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Generalized Kantorovich type Szász-Mirakjan operators based on q-integers

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Abstract. In this paper, we introduce a new type Kantorovich variant of Szász-Mirakjan operators based on q-integers. We study convergence properties by using Korovkin's theorem and estimate the rate of convergence by using modulus of continuity. We examine local and weighted approximation properties in terms of modulus of continuity. We obtain a direct estimate in terms of Lipschitz type maximal function of order α. Moreover, we give a quantitative Voronovskaja type theorem for these newly defined operators. Finally, we present numerical results and graphical representation.

1. Introduction

One of the most important research areas in applied mathematics is approximation of functions by positive linear operators. In particular, it has an important place in application areas such as numerical analysis and computer aided geometric design (CAGD). In this context, Bernstein polynomials play a significant role in approximation theory. Due to the rapid development of q-calculus, many authors have introduced various generalizations of Bernstein polynomials involving q-integers. In [8], Bernstein obtained a formative proof of the Weierstrass approximation theorem using a probabilistic approach. In 1987, Lupaş [6] introduced the first generalization of Bernstein polynomials involving q-integers. In 1996, Phillips [7] introduced another generalization of q-Bernstein polynomials. In the following years, several generalizations of some positive linear operators based on q-integers have been introduced and approximation properties have been studied.

In 1950, for $f \in C[0,\infty)$ Szász [21] defined the operators

$$S_\eta(f;\tau) = \sum_{k=0}^{\infty} p_{\eta,k}(\tau) f\left(\frac{k}{\eta}\right) \qquad \tau \in [0,\infty),\ \eta = 1,2,...,$$

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where $p_{\eta,k}(\tau) = e^{-\eta\tau}\dfrac{(\eta\tau)^k}{k!}$.

Szász-Mirakjan operators are among the most important operators that can be used to approximate the functions on unbounded intervals. Since they have simple and useful structure, they have been intensively studied in the recent years. Many authors in this area introduced and studied different generalizations of classical Szász-Mirakjan operators and Szász-Mirakjan operators based on the q-integers (see [9], [14], [15], [16], [17], [10], [11], [12], [13], [18], [20], [22], [23], [26], [28], [29], [30], [31], [32]).

On the other hand, Kantorovich type Szász-Mirakjan operators are defined as follows:

$$K_\eta(f;\tau) = \sum_{k=0}^{\infty} e^{-\eta\tau}\frac{(\eta\tau)^k}{k!}\int_0^1 f\left(\frac{k+\nu}{\eta}\right)d\nu.$$

With the help of the l-th order integration formula and the definition of the Kantorovich variant Szász-Mirakjan operators, an l-th order Kantorovich variant Szász-Mirakjan operator is defined recently by Sabancigil, Kara and Mahmudov [19] as follows:

$$K_\eta^l(f;\tau) = \sum_{k=0}^{\infty} p_{\eta,k}(\tau)\int_0^1 \dots \int_0^1 f\left(\frac{k+\nu_1+\dots+\nu_l}{\eta+l}\right)d\nu_1\dots d\nu_l,$$

where $p_{\eta,k}(\tau) = e^{-\eta\tau}\frac{(\eta\tau)^k}{k!}$, $\eta \in \mathbb{N}$, $\tau \geq 0$, f is a real valued continuous function defined on $[0,\infty)$. In this article, we define a new type generalized Kantorovich variant of Szász-Mirakjan operators based on the q-integers.

The paper is organized as follows. In Section 2, after giving standard notations to be used throughout the paper, we introduce a new type generalized Kantorovich variant of Szász-Mirakjan operators based on the q-integers. We evaluate the moments. $K_{\eta,q}^l(\nu^m;\tau)$ for $m = 0,1,2,3,4$ and the central moments $K_{\eta,q}^l((\nu-\tau)^m;\tau)$ for $m = 1,2,4$. In Section 3, We give basic convergence results by using Korovkin's theorem and estimation of the rate of convergence by using modulus of continuity. In Section 4, we discuss some local approximation properties of these operators and state some theorems about local approximation. In Section 5, we investigate weighted approximation properties of the new Kantorovich variant of Szász-Mirakjan operators based on q-integers. In Section 6, we provide the quantitative Voronovskaja-type asymptotic formula. In Section 7, we give graphical representation of the operators and discuss for some special cases.

2. Operators and their moments

In this section, we will remind the main concepts and notations about quantum-calculus.

For $\eta \in \mathbb{N}$, q-integer and q-factorial coefficients are defined by,

$$[\eta]_q := \begin{cases} \dfrac{1-q^\eta}{1-q} & \text{if} \quad 0<q<1, \\ \eta & \text{if} \quad q=1 \end{cases}, \quad [0]_q = 0,$$

and

$$[\eta]_q! := \begin{cases} [\eta]_q[\eta-1]_q\dots[1]_q & \text{if} \quad \eta \in \mathbb{N}-\{0\} \\ 1 & \text{if} \quad \eta = 0 \end{cases}.$$

q-binomial coefficient is defined as

$$\begin{bmatrix} \eta \\ k \end{bmatrix}_q := \frac{[\eta]_q!}{[k]_q![\eta-k]_q!}, \quad (k,\eta \in \mathbb{Z}^+ \text{ and } 0 \leq k \leq \eta).$$

Exponential function e^τ based on the q-integers is defined as (see [24])

$$e_q(\tau)=\sum_{k=0}^{\infty}\frac{\tau^k}{[k]_q!}=\frac{1}{(1-(1-q)\tau)_q^{\infty}},\quad |\tau|<\frac{1}{1-q},\quad \left|q\right|<1.$$

An alternative definition of the q-exponential function is given by

$$E_q(\tau)=\sum_{k=0}^{\infty}\frac{\tau^k}{[k]_q!}q^{\frac{k(k-1)}{2}}=(1-(1-q)\tau)_q^{\infty},\qquad \left|q\right|<1.$$

Detailed information on q-calculus can be found in [24].

Let $B_m[0,\infty)=\left\{f:\left|f(\tau)\right|\leq M_f(1+\tau^m),\ \tau\in[0,\infty)\,,\ m>0 \text{ and } M_f \text{ is a constant depending on } f\right\}$,

$$C_m[0,\infty)=\left\{f\in B_m[0,\infty)\cap C[0,\infty):\left\|f\right\|_m:=\sup_{\tau\in[0,\infty)}\frac{\left|f(\tau)\right|}{1+\tau^m}<\infty\right\},$$
$$C_m^*[0,\infty)=\left\{f\in C_m[0,\infty):\lim_{\tau\to\infty}\frac{\left|f(\tau)\right|}{1+\tau^m}<\infty\right\}.$$

The spaces mentioned above are equipped with the following norm

$$\left\|f\right\|_m=\sup_{\tau\in[0,\infty)}\frac{\left|f(\tau)\right|}{1+\tau^m}.$$

In [14], N.I. Mahmudov introduced and studied the following Szász-Mirakjan operators based on the q-integers.

Lemma 2.1. *[14] Let $\eta\in\mathbb{N}$ and $0<q<1$. For $f:[0,\infty)\longrightarrow\mathbb{R}$, q-Szász-Mirakjan operators are defined by*

$$S_{\eta,q}(f;\tau)=\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-2}[\eta]_q}\right)s_{\eta,k}(q,\tau) \tag{1}$$

where $s_{\eta,k}(q,\tau)=q^{\frac{k(k-1)}{2}}\frac{[\eta]_q^k\tau^k}{[k]_q!}\frac{1}{E_q([\eta]_q\tau)}$.

Lemma 2.2. *[14] Let $0<q<1$, we have*

$$S_{\eta,q}(1;\tau)=1,$$
$$S_{\eta,q}(\nu;\tau)=q\tau,$$
$$S_{\eta,q}\left(\nu^2;\tau\right)=q\tau^2+\frac{q^2}{[\eta]_q}\tau,$$
$$S_{\eta,q}\left(\nu^3;\tau\right)=\tau^3+\frac{(2q^2+q)}{[\eta]_q}\tau^2+\frac{q^3}{[\eta]_q^2}\tau,$$
$$S_{\eta,q}\left(\nu^4;\tau\right)=\frac{\tau^4}{q^2}+(3q+2+\frac{1}{q})\frac{\tau^3}{[\eta]_q}+(3q^3+3q^2+q)\frac{\tau^2}{[\eta]_q^2}+\frac{q^4\tau}{[\eta]_q^3}.$$

Inspired by Mahmudov's operators $S_{\eta,q}(f;\tau)$, we intoduce a new type generalized Kantorovich kind of the Szász-Mirakjan operators as follows:

Definition 2.3. *Let $0 < q < 1$, $l \in \mathbb{Z}^+$ and $\eta \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, a new generalized Kantorovich variant of the Szász-Mirakjan operators is defined as follows:*

$$K^l_{\eta,q}(f;\tau) = \sum_{k=0}^{\infty} s_{\eta,k}(q,\tau) \int_0^1 \dots \int_0^1 f\left(\frac{q^{(1-k)}[k]_q + \nu_1 + \dots + \nu_l}{[\eta]_q}\right) d\nu_1 \dots d\nu_l, \tag{2}$$

where $s_{\eta,k}(q,\tau) = q^{\frac{k(k-1)}{2}} \frac{[\eta]_q^k \tau^k}{[k]_q!} \frac{1}{E_q([\eta]_q \tau)}$.

