,i) =& G\left(\left(|x| \wedge \Delta^{-1/30}\right)\frac{x}{|x|}, \left(|y| \wedge \Delta^{-1/30}\right)\frac{y}{|y|}, i\right). \end{aligned}$$

Then, define the truncated EM method

$$\begin{cases} Y(t_{n+1}) = 1, \quad n = -M-1, -M, \cdots, -1, \\ Y(t_{n+1}) = Y(t_n) + +D(Y(t_{n+1-M}), r_n) - D(Y(t_{n-M}), r_n) \\ \qquad\qquad + F_\Delta(Y(t_n), Y(t_{n-M}), r_n)h + G_\Delta(Y(t_n), Y(t_{n-M}), r_n)\Delta w_n, \quad n = 0, 1, \cdots, N-1. \end{cases}$$

Table 2: Mean-square convergence order of balanced Euler method and truncated Euler-Maruyama method for NSDDE-MS (117)

	balanced Euler		truncated Euler-Maruyama	
stepsize	Error	order	Error	order
$2^5\Delta t$	0.1720	-	0.1495	-
$2^6\Delta t$	0.2102	0.2891	0.1886	0.3348
$2^7\Delta t$	0.2680	0.3507	0.2458	0.3819
$2^8\Delta t$	0.3680	0.4574	0.3204	0.3825

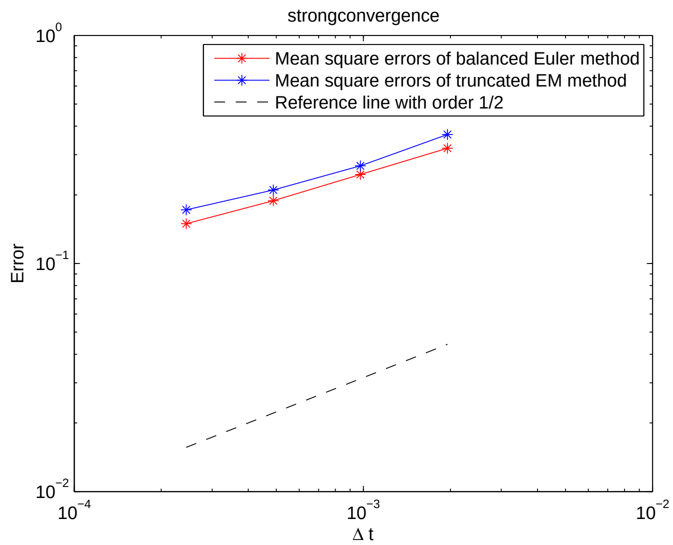

Figure 2: Mean-square convergence order of balanced Euler method and truncated Euler-Maruyama method for NSDDE-MS (117).

The strong convergence order of balanced Euler for NSDDE-MS (117) is shown in Table (2) and Figure (2). From this, we can see that mean-square convergence order of balanced Euler method for NSDDE-MS (117) is 1/2 while mean-square convergence order of truncated EM method for NSDDE-MS (117) is close to 1/2 which was considered in [21].

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