Filomat 38:14 (2024), 5179–5207
https://doi.org/10.2298/FIL2414179B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New fractional refinements of harmonic Hermite-Hadamard-Mercer type inequalities via support line

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Abstract. In this research, we first provide new and refined fractional integral Mercer inequalities for harmonic convex functions by deploying the idea of line of support. Thus, these refinements allow us to develop new extensions for integral inequalities pertaining harmonic convex functions. We also provide some new fractional auxiliary equalities in Mercer sense. By employing Mercer's harmonic convexity on them, we exhibit new fractional Mercer variants of trapezoid and midpoint type inequalities. We prove new Hermite-Hadamard (**H-H**) type inequalities with special functions involving fractional integral operators. For the development of these new integral inequalities, we use Power-mean, Hölder's and improved Hölder integral inequalities. We unveiled complicated integrals into simple forms by involving hypergeometric functions. Visual illustrations demonstrate the accuracy and supremacy of the offered technique. As an application, new bounds regarding hypergeometric functions as well as special means of $\mathbb{R}$ (real numbers) and quadrature rule are exemplified to show the applicability and validity of the offered technique.

1. Introduction

The study of convex functions always gives stunning and magnificent sight of the beauty in advanced mathematics. The mathematicians always pay attention and work hard in this direction to explore a large variety of results that are fruitful and notable for applications. Inequalities in science and engineering have seen a remarkable theoretical and practical development in recent years. One of the primary applications of convex functions is in the construction of inequalities with useful applications (see [1–4]). The classical convex mapping is defined as:

Definition 1.1. *[1] A function $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ is called convex ($\psi \in H(J)$), if*

$$\psi(\ell\sigma_1 + (1-\ell)\sigma_2) \leq \ell\psi(\sigma_1) + (1-\ell)\psi(\sigma_2), \tag{1}$$

for each $\sigma_1, \sigma_2 \in J$ and $\ell \in [0,1]$ holds.

2020 *Mathematics Subject Classification*. 26D15, 26D10, 26A51, 26A15.
Keywords. Harmonically convex function, Hermite-Hadamard inequality, Jensen-Mercer inequality, Fractional integrals, Quadrature inequalities.
Received: 29 May 2023; Revised: 26 July 2023; Accepted: 20 November 2023
Communicated by Miodrag Spalević
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The classical (**H-H**) inequality is a well-established inequality in the theory of convex functions with geometrical interpretation and many applications. It states that if ψ is a convex function on the interval $[\sigma_1, \sigma_2]$, then:

$$\psi\left(\frac{\sigma_1+\sigma_2}{2}\right) \le \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(\epsilon)d\epsilon \le \frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}.$$

The inequality holds in reversed direction if the function is concave on $[\sigma_1, \sigma_2]$.
The (**H-H**) is a powerful tool for inequalities involving convex functions, and it can be used to prove other important inequalities, such as the Jensen inequality. In [5], the authors discuss the importance of convex functions in mathematics and other fields. They also provide a detailed overview of the (**H-H**) inequality and its applications. In [6, 7], authors focuses on refinements of the (**H-H**) inequality. They introduce a new type of weighted integral that can be used to improve the lower bound for the integral mean of a convex function.

The following characteristics of convex mappings are employed for further main findings.

Definition 1.2. *[1] A mapping ψ on interval J has a support at point $\epsilon_0 \in J$, if there exists an affine function $A(\epsilon) = \psi(\epsilon_0) + c(\epsilon - \epsilon_0)$ such that $A(\epsilon) \le \psi(\epsilon)$ for all $\epsilon \in J$. The graph of support function A is support line for function ψ at point ϵ_0.*

Theorem 1.3. *[1] $\psi : (\sigma_1, \sigma_2) \to \mathbb{R}$ is a convex function iff there is at least one line of support for ψ at each $\epsilon_0 \in (\sigma_1, \sigma_2)$.*

Harmonic convex functions are a significant development of convex functions. Many work have been devoted to generalising harmonic convex functions and finding (**H-H**) type inequalities for them. Several interesting and important inequalities can be derived from harmonic convex functions. In the papers [8–10] one can see many excellent inequalities for harmonically convex functions.