Note that, if we take $l = 1$ and $q = 1$, the operators return back to the classical Kantorovich type Szasz-Mirakjan operators $K^l_\eta(f;\tau)$.

Lemma 2.4. *We have the following two formulas:*

$$\left(\frac{q^{(1-k)}[k]_q + \nu_1 + \dots + \nu_l}{[\eta]_q}\right)^m = \sum_{j_0+\dots+j_l=m} \binom{m}{j_0,\dots,j_l} \frac{q^{(1-k)j_0}[k]_q^{j_0} \nu_1^{j_1} \dots \nu_l^{j_l}}{[\eta]_q^m}$$

and

$$\int_0^1 \dots \int_0^1 \frac{q^{(1-k)j_0}[k]_q^{j_0} \nu_1^{j_1} \dots \nu_l^{j_l}}{[\eta]_q^m} d\nu_1 \dots d\nu_l = \frac{q^{(1-k)j_0}[k]_q^{j_0}}{[\eta]_q^m (j_1+1)\dots(j_l+1)}.$$

In approximation theory the moments and central moments of the operators are very important in determining the rate of convergence. In the next lemma we derive a recurrence formula for $K^l_{\eta,q}(\nu^m;\tau)$ that will be used to find the moments and central moments of the operators $K^l_{\eta,q}(f;\tau)$.

Lemma 2.5. *Let $0 < q < 1$, $l \in \mathbb{Z}^+$, $m \in \mathbb{Z}^+ \cup \{0\}$ and $\eta \in \mathbb{N}$. For the operators $K^l_{\eta,q}(f;\tau)$, we have*

$$K^l_{\eta,q}(\nu^m;\tau) = \sum_{j_0+\dots+j_l=m} \binom{m}{j_0,\dots,j_l} \frac{[\eta]_q^{j_0}}{[\eta]_q^m q^{j_0}(j_1+1)\dots(j_l+1)} S_{\eta,q}(\nu^{j_0};\tau) \tag{3}$$

where

$$S_{\eta,q}(f;\tau) = \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-2}[\eta]_q}\right) s_{\eta,k}(q,\tau). \text{ (see [14]).}$$

Proof. By using the definition of the operators $K^l_{\eta,q}(f;\tau)$, we have

$$\begin{aligned}
K^l_{\eta,q}(\nu^m;\tau) &= \sum_{k=0}^{\infty} s_{\eta,k}(q,\tau) \int_0^1 \dots \int_0^1 \left(\frac{q^{(1-k)}[k]_q + \nu_1 + \dots + \nu_l}{[\eta]_q}\right)^m d\nu_1 \dots d\nu_l \\
&= \sum_{k=0}^{\infty} s_{\eta,k}(q,\tau) \sum_{j_0+\dots+j_l=m} \binom{m}{j_0,\dots,j_l} \int_0^1 \dots \int_0^1 \frac{q^{(1-k)j_0}[k]_q^{j_0} \nu_1^{j_1} \dots \nu_l^{j_l}}{[\eta]_q^m} d\nu_1 \dots d\nu_l \\
&= \sum_{k=0}^{\infty} s_{\eta,k}(q,\tau) \sum_{j_0+\dots+j_l=m} \binom{m}{j_0,\dots,j_l} \frac{q^{(1-k)j_0}[k]_q^{j_0}}{[\eta]_q^m (j_1+1)\dots(j_l+1)} \\
&= \sum_{j_0+\dots+j_l=m} \binom{m}{j_0,\dots,j_l} \frac{[\eta]_q^{j_0}}{[\eta]_q^m q^{j_0}(j_1+1)\dots(j_l+1)} \sum_{k=0}^{\infty} \frac{[k]_q^{j_0}}{q^{(k-2)j_0}[\eta]_q^{j_0}} s_{\eta,k}(q,\tau)
\end{aligned}$$

$$= \sum_{j_0+\ldots+j_l=m} \binom{m}{j_0, \ldots, j_l} \frac{[\eta]_q^{j_0}}{[\eta]_q^m q^{j_0} (j_1+1)\ldots(j_l+1)} S_{\eta,q}(\nu^{j_0}; \tau).$$

□

In the following two lemmas we give explicit formulas for the moments $K_{\eta,q}^l(\nu^j;\tau)$ for $j = 0,1,2,3,4$ and for the central moments $K_{\eta,q}^l((\nu-\tau)^j;\tau)$ for $j = 1,2,4$.

Lemma 2.6. *Let $0 < q < 1$, $l \in \mathbb{Z}^+$ and $\eta \in \mathbb{N}$. We have*

$$\begin{aligned}
K_{\eta,q}^l(1;\tau) &= 1,\\
K_{\eta,q}^l(\nu;\tau) &= \frac{l}{2[\eta]_q} + \tau,\\
K_{\eta,q}^l(\nu^2;\tau) &= \frac{3l^2+l}{12[\eta]_q^2} + \frac{(l+1)}{[\eta]_q}\tau + \frac{1}{q}\tau^2,\\
K_{\eta,q}^l(\nu^3;\tau) &= \frac{l^3-l^2+2l}{8[\eta]_q^3} + \frac{\left(3l^2+7l+4\right)}{4[\eta]_q^2}\tau\\
&+ \frac{3lq^2+4q^2+2}{2q^3[\eta]_q}\tau^2 + \frac{1}{q^3}\tau^3,\\
K_{\eta,q}^l(\nu^4;\tau) &= \frac{185l^2-102l-50l^3+15l^4}{240[\eta]_q^4} + \frac{l^3+2l^2+7l+2}{2[\eta]_q^3}\tau\\
&+ \frac{3q^3l^2+4lq^2+9lq^3+2(q+3q^2+3q^3)}{2q^4[\eta]_q^2}\tau^2\\
&+ \frac{\left(3q+2+\frac{1}{q}+2lq\right)}{q^4[\eta]_q}\tau^3 + \frac{1}{q^6}\tau^4.
\end{aligned}$$

Proof. By using the recurrence formula (3) and by direct calculations we get

$$\begin{aligned}
K_{\eta,q}^l(\nu^2;\tau) &= \sum_{j_0+\ldots+j_l=2} \binom{2}{j_0,\ldots j_l} \frac{[\eta]^{j_0}}{[\eta]^2 q^{j_0}(j_1+1)\ldots(j_l+1)} S_{\eta,q}(\nu^{j_0};\tau)\\
&= \binom{l}{2}\frac{1}{2[\eta]^2} + \binom{l}{1}\frac{1}{3[\eta]^2} + \binom{l}{1}\frac{1}{[\eta]}\tau + \frac{1}{q^2}\left(q\tau^2 + \frac{q^2}{[\eta]_q}\tau\right)\\
&= \frac{3l^2+l}{12[\eta]^2} + \frac{(l+1)}{[\eta]}\tau + \frac{1}{q}\tau^2.
\end{aligned}$$

and

$$\begin{aligned}
K_{\eta,q}^l(\nu^3;\tau) &= \sum_{j_0+\ldots+j_l=3} \binom{3}{j_0,\ldots j_l} \frac{[\eta]^{j_0}}{q^{j_0}[\eta]^3(j_1+1)\ldots(j_l+1)} S_{\eta,q}(\nu^{j_0};\tau)\\
&= \binom{l}{3}\frac{3}{4[\eta]^3} + \binom{l}{2}\frac{1}{2[\eta]^3} + \binom{l}{1}\frac{1}{4[\eta]^3}\\
&+ \binom{l}{2}\frac{3}{2[\eta]^2}\tau + \binom{l}{1}\frac{1}{[\eta]^2}\tau + \binom{l}{1}\frac{3}{2q^2[\eta]}\left(q\tau^2 + \frac{q^2}{[\eta]_q}\tau\right)
\end{aligned}$$

$$+\frac{1}{q^3}\left(\tau^3+\frac{(2q^2+q)}{[\eta]_q}\tau^2+\frac{q^3}{[\eta]_q^2}\tau\right)$$
$$=\frac{l^3-l^2+2l}{8\,[\eta]_q^3}+\frac{\left(3l^2+7l+4\right)}{4\,[\eta]_q^2}\tau+\frac{3lq^2+4q^2+2}{2q^3\,[\eta]_q}\tau^2+\frac{1}{q^3}\tau^3.$$

$K_{\eta,q}^l(\nu^4;\tau)$ can be calculated by using a similar procedure. □

Lemma 2.7. *Let $0<q<1$, $l\in\mathbb{Z}^+$ and $\eta\in\mathbb{N}$. For every $\tau\in[0,\infty)$ we have the following equalities:*

$$K_{\eta,q}^l((\nu-\tau);\tau)=\frac{l}{2\,[\eta]_q} \tag{4}$$

$$K_{\eta,q}^l((\nu-\tau)^2;\tau)=\frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}\tau+\frac{(1-q)}{q}\tau^2 \tag{5}$$

$$\begin{aligned}K_{\eta,q}^l((\nu-\tau)^4;\tau)&=\frac{185l^2-102l-50l^3+15l^4}{240\,[\eta]_q^4}\\&+\frac{3l^2+5l+2}{2\,[\eta]_q^3}\tau\\&+\left(\frac{3q^3l^2+4lq^2+9lq^3+2(q+3q^2+3q^3)-3q^4l^2-13q^4l-8q^4}{2q^4\,[\eta]_q^2}\right)\tau^2\\&+\left(\frac{3q+2+\frac{1}{q}+2lq-2q\left(3lq^2+4q^2+2\right)+6q^4\,(l+1)-2q^4l}{q^4\,[\eta]_q}\right)\tau^3\\&+\left(\frac{1-4q^3-3q^6+6q^5}{q^6}\right)\tau^4.\end{aligned}$$

Proof. By using the linearity property of the operators $K_{\eta,q}^l(f;\tau)$ and Lemma 2.6, we can prove all three equalities with the same method. Thus, we give the proof for only $K_{\eta,q}^l((\nu-\tau)^2;\tau)$.

$$\begin{aligned}K_{\eta,q}^l((\nu-\tau)^2;\tau)&=K_{\eta,q}^l(\nu^2;\tau)-2\tau K_{\eta,q}^l(\nu;\tau)+\tau^2K_{\eta,q}^l(1;\tau)\\&=\frac{3l^2+l}{12\,[\eta]_q^2}+\frac{(l+1)}{[\eta]_q}\tau+\frac{1}{q}\tau^2-2\tau\left(\frac{l}{2\,[\eta]_q}+\tau\right)+\tau^2\\&=\frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}\tau+\frac{(1-q)}{q}\tau^2.\end{aligned}$$

□

In approximation theory, one of the leading problems is estimating the rate of convergence for the sequences of positive linear operators. Voronovskaja kind identities are among the most useful formulas for examining their behavior asymptotically. In the next lemma, we calculate some limits that later on will be used for the proof of the Voronovskaja type theorem for the operators $K_{\eta,q}^l(f;\tau)$.

Lemma 2.8. *Assume that $q_\eta\in(0,1)$, $q_\eta\to1$ and $q_\eta^\eta\to b$ as $\eta\to\infty$. For every $\tau\in[0,\infty)$ we have the following limits:*

$$\lim_{\eta\to\infty}[\eta]_{q_\eta}K_{\eta,q_\eta}^l(\nu-\tau;\tau)=\frac{l}{2},$$

$$\lim_{\eta\to\infty}[\eta]_{q_\eta} K^l_{\eta,q_\eta}((\nu-\tau)^2;\tau) = \tau + \tau^2(1-b),$$
$$\lim_{\eta\to\infty}[\eta]^2_{q_\eta} K^l_{\eta,q_\eta}((\nu-\tau)^4;\tau) = 3\tau^2 + 2(1-b)\tau^3 + 3(1-b)^2\tau^4.$$

Proof. By using Lemma 2.7, we have

$$\begin{aligned}\lim_{\eta\to\infty}[\eta]_{q_\eta} K^l_{\eta,q_\eta}(\nu-\tau;\tau) &= \lim_{\eta\to\infty}[\eta]_{q_\eta}\left(\frac{l}{2[\eta]_{q_\eta}}\right)\\ &= \frac{l}{2},\end{aligned}$$

$$\begin{aligned}\lim_{\eta\to\infty}[\eta]_{q_\eta} K^l_{\eta,q_\eta}((\nu-\tau)^2;\tau) &= \lim_{\eta\to\infty}\left(\frac{3l^2+l}{12}\frac{1-q_\eta}{1-q_\eta^\eta} + \tau + \frac{1-q_\eta^\eta}{1-q_\eta}\frac{\left(1-q_\eta\right)}{q_\eta}\tau^2\right)\\ &= \tau + \tau^2(1-b).\end{aligned}$$

Now since

$$\begin{aligned}\lim_{\eta\to\infty}[\eta]^2_{q_\eta} K^l_{\eta,q_\eta}((\nu-\tau)^4;\tau) &= \lim_{\eta\to\infty}\frac{185l^2-102l-50l^3+15l^4}{240[\eta]^2_{q_\eta}}\\ &+ \lim_{\eta\to\infty}\frac{3l^2+5l+2}{2[\eta]_{q_\eta}}\tau\\ &+ \lim_{\eta\to\infty}\left(\frac{3q_\eta^3l^2+4lq_\eta^2+9lq_\eta^3+2(q_\eta+3q_\eta^2+3q_\eta^3)-3q_\eta^4l^2-13q_\eta^4l-8q_\eta^4}{2q_\eta^4}\right)\tau^2\\ &+ \lim_{\eta\to\infty}[\eta]_{q_\eta}\left(\frac{3q_\eta+2+\frac{1}{q_\eta}+2lq_\eta-2q_\eta\left(3lq_\eta^2+4q_\eta^2+2\right)+6q_\eta^4(l+1)-2q_\eta^4l}{q_\eta^4}\right)\tau^3\\ &+ \lim_{\eta\to\infty}[\eta]^2_{q_\eta}\left(\frac{1-4q_\eta^3-3q_\eta^6+6q_\eta^5}{q_\eta^6}\right)\tau^4.\\ &= I + II + III + IV + V.\end{aligned}$$

Let us calculate the limits I, II, III, IV and V, respectively.

$$I = \lim_{\eta\to\infty}\frac{185l^2-102l-50l^3+15l^4}{240[\eta]^2_{q_\eta}} = 0,$$

$$II = \lim_{\eta\to\infty}\frac{3l^2+5l+2}{2[\eta]_{q_\eta}}\tau = 0,$$

$$\begin{aligned}III &= \lim_{\eta\to\infty}\left(\frac{3q_\eta^3l^2+4lq_\eta^2+9lq_\eta^3+2(q_\eta+3q_\eta^2+3q_\eta^3)-3q_\eta^4l^2-13q_\eta^4l-8q_\eta^4}{2q_\eta^4}\right)\tau^2\\ &= 3\tau^2,\end{aligned}$$

$$IV = \lim_{\eta\to\infty}[\eta]_{q_\eta}\left(\frac{3q_\eta+2+\frac{1}{q_\eta}+2lq_\eta-2q_\eta\left(3lq_\eta^2+4q_\eta^2+2\right)+6q_\eta^4(l+1)-2q_\eta^4l}{q_\eta^4}\right)\tau^3$$

$$= \lim_{\eta\to\infty} \frac{1-q_\eta^\eta}{1-q_\eta} \left(\frac{\left(q_\eta - 1\right)\left(-3q_\eta - 2q_\eta^2 - 2q_\eta^3 + 6q_\eta^4 - 2lq_\eta^2 - 2lq_\eta^3 + 4lq_\eta^4 - 1\right)}{q_\eta^4} \right) \tau^3$$

$$= \lim_{\eta\to\infty} (1-q_\eta^\eta) \left(\frac{-\left(-3q_\eta - 2q_\eta^2 - 2q_\eta^3 + 6q_\eta^4 - 2lq_\eta^2 - 2lq_\eta^3 + 4lq_\eta^4 - 1\right)}{q_\eta^4} \right) \tau^3 = 2(1-b)\tau^3,$$

$$V = \lim_{\eta\to\infty} [\eta]_{q_\eta}^2 \frac{\left(1 - 4q_\eta^3 + 6q_\eta^5 - 3q_\eta^6\right)}{q_\eta^6} \tau^4 = \frac{\left(1-q_\eta^\eta\right)^2}{\left(1-q_\eta\right)^2} \frac{\left(2q_\eta + 3q_\eta^2 - 3q_\eta^4 + 1\right)\left(q_\eta - 1\right)^2}{q_\eta^6} \tau^4$$

$$= \frac{\left(1-q_\eta^\eta\right)^2 \left(2q_\eta + 3q_\eta^2 - 3q_\eta^4 + 1\right)}{q_\eta^6} \tau^4 = 3\,(1-b)^2\,\tau^4.$$

Thus, we obtain

$$\lim_{\eta\to\infty} [\eta]_{q_\eta}^2 K_{\eta,q_\eta}^l((\nu-\tau)^4;\tau) = I + II + III + IV + V$$
$$= 3\tau^2 + 2(1-b)\tau^3 + 3(1-b)^2\tau^4.$$

□

3. Direct Approximation Results

In this section, we present a Korovkin type approximation theorem for the operators $K_{\eta,q}^l(f;\tau)$.