Definition 1.4. *[8] A mapping $\psi : J \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is harmonically convex ($\psi \in H_K(J)$), if*

$$\psi\Big(\frac{\sigma_1\sigma_2}{\ell\sigma_1+(1-\ell)\sigma_2}\Big) \le \ell\psi(\sigma_2) + (1-\ell)\psi(\sigma_1), \tag{2}$$

for all $\sigma_1, \sigma_2 \in J$ and $\ell \in [0, 1]$.

Proposition 1.5. *[8] If $\psi : (0, \infty) \to \mathbb{R}$ is convex and nondecreasing on $(0, \infty)$, then $\psi \in H_K(J)$.*

Dragomir in [9] gave an important characterization of harmonic convex function as:

Remark 1.6. *[9] Let $[\sigma_1, \sigma_2] \subset J \subseteq (0, \infty)$, if function $\mathfrak{h} : [\frac{1}{\sigma_2}, \frac{1}{\sigma_1}] \to \mathbb{R}$ defined as $\mathfrak{h}(\epsilon) = \psi(\frac{1}{\epsilon})$, then ψ is harmonically convex on $[\sigma_1, \sigma_2]$ iff $\mathfrak{h}$ is convex on $[\frac{1}{\sigma_2}, \frac{1}{\sigma_1}]$.*

(**H-H**) inequality is the utmost important and extensively used result in inequality theory. In [8], Işcan gave (**H-H**) type inequalities for harmonically convex mapping stated as:

Theorem 1.7. *Let $\psi \in H_K(J)$ and $\sigma_1, \sigma_2 \in J$ with $\sigma_1 < \sigma_2$. If $\psi \in L[\sigma_1, \sigma_2]$, then*

$$\psi\left(\frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right) \le \frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon \le \frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}, \tag{3}$$

A huge literature represent improvements and refinements of (**H-H**) integral inequalities pertaining harmonic convex functions see [11–13].
Recently, Dragomir in [14], gave the notable Jensen type inequality for harmonic convex function $\psi \in H_K(J)$ on the interval $J \subseteq (0, \infty)$, as:

$$\psi\Big(\frac{1}{\sum_{j=1}^{k}\frac{\ell_j}{\chi_j}}\Big) \le \sum_{j=1}^{k}\ell_j\psi(\chi_j), \tag{4}$$

holds for all $\chi_j \in J$, $\ell_j \in [0,1]$ such that $\sum_{j=1}^{k} \ell_j = 1$ for $(j = 1, 2, 3, ..., k)$.

In 2020, a new variant of Jensen-type inequality was introduced by Baloch et al. in [15] as: if $\psi \in H_K(J)$ on the interval $J = [\sigma_1, \sigma_2] \subseteq (0, \infty)$, then

$$\psi\Big(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \sum_{j=1}^{k} \frac{\ell_j}{\chi_j}}\Big) \leq \psi(\sigma_1) + \psi(\sigma_2) - \sum_{j=1}^{k} \ell_j \psi(\chi_j), \tag{5}$$

for all $\chi_j \in [\sigma_1, \sigma_2]$, $\ell_j \in [0,1]$ and $(j = 1, 2, 3, ..., k)$. Moreover in [15], they also gave a new variant of Hermite-Hadamard-Mercer (**H-H-M**) inequalities by utilizing (5) for $\psi \in H_K([\sigma_1, \sigma_2])$ and $\lambda_1, \lambda_2 \in [\sigma_1, \sigma_2]$ as:

$$\begin{aligned} \psi\Big(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}}\Big) &\leq \psi(\sigma_1) + \psi(\sigma_2) - \int_0^1 \psi\left(\frac{\lambda_1\lambda_2}{\ell\lambda_1 + (1-\ell)\lambda_2}\right) d\ell \\ &\leq \psi(\sigma_1) + \psi(\sigma_2) - \psi\Big(\frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\Big). \end{aligned} \tag{6}$$

Theorem 1.8. *[16] Let $p,\ q > 1$ are conjugate exponents. If ψ, Υ are real functions defined on $J = [\sigma_1, \sigma_2]$ and if $|\psi|^p, |\Upsilon|^q$ are integrable functions on interval J, then we have*

$$\begin{aligned} &\int_{\sigma_1}^{\sigma_2} |\psi(\epsilon)\Upsilon(\epsilon)| d\epsilon \\ &\leq \frac{1}{\sigma_2 - \sigma_1} \Big\{ \Big(\int_{\sigma_1}^{\sigma_2} (\sigma_2 - \epsilon) |\psi(\epsilon)|^p d\epsilon \Big)^{\frac{1}{p}} \Big(\int_{\sigma_1}^{\sigma_2} (\sigma_2 - \epsilon) |\Upsilon(\epsilon)|^q d\epsilon \Big)^{\frac{1}{q}} \\ &+ \Big(\int_{\sigma_1}^{\sigma_2} (\epsilon - \sigma_1) |\psi(\epsilon)|^p d\epsilon \Big)^{\frac{1}{p}} \Big(\int_{\sigma_1}^{\sigma_2} (\epsilon - \sigma_1) |\Upsilon(\epsilon)|^q d\epsilon \Big)^{\frac{1}{q}} \Big\} \\ &\leq \Big(\int_{\sigma_1}^{\sigma_2} |\psi(\epsilon)|^p d\epsilon \Big)^{\frac{1}{p}} \Big(\int_{\sigma_1}^{\sigma_2} |\Upsilon(\epsilon)|^q d\epsilon \Big)^{\frac{1}{q}}. \end{aligned}$$

Fractional analysis is one of the most attractive math topics due to its implementations. The fractional operators of integral and derivative serve to improve the links between mathematics and other fields, by giving answers that are more directly connected to real-world situations. In recent decades, a strong direction of development in fractional calculus has caught the attention of researchers from numerous disciplines to examine several possible approaches to construct fractional integrals. Fractional integral and derivative operators have developed over time [17–19]. R. P. Agrwal and D. Baleanu, two well recognized scholars, present the most up-to-date, short summary of fractional calculus in their research paper "Fractional calculus in the sky" [20].

Tan et al. studied multi-parameterized inequalities involving the tempered fractional integral operators [21]. They derived new inequalities for tempered fractional integrals via convex functions, and also applied these inequalities to solve some problems in the theory of fractional calculus. Furthermore, Du et al. studied Bullen-type inequalities via generalized fractional integrals along with their applications [22]. Du et al. in [23, 24] presented fractional integral inclusions relations having exponential kernels via interval-valued convex and co-ordinated convex mappings. They established new inclusion relations for fractional double integrals of interval-valued convex functions, and they also applied these inequalities to solve some problems in the theory of interval convex analysis. For more important results pertaining fractional integral inequalities by different techniques (see [25–28]). In [17], Kilbas et al. presented well-known Riemann-Liouville fractional operator defined as:

Definition 1.9. *[17] Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $\sigma_1 < \sigma_2$ and $\psi \in L[\sigma_1, \sigma_2]$, the Riemann-Liouville fractional operators are:*

$$\mathcal{J}^{\vartheta}_{\sigma_{1+}}\psi(\epsilon) = \frac{1}{\Gamma(\vartheta)}\int_{\sigma_1}^{\epsilon}(\epsilon-\ell)^{\vartheta-1}\psi(\ell)d\ell.\ \epsilon > \sigma_1,$$
$$\mathcal{J}^{\vartheta}_{\sigma_{2-}}\psi(\epsilon) = \frac{1}{\Gamma(\vartheta)}\int_{\epsilon}^{\sigma_2}(\ell-\epsilon)^{\vartheta-1}\psi(\ell)d\ell,\ \epsilon < \sigma_2,$$

where $\vartheta > 0$ and $\Gamma(\vartheta) = \int_0^\infty e^{-\ell}\ell^{\vartheta-1}d\ell$.

In recent decades, a strong direction of development in fractional calculus has caught the attention of researchers from numerous disciplines to examine several possible approaches to construct fractional integrals. In [29], İşcan and S. Wu presented (**H-H**) type inequalities for $\psi \in H_K(J)$ in fractional integral forms as follows:

Theorem 1.10. *Let $\psi : J = [\sigma_1, \sigma_2] \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $\psi \in L[\sigma_1, \sigma_2]$ with $0 < \sigma_1 < \sigma_2$. If $\psi \in H_K(J)$ on the interval $J = [\sigma_1, \sigma_2]$, then*

$$\psi\left(\frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right) \leq \frac{\Gamma(\vartheta+1)}{2}\left(\frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\right)^{\vartheta}\left[\mathcal{J}^{\vartheta}_{\frac{1}{\sigma_1}-}(\psi\circ\mathfrak{h})(\frac{1}{\sigma_2}) + \mathcal{J}^{\vartheta}_{\frac{1}{\sigma_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\sigma_1})\right]$$
$$\leq \frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}.$$

We need some important special function used for our study.