Theorem 3.1. *Consider the squence $q_\eta \in (0,1)$. For each $f \in C_2^*[0,\infty)$, $K_{\eta,q_\eta}^l(f;\tau)$ converges to f uniformly on $[0,B]$ if and only if $\lim_{\eta\to\infty} q_\eta = 1$.*

Proof. Suppose that $\lim_{\eta\to\infty} q_\eta = 1$ and $B > 0$ is fixed. Let us consider the lattice homomorphism $T_B : C[0,\infty) \to C[0,B]$ defined by

$$T_B(f) := f_{|[0,B]}.$$

We can obviously see that

$$T_B(K_{\eta,q_\eta}^l(1)) = T_B(1),\quad T_B(K_{\eta,q_\eta}^l(\nu)) \to T_B(\nu) \text{ and } T_B(K_{\eta,q_\eta}^l(\nu^2)) \to T_B(\nu^2)$$

uniformly on $[0,B]$. With the proposition 4.2.5, (6) of [2] we can state that $C_2^*[0,\infty)$ is isomorphic to $C[0,1]$ and the set $\left\{1,\nu,\nu^2\right\}$ is a Korovkin set in $C_2^*[0,\infty)$. So the universal Korovkin-type property (property (vi) of Thm. 4.1.4 in [2]) implies that

$$K_{\eta,q_\eta}^l(f;\tau) \to f(\tau) \text{ uniformly on } [0,B] \text{ as } \eta\to\infty$$

provided $f \in C_2^*[0,\infty)$ and $B > 0$.

For the proof of the converse result, we use contradiction method. Suppose that $\lim_{\eta\to\infty} q_\eta \neq 1$. Then it must have a subsequence $q_{\eta_k} \in (0,1)$ such that $q_{\eta_k} \to \beta \in [0,1)$ as $k\to\infty$.

Thus

$$\frac{1}{[\eta_k]_{q_{\eta_k}}} = \frac{1-q_{\eta_k}}{1-\left(q_{\eta_k}\right)^{\eta_k}} \to 1-\beta \text{ as } k\to\infty$$

and we get

$$\begin{aligned} K^{l}_{\eta_k,q_{\eta_k}}(\nu;\tau)-\tau &= \frac{l}{2\,[\eta]_{q_{\eta_k}}}+\tau-\tau \\ &\to \frac{l}{2}\,(1-\beta)\neq 0. \end{aligned}$$

This gives us a contradiction. Thus $\lim_{\eta\to\infty} q_\eta = 1$ as $\eta\to\infty$. □

Theorem 3.2. *Let* $0<q<1$, $f\in C_2\,[0,\infty)$ *and* $\omega_{j+1}(f,\delta)=\sup\left\{\left|f(\nu)-f(\tau)\right| : |\nu-\tau|\leq\delta,\ \tau,\nu\in[0,j+1]\right\}$ *be the modulus of continuity of f on the closed interval* $[0,j+1]$ *where* $j>0$. *Then we have*

$$\left\|K^{l}_{\eta,q}(f;\tau)-f(\tau)\right\|_{C[0,j]}\leq 4M_f\left(1+j^2\right)\alpha_\eta(j)+2\omega_{j+1}\left(f;\sqrt{\alpha_\eta(j)}\right), \tag{6}$$

where $\alpha_\eta(j)=\frac{3l^2+l}{12[\eta]_q^2}+\frac{1}{[\eta]_q}j+\frac{(1-q)}{q}j^2$.

Proof. For $\tau\in[0,j]$ and $\nu\geq 0$, we have

$$\left|f(\nu)-f(\tau)\right|\leq 4M_f\left(1+j^2\right)(\nu-\tau)^2+\left(1+\frac{|\nu-\tau|}{\delta}\right)\omega_{j+1}\,(f;\delta)\ \ \text{(see Equation 3.3 in [1]).}$$

By using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left|K^{l}_{\eta,q}(f;\tau)-f(\tau)\right| &\leq K^{l}_{\eta,q}\left(\left|f(\nu)-f(\tau)\right|;\tau\right) \\ &\leq 4M_f\left(1+j^2\right)K^{l}_{\eta,q}\left((\nu-\tau)^2;\tau\right)+\left(1+K^{l}_{\eta,q}\left(\frac{|\nu-\tau|}{\delta};\tau\right)\right)\omega_{j+1}\,(f;\delta) \\ &\leq 4M_f\left(1+j^2\right)K^{l}_{\eta,q}\left((\nu-\tau)^2;\tau\right)+\omega_{j+1}\,(f;\delta)\left(1+\frac{1}{\delta}\left(K^{l}_{\eta,q}\left((\nu-\tau)^2;\tau\right)\right)^{\frac{1}{2}}\right). \end{aligned}$$

For $\tau\in[0,j]$, using Lemma 2.7,

$$\begin{aligned} K^{l}_{\eta,q}\left((\nu-\tau)^2;\tau\right) &= \frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}\tau+\frac{(1-q)}{q}\tau^2 \\ &\leq \frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}j+\frac{(1-q)}{q}j^2=\alpha_\eta(j). \end{aligned}$$

Thus we get

$$\left|K^{l}_{\eta,q}(f;\tau)-f(\tau)\right|\leq 4M_f\left(1+j^2\right)\alpha_\eta(j)+\omega_{j+1}\,(f;\delta)\left(1+\frac{1}{\delta}\left(\alpha_\eta(j)\right)^{\frac{1}{2}}\right).$$

Now, if we choose $\delta=\sqrt{\alpha_\eta(j)}$, we obtain the required result. □

4. Local Approximation

In this section, we study local approximation properties of the operators $K^{l}_{\eta,q}\,(f;\tau)$. Let $C_B\,[0,\infty)$ denote the space of all bounded, real valued continuous functions on $[0,\infty)$. This space is equipped with the norm

$$\left\|f\right\|=\sup_{\tau\in[0,\infty)}\left|f\,(\tau)\right|.$$

On the other hand, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{ \left\| f-g \right\| + \delta \left\| g'' \right\| \right\},\ \delta \geq 0,$$

where $C_B^2[0,\infty) := \left\{ g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty) \right\}$. By Theorem 2.4 in [5], there exists an absolute constant $L > 0$ such that

$$K_2(f;\delta) \leq L\omega_2\left(f;\sqrt{\delta}\right), \tag{7}$$

where $\omega_2(f;\delta)$ is the second order modulus of smoothness defined as

$$\omega_2(f;\delta) = \sup_{0<\omega\leq\delta} \sup_{\tau\in[0,\infty)} \left| f(\tau+2\omega) - 2f(\tau+\omega) + f(\tau) \right|.$$

In the following theorem we give a local approximation for the operators $K_{\eta,q}^l(f;\tau)$ in terms of the first modulus of continuity and the second modulus of smoothness.