Definition 1.11. *[30]*

1. *The Beta function has real number domain and defined as:*

$$\beta(\sigma_1, \sigma_2) = \frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1+\sigma_2)} = \int_0^1 \ell^{\sigma_1-1}(1-\ell)^{\sigma_2-1}d\ell,\ \sigma_1, \sigma_2 > 0. \tag{7}$$

2. *The Hypergeometric function is defined as:*

$${}_2\mathfrak{F}_1(\sigma_1, \sigma_2; \mathfrak{c}; \mathfrak{z}) = \frac{1}{\beta(\sigma_2, \mathfrak{c}-\sigma_2)}\int_0^1 \ell^{\sigma_2-1}(1-\ell)^{\mathfrak{c}-\sigma_2-1}(1-\mathfrak{z}\ell)^{-\sigma_1}d\ell, \tag{8}$$

where $\mathfrak{c} > \sigma_2 > 0, |\mathfrak{z}| < 1$.

Lemma 1.12. *[31] For $0 < \vartheta \leq 1$ and $0 < \sigma_1 \leq \sigma_2$ we have*

$$\left|\sigma_1{}^{\vartheta} - \sigma_2{}^{\vartheta}\right| \leq (\sigma_2-\sigma_1)^{\vartheta}.$$

The aim of this analysis is to utilize fractional calculus to derive novel refinements of Mercer type inequalities for harmonically convex mapping. We employ the concept of line of support to obtain new fractional (**H-H-M**) type inequalities. Also our objective is to acquire some fractional trapezoid and midpoint type inequalities by utilising right and left Riemann-Liouville fractional integral operators via Mercer approach. In the case when we take $\vartheta = 1$ in the obtained results, inequalities of the classical Mercer type and their different refinements are derived too. Finally, the reported results were verified by diminished results and implementations.

2. Fractional Refinements of (H-H-M) type Inequalities

In this section, we develop several fractional refinements of (**H-H-M**) type inequalities.

Theorem 2.1. *Under the same assumptions as in Theorem 1.10 with $\vartheta > 0$, we get following two inequalities:*

$$\psi\Big(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\Big) \leq \psi(\sigma_1)+\psi(\sigma_2)-\Gamma(\vartheta+1)\Big(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\Big)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_1})$$
$$\leq \psi(\sigma_1)+\psi(\sigma_2)-\psi\Big(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\Big), \tag{9}$$

and

$$\psi\Big(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\Big) \leq \Gamma(\vartheta+1)\Big(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\Big)^{\vartheta}\mathcal{J}^{\vartheta}_{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)-}(\psi\circ\mathfrak{h})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)$$
$$\leq \frac{\vartheta\psi\left(\frac{1}{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)}\right)+\psi\left(\frac{1}{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)}\right)}{\vartheta+1}$$
$$\leq \psi(\sigma_1)+\psi(\sigma_2)-\frac{\psi(\lambda_1)+\vartheta\psi(\lambda_2)}{\vartheta+1}, \tag{10}$$

for all $\lambda_1,\lambda_2\in[\sigma_1,\sigma_2]$ and $\mathfrak{h}(\epsilon)=\psi(\frac{1}{\epsilon})$, $\epsilon\in[\frac{1}{\sigma_2},\frac{1}{\sigma_1}]$.

Proof. Since ψ is harmonic convex on $[\sigma_1,\sigma_2]$ by using Remark 1.6, $\mathfrak{h}(\epsilon)=\psi(\frac{1}{\epsilon})$ is convex on $[\frac{1}{\sigma_2},\frac{1}{\sigma_1}]$. Hence, using 1.3, there is at least one line of support.

$$A(\epsilon)=\mathfrak{h}(\epsilon_0)+c\,(\epsilon-\epsilon_0)\leq\mathfrak{h}(\epsilon), \tag{11}$$

we put $\epsilon_0=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}$ and $\epsilon=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}$ in 11.

$$A(\epsilon)=\mathfrak{h}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right)+c\left(\epsilon-\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right)\leq\mathfrak{h}(\epsilon),$$

for all $\epsilon\in[\frac{1}{\sigma_1},\frac{1}{\sigma_2}]$ and $c\in\left[\mathfrak{h}'_-\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_2+\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\right),\mathfrak{h}'_+\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_2+\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\right)\right]$.