Theorem 4.1. *Let $f \in C_B[0,\infty)$ .Then, for every $\tau \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$\left| K_{\eta,q}^l(f;\tau) - f(\tau) \right| \leq M\omega_2\left(f;\sqrt{\delta_\eta(\tau)}\right) + \omega\left(f;\beta_\eta(\tau)\right),$$

where

$$\delta_\eta(\tau) = K_{\eta,q}^l((\nu-\tau)^2;\tau) + \left(K_{\eta,q}^l((\nu-\tau);\tau)\right)^2 = \frac{3l^2+l}{12\left[\eta\right]_q^2} + \frac{1}{\left[\eta\right]_q}\tau + \frac{(1-q)}{q}\tau^2 + \left(\frac{l}{2\left[\eta\right]_q}\right)^2$$

and

$$\beta_\eta(\tau) = \left| K_{\eta,q}^l((\nu-\tau);\tau) \right| = \frac{l}{2\left[\eta\right]_q}.$$

Proof. Let

$$^*K_{\eta,q}^l(f;\tau) = K_{\eta,q}^l(f;\tau) + f(\tau) - f\left(\rho_\eta(\tau)\right),$$

where $f \in C_B[0,\infty]$, $\rho_\eta(\tau) = K_{\eta,q}^l((\nu-\tau);\tau) + \tau = \frac{l}{2[\eta]_q} + \tau$. Note that $^*K_{\eta,q}^l((\nu-\tau);\tau) = 0$. With the help of the Taylor's formula we can write

$$g(\nu) = g(\tau) + g'(\tau)(\nu-\tau) + \int_\tau^\nu (\nu-s)g''(s)ds,\ \ g \in C_B^2[0,\infty).$$

Applying $^*K_{\eta,q}^l$ to the left side and to the right side of the last equation, we obtain

$$\begin{aligned}
^*K_{\eta,q}^l(g;\tau) - g(\tau) &= {}^*K_{\eta,q}^l((\nu-\tau)g'(\tau);\tau) + {}^*K_{\eta,q}^l\left(\int_\tau^\nu (\nu-s)g''(s)ds;\tau\right) \\
&= g'(\tau)^*K_{\eta,q}^l((\nu-\tau);\tau) + K_{\eta,q}^l\left(\int_\tau^\nu (\nu-s)g''(s)ds;\tau\right) - \int_\tau^{\rho_\eta(\tau)} \left(\rho_\eta(\tau)-s\right)g''(s)ds \\
&= K_{\eta,q}^l\left(\int_\tau^\nu (\nu-s)g''(s)ds;\tau\right) - \int_\tau^{\rho_\eta(\tau)} \left(\rho_\eta(\tau)-s\right)g''(s)ds.
\end{aligned}$$

On the other hand,

$$\left|\int_{\tau}^{\nu}(\nu-s)g''(s)ds\right|\leq\int_{\tau}^{\nu}(\nu-s)\left|g''(s)\right|ds\leq\left\|g''\right\|\int_{\tau}^{\nu}(\nu-s)ds\leq\left\|g''\right\|(\nu-s)^{2}$$

and

$$\left|\int_{\tau}^{\rho_{\eta}(\tau)}\left(\rho_{\eta}(\tau)-s\right)g''(s)ds\right|\leq\left\|g''\right\|(\rho_{\eta}(\tau)-\tau)^{2}=\left\|g''\right\|\left(K_{\eta,q}^{l}(\nu-\tau;\tau)\right)^{2}.$$

which implies

$$\begin{aligned}\left|{}^{*}K_{\eta,q}^{l}(g;\tau)-g(\tau)\right|&\leq\left|K_{\eta,q}^{l}\left(\int_{\tau}^{\nu}(\nu-s)g''(s)ds;\tau\right)\right|+\left|\int_{\tau}^{\rho_{\eta}(\tau)}\left(\rho_{\eta}(\tau)-s\right)g''(s)ds\right|\\&\leq\left\|g''\right\|\left\{K_{\eta,q}^{l}((\nu-\tau)^{2};\tau)+(K_{\eta,q}^{l}(\nu-\tau);\tau)^{2}\right\}\\&=\left\|g''\right\|\delta_{\eta}(\tau).\end{aligned}\tag{8}$$

We also have

$$\left|{}^{*}K_{\eta,q}^{l}(f;\tau)\right|\leq\left|K_{\eta,q}^{l}(f;\tau)\right|+\left|f(\tau)\right|+\left|f(\rho_{\eta}(\tau))\right|\leq K_{\eta,q}^{l}\left(\left|f\right|;\tau\right)+2\left\|f\right\|\leq3\left\|f\right\|.$$

Using (8) and the fact that ${}^{*}K_{\eta,q}^{l}$ is uniformly bounded, we obtain

$$\begin{aligned}\left|K_{\eta,q}^{l}(f;\tau)-f(\tau)\right|&\leq\left|{}^{*}K_{\eta,q}^{l}(f-g;\tau)\right|+\left|{}^{*}K_{\eta,q}^{l}(g;\tau)-g(\tau)\right|+\left|f(\tau)-g(\tau)\right|+\left|f\left(\rho_{\eta}(\tau)\right)-f(\tau)\right|\\&\leq4\left\|f-g\right\|+\left\|g''\right\|\delta_{\eta}(\tau)+\omega\left(f,\beta_{\eta}(\tau)\right).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in C_{B}^{2}[0,\infty)$, we obtain

$$\left|K_{\eta,q}^{l}(f;\tau)-f(\tau)\right|\leq4K_{2}\left(f;\delta_{\eta}(\tau)\right)+\omega\left(f,\beta_{\eta}(\tau)\right),$$

which together with (7) completes the proof of the theorem. □

Theorem 4.2. *Assume that $\alpha\in(0,1]$ and $A\subset[0,\infty)$. Then, if $f\in C_{B}[0,\infty)$ is locally Lip(α); that is the condition*

$$\left|f(y)-f(\tau)\right|\leq L\left|y-\tau\right|^{\alpha},\ y\in A\text{ and }\tau\in[0,\infty)\tag{9}$$

holds, then, for each $\tau\in[0,\infty)$, we have

$$\left|K_{\eta,q}^{l}(f;\tau)-f(\tau)\right|\leq L\left\{\lambda_{\eta,l}^{\frac{\alpha}{2}}(\tau)+2(d(\tau,A))^{\alpha}\right\},$$

where

$$\lambda_{\eta,l}(\tau)=\frac{3l^{2}+l}{12\,[\eta]_{q}^{2}}+\frac{1}{[\eta]_{q}}\tau+\frac{(1-q)}{q}\tau^{2},$$

L is a constant which depends on α and f and $d(\tau,A)$ is the distance between τ and A defined as

$$d(\tau,A)=\inf\left\{|\nu-\tau|:\nu\in A\right\}.$$

Proof. Suppose that $\bar{A}$ is the closure of A in $[0,\infty)$. Then, there is a point $\tau_{0}\in\bar{A}$ such that $|\tau-\tau_{0}|=d(\tau,A)$. By the triangle inequality

$$\left|f(\nu)-f(\tau)\right|\leq\left|f(\nu)-f(\tau_{0})\right|+\left|f(\tau)-f(\tau_{0})\right|$$

and by (9) we get

$$\begin{aligned}\left|K^l_{\eta,q}(f;\tau)-f(\tau)\right| &\le K^l_{\eta,q}\left(\left|f(\nu)-f(\tau_0)\right|;\tau\right)+K^l_{\eta,q}\left(\left|f(\tau)-f(\tau_0)\right|;\tau\right)\\ &\le L\left\{K^l_{\eta,q}\left(|\nu-\tau_0|^\alpha;\tau\right)+|\tau-\tau_0|^\alpha\right\}\\ &\le L\left\{K^l_{\eta,q}\left(|\nu-\tau|^\alpha+|\tau-\tau_0|^\alpha;\tau\right)+|\tau-\tau_0|^\alpha\right\}\\ &\le L\left\{K^l_{\eta,q}\left(|\nu-\tau|^\alpha;\tau\right)+2\,|\tau-\tau_0|^\alpha\right\}.\end{aligned}$$

Now by using the Hölder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we get

$$\begin{aligned}\left|K^l_{\eta,q}(f;\tau)-f(\tau)\right| &\le L\left\{\left[K^l_{\eta,q}\left(|\nu-\tau|^{\alpha p};\tau\right)\right]^{\frac{1}{p}}\left[K^l_{\eta,q}\left(1^q;\tau\right)\right]^{\frac{1}{q}}+2\,(d(\tau,A))^\alpha\right\}\\ &= L\left\{\left[K^l_{\eta,q}\left(|\nu-\tau|^2;\tau\right)\right]^{\frac{\alpha}{2}}+2\,(d(\tau,A))^\alpha\right\}\\ &= L\left\{\left[\frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}\tau+\frac{(1-q)}{q}\tau^2\right]^{\frac{\alpha}{2}}+2\,(d(\tau,A))^\alpha\right\}\\ &= L\left\{\lambda_{\eta,l}(\tau)^{\frac{\alpha}{2}}+2\,(d(\tau,A))^\alpha\right\}\end{aligned}$$

and the proof is completed. □

5. Weighted Approximation

In this section, we study weighted approximation theorems for the operators $K^l_{\eta,q}(f;\tau)$.