$$\Rightarrow\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)+c\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}-\frac{1}{\sigma_1}-\frac{1}{\sigma_2}+\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right)$$
$$\leq\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right). \tag{12}$$

Multiplying above inequality (12) with $\vartheta\ell^{\vartheta-1}$ and integrating w.r.t 'ℓ' over $[0,1]$, we obtain

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)+c\left(-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}+\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right)$$
$$\leq\vartheta\int_0^1\ell^{\vartheta-1}\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)d\ell$$

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)\leq\vartheta\int_0^1\ell^{\vartheta-1}\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)d\ell. \tag{13}$$

Applying Mercer's inequality, we have

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)\le\vartheta\int_0^1\ell^{\vartheta-1}\Big[\psi(\sigma_1)+\psi(\sigma_2)-\ell\psi(\lambda_2)-(1-\ell)\psi(\lambda_1)\Big]d\ell. \tag{14}$$

Since ψ is harmonic convex, we have $-(\ell\psi(\lambda_2)+(1-\ell)\psi(\lambda_1))\le-\psi\left(\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}\right)$ and (14) becomes

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)\le\psi(\sigma_1)+\psi(\sigma_2)-\vartheta\int_0^1\ell^{\vartheta-1}\psi\left(\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}\right)d\ell. \tag{15}$$

Substitute $\frac{1}{\mathfrak{u}}=\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}$ in (15), we obtain

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)\le\psi(\sigma_1)+\psi(\sigma_2)-\vartheta\int_{\frac{1}{\lambda_2}}^{\frac{1}{\lambda_1}}\frac{\left(\frac{1}{\lambda_1}-\mathfrak{u}\right)^{\vartheta-1}}{\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)^{\vartheta-1}}\psi(\frac{1}{\mathfrak{u}})\frac{d\mathfrak{u}}{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}$$

$$\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right)\le\psi(\sigma_1)+\psi(\sigma_2)-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_1}). \tag{16}$$

Now for second inequality of (9). Put $\epsilon_0=\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}$ and $\epsilon=\frac{\ell\lambda_1+(1-\ell)\lambda_2}{\lambda_1\lambda_2}$ in (11) we get:

$$\mathfrak{h}\left(\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right)+c\left[\frac{\ell\lambda_1+(1-\ell)\lambda_2}{\lambda_1\lambda_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right]\le\mathfrak{h}\left(\frac{\ell\lambda_1+(1-\ell)\lambda_2}{\lambda_1\lambda_2}\right)$$

$$\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right)+c\left[\frac{\ell\lambda_1+(1-\ell)\lambda_2}{\lambda_1\lambda_2}-\frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}\right]\le\psi\left(\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}\right),$$

for all $\ell\in[0,1]$. Multiplying above inequality with $\vartheta\ell^{\vartheta-1}$ and integrating over $[0,1]$ w.r.t 'ℓ', we have:

$$\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right)\le\vartheta\int_0^1\ell^{\vartheta-1}\psi\left(\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}\right)d\ell.$$

Put $\frac{1}{\mathfrak{u}}=\frac{\lambda_1\lambda_2}{\ell\lambda_1+(1-\ell)\lambda_2}$, we obtain

$$\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right)\le\vartheta\int_{\frac{1}{\lambda_2}}^{\frac{1}{\lambda_1}}\frac{\left(\frac{1}{\lambda_1}-\mathfrak{u}\right)^{\vartheta-1}}{\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)^{\vartheta-1}}\psi(\frac{1}{\mathfrak{u}})\frac{d\mathfrak{u}}{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}$$

$$\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right)\le\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_1})$$

$$-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_1})\le-\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right). \tag{17}$$

Adding $\psi(\sigma_1)+\psi(\sigma_2)$ on both sides of 17, we have

$$\psi(\sigma_1)+\psi(\sigma_2)-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_1})$$
$$\le\psi(\sigma_1)+\psi(\sigma_2)-\psi\left(\frac{(\vartheta+1)\lambda_1\lambda_2}{\vartheta\lambda_1+\lambda_2}\right), \tag{18}$$

and on combining (16) and (18), we obtain (9). This completes the proof.