In [14], Mahmudov gave the following explicit formula for the moments $S_{\eta,q}(\nu^{j_0};\tau)$, which is the q-analogue of the finding of M. Becker, see [25] Lemma 3.

Lemma 5.1. *[4] For $0<q<1$ and $j_0\in\mathbb{N}$, there holds*

$$S_{\eta,q}(\nu^{j_0};\tau)=\sum_{j=1}^{j_0}S_q(j_0,j)\frac{\tau^j}{[\eta]_q^{j_0-j}}. \tag{10}$$

Here $S_q(j_0,j)$ are q-Stirling numbers of the second kind which are described by the following recurrence formula

$S_q(j_0+1,j)=[j]\,S_q(j_0,j)+S_q(j_0,j-1),\quad j_0\ge 0,\ j\ge 1,$
where $S_q(0,0)=1,\ S_q(j_0,0)=0,\ j_0>0,\ S_q(j_0,j)=0,\ j_0<j.$

Lemma 5.2. *Let $m\in\mathbb{N}\cup\{0\}$, $q\in(0,1)$ and $l\in\mathbb{Z}^+$ be fixed. We have*

$$\left\|K^l_{\eta,q}(1+\nu^m;\tau)\right\|_m\le C_m(q,l),\ \eta\in\mathbb{N}, \tag{11}$$

where $C_m(q,l)$ is a positive constant. Moreover, we have

$$\left\|K^l_{\eta,q}(f;\tau)\right\|_m\le C_m(q,l)\left\|f\right\|_m,\ \ \eta\in\mathbb{N}, \tag{12}$$

where $f\in C^*_m[0,\infty)$. Thus, for any $m\in\mathbb{N}\cup\{0\}$, $K^l_{\eta,q}:C^*_m[0,\infty)\to C^*_m[0,\infty)$ is a linear positive operator.

Proof. If $m=0$, inequality (11) is obvious.

If $m\ge 1$, using (10) and Lemma 2.6, we get

$$\frac{1}{\tau^m+1}K^l_\eta(1+\nu^m;\tau)=\frac{1}{\tau^m+1}+\frac{1}{\tau^m+1}\sum_{j_0+\ldots+j_l=m}\binom{m}{j_0,\ldots j_l}\frac{[\eta]^{j_0}}{[\eta]^m\,q^{j_0}(j_1+1)\ldots(j_l+1)}\sum_{j=1}^{j_0}S_q(j_0,j)\frac{\tau^j}{\eta^{j_0-j}}$$

$$\leq 1 + k_m(q,l) = C_m(q,l).$$

Here $C_m(q,l)$ is a positive constant which depends on q, m and l, so 11 follows from this. On the other hand

$$\left\|K^l_{\eta,q}(f)\right\|_m \leq \left\|f\right\|_m \left\|K^l_{\eta,q}(1+\nu^m)\right\|_m$$

for every $f \in C^*_m[0,\infty)$. Now if we apply (11), we obtain

$$\left\|K^l_{\eta,q}(f;\tau)\right\|_m \leq C_m(q,l)\left\|f\right\|_m.$$

□

Theorem 5.3. *Let $q = q_\eta \in (0,1)$, $q_\eta \to 1$ and $q^\eta_\eta \to b$ as $\eta \to \infty$. Then for each $f \in C^*_2[0,\infty)$, one has*

$$\lim_{\eta\to\infty}\left\|K^l_{\eta,q_\eta}(f;\tau) - f(\tau)\right\|_2 = 0.$$

Proof. To prove this theorem, we are going to use Korovkin type theorem for weighted approximation ([3]) and Lemma 2.6. Thus, it will be sufficient to verify the following condition for $m = 0,1,2$:

$$\lim_{\eta\to\infty}\left\|K^l_{\eta,q_\eta}(\nu^m;\tau) - \tau^m\right\|_2 = 0.$$

Since $K^l_{\eta,q_\eta}(1;\tau) = 1$, it is obvious, for $m = 0$.

For $m = 1$, we have

$$\begin{aligned}
\lim_{\eta\to\infty}\left\|K^l_{\eta,q_\eta}(\nu;\tau) - \tau\right\|_2 &= \lim_{\eta\to\infty}\sup_{\tau\geq 0}\frac{\left|K^l_{\eta,q_\eta}(\nu;\tau) - \tau\right|}{1+\tau^2}\\
&= \lim_{\eta\to\infty}\sup_{\tau\geq 0}\frac{1}{1+\tau^2}\left|\frac{l}{2\,[\eta]_{q_\eta}} + \tau - \tau\right|\\
&\leq \lim_{\eta\to\infty}\frac{l}{2\,[\eta]_{q_\eta}}\sup_{\tau\geq 0}\frac{1}{1+\tau^2}\\
&\leq \lim_{\eta\to\infty}\frac{l}{2\,[\eta]_{q_\eta}},
\end{aligned}$$

and for $m = 2$, we have

$$\begin{aligned}
\lim_{\eta\to\infty}\left\|K^l_{\eta,q_\eta}(\nu;\tau) - \tau\right\|_2 &= \lim_{\eta\to\infty}\sup_{\tau\geq 0}\frac{\left|K^l_{\eta,q_\eta}(\nu^2;\tau) - \tau^2\right|}{1+\tau^2}\\
&= \lim_{\eta\to\infty}\sup_{\tau\geq 0}\frac{1}{1+\tau^2}\left|\frac{3l^2+l}{12\,[\eta]^2_{q_\eta}} + \frac{(l+1)}{[\eta]_{q_\eta}}\tau + \frac{1}{q_\eta}\tau^2 - \tau^2\right|\\
&\leq \lim_{\eta\to\infty}\left(\left(\frac{1}{q_\eta} - 1\right)\sup_{\tau\geq 0}\frac{\tau^2}{1+\tau^2} + \frac{(l+1)}{[\eta]_{q_\eta}}\sup_{\tau\geq 0}\frac{\tau}{1+\tau^2} + \frac{3l^2+l}{12\,[\eta]^2_{q_\eta}}\sup_{\tau\geq 0}\frac{1}{1+\tau^2}\right)\\
&\leq \lim_{\eta\to\infty}\left(\left(\frac{1}{q_\eta} - 1\right) + \frac{(l+1)}{[\eta]_{q_\eta}} + \frac{3l^2+l}{12\,[\eta]^2_{q_\eta}}\right),
\end{aligned}$$

which implies that

$$\lim_{\eta\to\infty}\left\|K^l_\eta(\nu^m;\tau) - \tau^m\right\|_2 = 0, \quad m = 0,1,2.$$

□

In the following theorem, we give an estimation in terms of the weighted modulus of continuity. The weighted modulus of continuity is defined as

$$\Omega_m(f,\delta)=\sup_{\tau\geq 0,\,0<\omega\leq\delta}\frac{\left|f(\tau+\omega)-f(\tau)\right|}{1+(\tau+\omega)^m}, \tag{13}$$

where $f\in C_2^*[0,\infty)$,

Lemma 5.4. *[27] If $f\in C_m^*[0,\infty)$, $m\in\mathbb{N}$, then*

(i) $\Omega_m(f,\delta)$ is a monotone increasing function of δ,
(ii) $\lim\limits_{\delta\to 0}\Omega_m(f,\delta)=0$,
(iii) for any $\rho\in[0,\infty)$, $\Omega_m(f,\rho\delta)\leq(1+\rho)\Omega_m(f,\delta)$.

Theorem 5.5. *For $f\in C_m^*[0,\infty)$, we have*

$$\left\|K_{\eta,q}^l(f)-f\right\|_{m+1}\leq N\Omega_m\left(f,\left(1/\sqrt{q\,[\eta]_q}\right)\right),$$

where N is a constant independent of η and f.