Now we prove (10). Let $\epsilon = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1} \Rightarrow \ell = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\epsilon - (\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1})\right]$ and (13) becomes

$$\begin{aligned}
&\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right) \\
&\leq \vartheta\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta} \int_{\frac{1}{\sigma_1}+\frac{1}{\sigma_1}-\frac{1}{\lambda_1}}^{\frac{1}{\sigma_1}+\frac{1}{\sigma_1}-\frac{1}{\lambda_2}} \left[\left(\mathfrak{u} - (\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1})\right)\right]^{\vartheta-1} \psi(\frac{1}{\mathfrak{u}})d\mathfrak{u} \\
&= \Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta} \mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2})-}(\psi\circ\mathfrak{h})\Big(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}\Big)
\end{aligned}$$

$$\begin{aligned}
&\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\vartheta\lambda_1+\lambda_2}{(\vartheta+1)\lambda_1\lambda_2}}\right) \\
&\leq \Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta} \mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2})-}(\psi\circ\mathfrak{h})\Big(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}\Big).
\end{aligned} \tag{19}$$

Now we prove other two inequalities of (10). By using the harmonic convexity of ψ on $[\sigma_1, \sigma_2]$, we have:

$$\begin{aligned}
&\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right) = \psi\left(\frac{1}{\ell\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right) + (1-\ell)\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}\right)}\right) \\
&\Rightarrow \psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right) \leq \ell\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right) + (1-\ell)\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right) \\
&\leq \psi(\sigma_1) + \psi(\sigma_2) - \ell\psi(\lambda_2) - (1-\ell)\psi(\lambda_1).
\end{aligned}$$

Multiplying above inequality with $\vartheta\ell^{\vartheta-1}$ and integrating over $[0, 1]$ with respect to ℓ, we have:

$$\begin{aligned}
&\vartheta\int_0^1 \ell^{\vartheta-1}\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right)d\ell \\
&\leq \vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right)\int_0^1 \ell^{\vartheta}d\ell + \vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right)\int_0^1 (\ell^{\vartheta-1} - \ell^{\vartheta})d\ell \\
&\leq \psi(\sigma_1) + \psi(\sigma_2) - \vartheta\psi(\lambda_2)\int_0^1 \ell^{\vartheta}d\ell - \vartheta\psi(\lambda_1)\int_0^1 (\ell^{\vartheta-1} - \ell^{\vartheta})d\ell
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \vartheta\int_0^1 \ell^{\vartheta-1}\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right)d\ell \\
&\leq \frac{\vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) + \psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{\vartheta+1} \\
&\leq \psi(\sigma_1) + \psi(\sigma_2) - \frac{\vartheta\psi(\lambda_2) + \psi(\lambda_1)}{\vartheta+1},
\end{aligned} \tag{20}$$

by changing variable, (20) becomes

$$\begin{aligned}
&\vartheta\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\int_{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}^{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\left[\left(\mathfrak{u}-(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)\right]^{\vartheta-1}\psi(\frac{1}{\mathfrak{u}})d\mathfrak{u}\\
&\leq\frac{\vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{\vartheta+1}\leq\psi(\sigma_1)+\psi(\sigma_2)-\frac{\vartheta\psi(\lambda_2)+\psi(\lambda_1)}{\vartheta+1}\\
&\Rightarrow\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2})-}(\psi\circ\mathfrak{h})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\\
&\leq\frac{\vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{\vartheta+1}\leq\psi(\sigma_1)+\psi(\sigma_2)-\frac{\vartheta\psi(\lambda_2)+\psi(\lambda_1)}{\vartheta+1}. \qquad (21)
\end{aligned}$$

By combining (19) and (21), we obtain (10). □

Remark 2.2. *If we put $\vartheta = 1$ in Theorem 2.1, we get (**H-H-M**) inequalities* (6) *and*

$$\begin{aligned}
&\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}}\right)\leq\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon\\
&\leq\frac{\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)}{2}\leq\psi(\sigma_1)+\psi(\sigma_2)-\frac{\psi(\lambda_1)+\psi(\lambda_2)}{2}, \qquad (22)
\end{aligned}$$

*for all $\lambda_1, \lambda_2 \in [\sigma_1, \sigma_2]$. It is pertinent to mention that (**H-H-M**) inequalities* (22) *are better than the inequalities proved by Baloch et al. in [32]. As* (22) *are refinements of these inequalities in [32].*

Remark 2.3. *If we put $\vartheta = 1, \sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in* (10)*, we have* (3) *(**H-H**) inequality appeared in [8].*

3. New Mercer Trapezoidal Type Equalities

Throughout the rest of the paper we assumed the following assumptions:
A_1 = Let $\psi : J = [\sigma_1, \sigma_2] \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $[\sigma_1, \sigma_2]$ with $0 < \sigma_1 \leq \sigma_2$.

$$\begin{aligned}
I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)&=\frac{\vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{\vartheta+1}\\
&-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2})-}(\psi\circ h)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right),\\
I_{\psi_2}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)&=\frac{\vartheta\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)}{\vartheta+1}\\
&-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})+}(\psi\circ h)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right),
\end{aligned}$$

where $\lambda_1, \lambda_2 \in J$ with $\lambda_1 < \lambda_2$.
New Mercer Trapezoidal type Lemmas for harmonic convex function are presented in this section.

Lemma 3.1. *If $\psi' \in \mathcal{L}[\sigma_1, \sigma_2]$ along with assumption A_1, then following identity for fractional integral holds:*

$$I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2) = \frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\int_0^1 \frac{1-(\vartheta+1)\ell^{\vartheta}}{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}\right)^2}\psi'\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)d\ell, \tag{23}$$

where $\mathfrak{h}(\epsilon) = \frac{1}{\epsilon}$, $\epsilon \in [\frac{1}{\sigma_2}, \frac{1}{\sigma_1}]$ and $\vartheta > 0$.

Proof. Let $A_\ell = \frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}$. It suffices to note that

$$\begin{aligned}I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2) &= \frac{\lambda_2-\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\int_0^1 \frac{1}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell - \frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\int_0^1 \frac{\ell^{\vartheta}}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell\\ &= I_1 - I_2.\end{aligned} \tag{24}$$

Integrating by parts we get

$$\begin{aligned}I_1 &= \frac{\lambda_2-\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\int_0^1 \frac{1}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell = -\frac{1}{\vartheta+1}\psi\left(\frac{1}{A_\ell}\right)\Bigg|_0^1\\ &= -\frac{1}{\vartheta+1}\left[\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_2}}\right)-\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_1}}\right)\right].\end{aligned} \tag{25}$$

Similarly, we get

$$\begin{aligned}I_2 &= \frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\int_0^1 \frac{\ell^{\vartheta}}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell = -\ell^{\vartheta}\psi\left(\frac{1}{A_\ell}\right)\Bigg|_0^1 + \vartheta\int_0^1 \ell^{\vartheta-1}\psi\left(\frac{1}{A_\ell}\right)d\ell\\ &= -\psi\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_2}\right)+\vartheta\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\int_{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_1}}^{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_2}}\left[\mathfrak{u}-\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_1}\right)\right]^{\vartheta-1}\psi(\frac{1}{\mathfrak{u}})d\mathfrak{u}\\ &= -\psi\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_2}\right)+\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2})-}(\psi\circ h)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right).\end{aligned} \tag{26}$$

Using 25 and 26 in 24, we get the equality 23. □

Remark 3.2. *If we put $\vartheta = 1$ in Lemma 3.1, then we obtain*

$$\begin{aligned}&\frac{\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{2} - \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon\\ &= \frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2}\int_0^1 \frac{1-2\ell}{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}\right)^2}\psi'\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)d\ell,\end{aligned}$$

which is new in literature.

Remark 3.3. *If we put $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in Lemma 3.1, then we obtain*

$$\begin{aligned}&\frac{\vartheta\psi(\lambda_1)+\psi(\lambda_2)}{\vartheta+1} - \Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}\mathcal{J}^{\vartheta}_{\frac{1}{\lambda_1}-}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_2})\\ &= \frac{\lambda_1\lambda_2(\lambda_2-\lambda_1)}{\vartheta+1}\int_0^1 \frac{1-(\vartheta+1)\ell^{\vartheta}}{(\ell\lambda_2+(1-\ell)\lambda_1)^2}\psi'\left(\frac{\lambda_1\lambda_2}{\ell\lambda_2+(1-\ell)\lambda_1}\right)d\ell,\end{aligned}$$

which is appeared in [33].

Remark 3.4. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Lemma 3.1, then we obtain*

$$\frac{\psi(\lambda_1) + \psi(\lambda_2)}{2} - \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon$$
$$= \frac{\lambda_2 - \lambda_1}{2\lambda_1\lambda_2}\int_0^1 \frac{1-2\ell}{(\ell\lambda_2 + (1-\ell)\lambda_1)^2}\psi'\left(\frac{\lambda_1\lambda_2}{\ell\lambda_2 + (1-\ell)\lambda_1}\right)d\ell,$$

which is appeared in [8].