Proof. Using (13) and the previous lemma, we may write

$$\begin{aligned}\left|f(\nu)-f(\tau)\right| &\leq (1+(\tau+|\nu-\tau|)^m)\left(\frac{|\nu-\tau|}{\delta}+1\right)\Omega_m(f,\delta)\\ &\leq (1+(2\tau+\nu)^m)\left(\frac{|\nu-\tau|}{\delta}+1\right)\Omega_m(f,\delta).\end{aligned}$$

Then by the Cauchy-Schwartz inequality, we obtain the following result.

$$\begin{aligned}\left|K_{\eta,q}^l(f;\tau)-f(\tau)\right| &\leq K_{\eta,q}^l\left|(f(\nu)-f(\tau)\right|;\tau)\\ &\leq \Omega_m(f,\delta)\left(K_{\eta,q}^l((1+(2\tau+\nu)^m);\tau)+K_{\eta,q}^l\left((1+(2\tau+\nu)^m)\frac{|\nu-\tau|}{\delta};\tau\right)\right)\\ &= \Omega_m(f,\delta)\left(K_{\eta,q}^l\left(1+(2\tau+\nu)^m;\tau\right)+\left(K_{\eta,q}^l\left((1+(2\tau+\nu)^m)^2;\tau\right)\right)^{1/2}\left(K_{\eta,q}^l\left(\frac{|\nu-\tau|^2}{\delta^2};\tau\right)\right)^{1/2}\right).\end{aligned}$$

Therefore, from Lemma 5.1 and 5.2, we have

$$K_{\eta,q}^l(1+(2\tau+\nu)^m;\tau)\leq C_m(q,l)\,(1+\tau^m),$$

$$\left(K_{\eta,q}^l\left((1+(2\tau+\nu)^m)^2;\tau\right)\right)^{1/2}\leq C_m^1(q,l)\,(1+\tau^m).$$

Also, by (5), we have

$$\begin{aligned}\left(K_{\eta,q}^l\left(\frac{|\nu-\tau|^2}{\delta^2};\tau\right)\right)^{1/2} &\leq \frac{1}{\delta}\sqrt{\frac{3l^2+l}{12\,[\eta]_q^2}+\frac{1}{[\eta]_q}\tau+\frac{(1-q)}{q}\tau^2}\\ &\leq \frac{2l\,(1+\tau)}{\delta\sqrt{q\,[\eta]_q}}.\end{aligned}$$

Combining all these results, we obtain

$$\left|K_{\eta,q}^l(f;\tau)-f(\tau)\right|\leq\Omega_m(f,\delta)\left(C_m(q,l)\,(1+\tau^m)+C_m^1(q,l)\frac{(1+\tau^m)\,2l\,(1+\tau)}{\delta\sqrt{q\,[\eta]_q}}\right)$$

$$= \Omega_m(f,\delta)\left(C_m(q,l)\,(1+\tau^m) + C_m^1(q,l)C_1 \frac{\left(1+\tau^{m+1}\right)}{\delta\sqrt{q\,[\eta]_q}}\right),$$

where

$$C_1 = \sup_{\tau\geq 0} \frac{2l\left(1+\tau^m+\tau+\tau^{m+1}\right)}{1+\tau^{m+1}}.$$

Now, in the last inequality, if we write $\left(1/\sqrt{q\,[\eta]_q}\right)$ instead of δ, we get the required result. □

6. Voronovskaja Type Theorem

Here in this section, we prove a Voronovskaja type theorem for the operators $K^l_{\eta,q_\eta}(f;\tau)$.

Theorem 6.1. *Assume that* $l \in \mathbb{Z}^+$, $q = q_\eta \in (0,1)$, $q_\eta \to 1$ *and* $q_\eta^\eta \to b$ *as* $\eta \to \infty$. *For any* $f \in C_2^*[0,\infty)$ *such that* $f^{\,'}, f^{\,''} \in C_2^*[0,\infty)$ *we have the following limit:*

$$\lim_{\eta\to\infty} [\eta]_{q_\eta} \left| K^l_{\eta,q_\eta}(f;\tau) - f(\tau)\right| = \frac{l}{2} f^{\,'}(\tau) + \frac{1}{2}\left(\tau + \tau^2(1-b)\right) f^{\,''}(\tau)$$

uniformly on any $[0,B]$, $B > 0$.

Proof. For $\tau \in [0,\infty)$, the Taylor's formula for f is given by

$$f(\nu) = f(\tau) + f^{\,'}(\tau)(\nu-\tau) + \frac{1}{2} f^{\,''}(\tau)(\nu-\tau)^2 + r(\nu,\tau)(\nu-\tau)^2 \tag{14}$$

where $r(\nu,\tau)$ is the Peano's form of the remainder and $r(.,\tau) \in C_2^*[0,\infty)$ and $\lim_{\nu\to\tau} r(\nu,\tau) = 0$. Applying the operator K^l_{η,q_η} to (14), we get

$$[\eta]_{q_\eta} \left| K^l_{\eta,q_\eta}(f;\tau) - f(\tau)\right| = [\eta]_{q_\eta} f^{\,'}(\tau) K^l_{\eta,q_\eta}(\nu-\tau;\tau) + \frac{1}{2}[\eta]_{q_\eta} f^{\,''}(\tau) K^l_{\eta,q_\eta}\left((\nu-\tau)^2;\tau\right) + [\eta]_{q_\eta} K^l_{\eta,q_\eta}\left(r(\nu,\tau)(\nu-\tau)^2;\tau\right).$$

Now, by using the Cauchy-Schwarz inequality for the last term of the right hand side of the last equality, we have

$$K^l_{\eta,q_\eta}\left(r(\nu,\tau)(\nu-\tau)^2;\tau\right) \leq \sqrt{K^l_{\eta,q_\eta}\left(r^2(\nu,\tau);\tau\right)}\sqrt{K^l_{\eta,q_\eta}\left((\nu-\tau)^4;\tau\right)}. \tag{15}$$

It can be observed that $r^2(\tau,\tau) = 0$ and $r^2(.,\tau) \in C_2^*[0,\infty)$. Then, from Theorem (3.1),

$$\lim_{\eta\to\infty} K^l_{\eta,q_\eta}\left(r^2(\nu,\tau);\tau\right) = r^2(\tau,\tau) = 0 \tag{16}$$

uniformly for $\tau \in [0,B]$.

Now, from Lemma 2.8, (15) and (16), we have

$$\lim_{\eta\to\infty} [\eta]_q \left| K^l_{\eta,q_\eta}(f;\tau) - f(\tau)\right| = \frac{l}{2} f^{\,'}(\tau) + \frac{1}{2}\left(\tau + \tau^2(1-b)\right) f^{\,''}(\tau).$$

□

Now, in the following section we give some graphical representation and some numerical results for $K^l_{\eta,q}$ obtained by using the Matlab program.

7. Graphical simulations

Example 7.1. *Let* $f(\tau) = \tau^3 - 4\tau^2 + 2$ *with* $\tau \in [0, 4]$*. Here we take* $q \in \{0.85, 0.90, 0.95\}$, $l = 2$ *and* $\eta = 100$*. Figure 1 shows the convergence of operators* $K^l_{\eta,q}$ *to* $f(\tau)$ *for increasing values of* q *and fixed* η*. The absolute error function* $E^l_{\eta,q}(f;\tau) = \left|K^l_{\eta,q}(f;\tau) - f(\tau)\right|$ *is illustrated in Figure 2. Some numerical values of* $E^l_{\eta,q}(f;\tau)$ *at certain points on the interval* $[0, 4]$ *for* $q \in \{0.85, 0.90, 0.95\}$, $l = 2$ *and* $\eta = 100$ *are given in Table 1.*

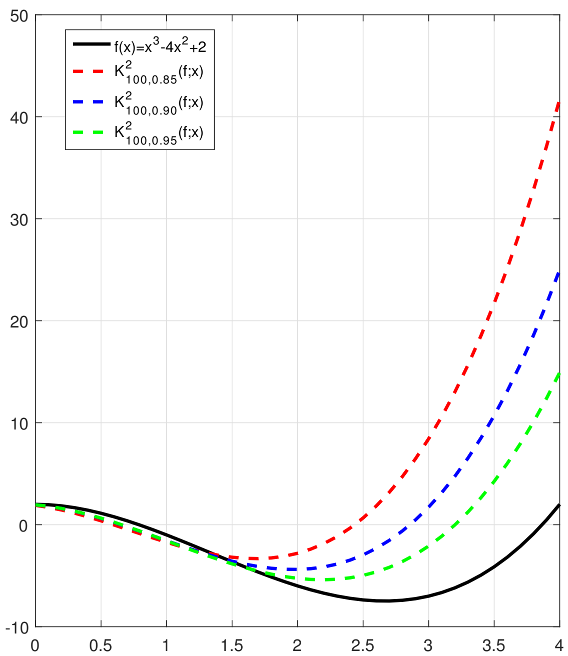

Figure 1 *Approximation to* f *by* $K^l_{\eta,q}(f;\tau)$ *for* $l = 2$, $\eta = 100$, $f(\tau) = \tau^3 - 4\tau^2 + 2$ *and* $q = 0.85, 0.90, 0.95$.

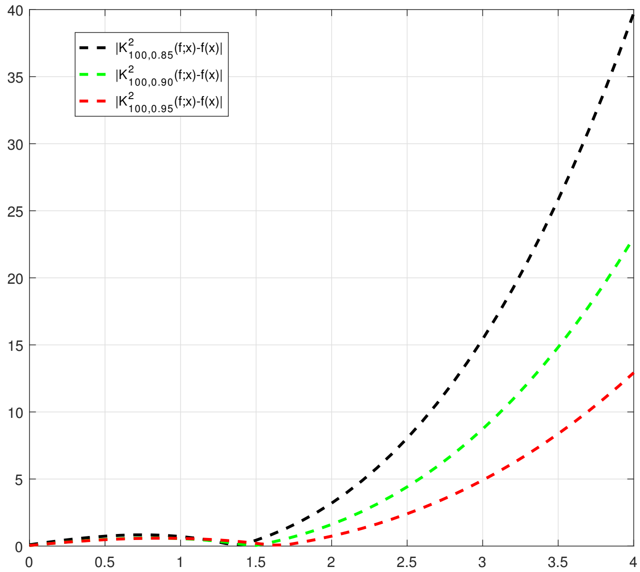

Figure 2 *Absolute error function* $E^l_{\eta,q}(f;\tau)$ *for* $q = 0.85, 0.90, 0.95$, $l = 2$, $f(\tau) = \tau^3 - 4\tau^2 + 2$ *and* $\eta = 100$.

Table 1*. Estimation of the absolute error function* $E^l_{\eta,q}(f;\tau)$ *with* $f(\tau) = \tau^3 - 4\tau^2 + 2$ *for some values of* τ *in* $[0, 4]$, $l = 2$ *and* $q \in \{0.85, 0.90, 0.95\}$.

τ	$\left\|K^2_{100,0.85}(f;\tau) - f(\tau)\right\|$	$\left\|K^2_{100,0.90}(f;\tau) - f(\tau)\right\|$	$\left\|K^2_{100,0.95}(f;\tau) - f(\tau)\right\|$
0	0.099937517	0.045169026	0.045169026
0.4	0.650770954	0.433854937	0.416276177
0.8	0.837417917	0.604724377	0.586989518
1.2	0.418598535	0.415028375	0.493430348
1.6	0.846967061	0.277982041	0.07171997
2	3.200558741	1.617055842	0.742020317
2.4	6.883456375	3.744941999	2.011669213
2.8	12.13693983	6.804389482	3.801105416
3.2	19.20228898	10.93814726	6.174207627
3.6	28.32078369	16.28896432	9.194854545
4	39.73370384	22.99958961	12.92692487

For fixed η and l, if we increase the value of q, the approximation becomes better, i.e for the largest value of q and fixed η and l, the error is minimum.

Example 7.2. *Consider* $f(\tau) = -\tau^3 - 4\tau^2 + 2$ *with* $\tau \in [0, 1]$. *Here we take* $\eta \in \{10, 100\}$, $l = 2$ *and* $q = 0.95$ *for* $K^l_{\eta,q}$. *Figure 3 demonstrates the convergence of operators* $K^l_{\eta,q}$ *to* $f(\tau)$ *for fixed* l, q *and increasing values of* η . *Moreover, the function of absolute error* $E^l_{\eta,q}(f;\tau) = \left|K^l_{\eta,q}(f;\tau) - f(\tau)\right|)$ *is illustrated in Figure 4. Then, some numerical values* *of* $E^{l}_{\eta,q}(f;\tau)$ *at certain points on the interval* $[0,4]$ *for* $\eta \in \{10, 100\}$, $l = 2$ *and* $q = 0.95$ *are given in Table 2*

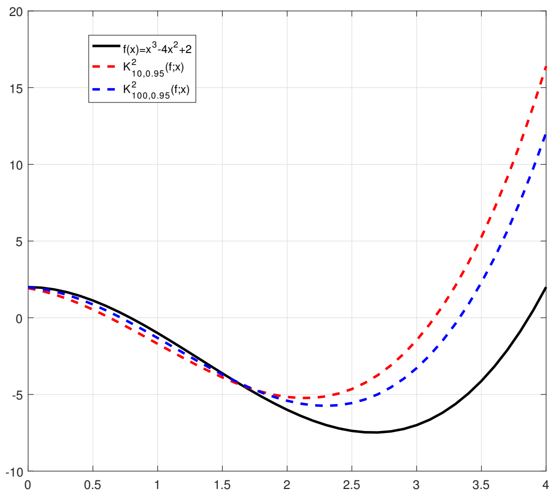

Figure 3 *Approximation to* f *by* $K^{l}_{\eta,q}(f;\tau)$ *for* $q = 0.95$, $l = 2$, $f(\tau) = \tau^3 - 4\tau^2 + 2$ *and* $\eta = 10, 100$.

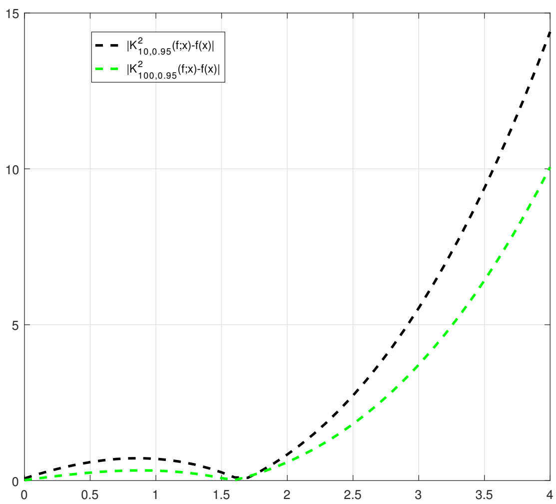

Figure 4 *Absolute error function* $E^{l}_{\eta,q}(f;\tau)$ *for* $l=2$, $q=0.95$, $f(\tau)=\tau^3-4\tau^2+2$ *and* $\eta=10,100$.

Table 2*. Estimation of the absolute error function* $E^{l}_{\eta,q}$ *with* $f(\tau)=\tau^3-4\tau^2+2$ *for some values of* τ *in* $[0,4]$, $l=2, q=0.95$ *and* $\eta\in\{10,100\}$

τ	$\left\lvert K^{2}_{10}(f;\tau)-f(\tau)\right\rvert$	$\left\lvert K^{2}_{100}(f;\tau)-f(\tau)\right\rvert$
0	0.069556239	0.011615179
0.4	0.517102276	0.217219633
0.8	0.714092577	0.323767624
1.2	0.596648441	0.26738045
1.6	0.10089117	0.015820587
2	0.837057936	0.589714187
2.4	2.281077578	1.518179049
2.8	4.295046454	2.865093873
3.2	6.942843264	4.694337359
3.6	10.28834671	7.069788206
4	14.39543548	10.05532511

As we can see from the error values in the table, the error is minimum for the largest value of η *and fixed values of* q *and* l.

8. Conclusion

In this paper, by using the q-analogue of integers and with the help of the concepts in q-calculus, we defined a new Kantorovich variant of Szasz-Mirakjan operators $K^{l}_{\eta,q}(f;\tau)$. We construct a recurrence formula and with the help of this formula we calculated the moments $K^{l}_{\eta,q}\left(\nu^j;\tau\right)$ for $j=0,1,2,3,4$ and we calculated the central moments $K^{l}_{\eta}\left((\nu-\tau)^j;\tau\right)$ for $j=1,2,4$. We studied local approximation properties of these operators. We established a Korovkin-type approximation theorem and a Voronovskaja-type theorem. We obtained the rate of convergence of the new Kantorovich variant of q-Szasz-Mirakjan operators via first modulus of continuity, second order modulus of smoothness and Peetre's K-functional. On the other hand by using the weighted modulus of continuity we investigated weighted approximation properties of the operators. At last we examined the operators for some special cases by giving graphical representation. For a future researh, we would like to study Kantorovich variants of the classical Szasz-Mirakjan Baskakov operators and q-Szasz-Mirakjan Baskakov Operators and extend these results via post quantum calculus.

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Competing interests

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Author's contributions

MK and PS prepared the orijinal draft. NM reviewed and edited the manuscript. All authors read and approved the final manuscript.

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