Lemma 3.5. *Under the same assumptions as in Lemma 3.1, we have following identity:*

$$I_{\psi_2}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2) = \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2(\vartheta + 1)}\int_0^1 \frac{(\vartheta+1)(1-\ell)^\vartheta - 1}{\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}\right)^2}\psi'\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right)d\ell, \tag{27}$$

where $\mathfrak{h}(\epsilon) = \frac{1}{\epsilon}$, $\epsilon \in [\frac{1}{\sigma_2}, \frac{1}{\sigma_1}]$ and $\vartheta > 0$.

Proof. Let $A_\ell = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}$. It suffices to note that

$$I_{\psi_2}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2) = \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}\int_0^1 \frac{(1-\ell)^\vartheta}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell - \frac{\lambda_2 - \lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\int_0^1 \frac{1}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell$$
$$= I_1 - I_2. \tag{28}$$

Integrating by parts we get

$$I_1 = \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}\int_0^1 \frac{(1-\ell)^\vartheta}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell = -(1-\ell)^\vartheta\psi\left(\frac{1}{A_\ell}\right)\Big|_0^1 - \vartheta\int_0^1 (1-\ell)^{\vartheta-1}\psi\left(\frac{1}{A_\ell}\right)d\ell$$
$$= \psi\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{g}{\lambda_1}\right) - \vartheta\left(\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\right)^\vartheta \int_{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_1}}^{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{g}{\lambda_2}}\left[\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{g}{\lambda_2} - \mathfrak{u}\right)\right]^{\vartheta-1}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon$$
$$= \psi\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{g}{\lambda_1}\right) - \Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\right)^\vartheta \mathcal{J}^\vartheta_{(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})+}(\psi \circ h)\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right). \tag{29}$$

Similarly, we get

$$I_2 = \frac{\lambda_2 - \lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\int_0^1 \frac{1}{A_\ell^2}\psi'\left(\frac{1}{A_\ell}\right)d\ell = -\frac{1}{\vartheta+1}\psi\left(\frac{1}{A_\ell}\right)\Big|_0^1$$
$$= -\frac{1}{\vartheta+1}\left[\psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{g}{\lambda_2}}\right) - \psi\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{g}{\lambda_1}}\right)\right]. \tag{30}$$

Using 29 and 30 in 28, we get the equality 27. □

Remark 3.6. *If we put $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in Lemma 3.5, then we obtain an equality appeared in [33].*

Remark 3.7. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Lemma 3.5, then we obtain an equality appeared in [8].*

4. Mercer Trapezoidal Type Fractional Integral Inequalities

Now we derive some novel trapezoidal type fractional integral inequalities by utilising above equalities.

Theorem 4.1. *If $|\psi'|^q \in H_K(J)$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $\psi' \in \mathcal{L}[\sigma_1, \sigma_2]$ along with assumption A_1, then the following inequality for fractional integral holds:*

$$|I_{\psi_1}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2)| \\ \leq \frac{\lambda_2 - \lambda_1}{(\vartheta + 1)\lambda_1\lambda_2} \mathbb{I}_1^{\frac{1}{p}}(\lambda_1, \lambda_2, \vartheta) \begin{bmatrix} \mathbb{I}_2(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_1)|^q + \mathbb{I}_2(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_3(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_2)|^q - \mathbb{I}_4(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_1)|^q \end{bmatrix}^{\frac{1}{q}}, \tag{31}$$

where

$$\mathbb{I}_1(\lambda_1, \lambda_2, \vartheta) = \frac{1}{\sqrt[\vartheta]{\vartheta + 1}(\vartheta p + 1)} + \frac{(\sqrt[\vartheta]{\vartheta + 1} - 1)^{\vartheta p + 1}}{\sqrt[\vartheta]{\vartheta + 1}(\vartheta p + 1)}$$

$$\mathbb{I}_2(\lambda_1, \lambda_2, \vartheta) = \left[\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 1; 2, 1 - \frac{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right)$$

$$\mathbb{I}_3(\lambda_1, \lambda_2, \vartheta) = \frac{1}{2}\left[\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 2; 3, 1 - \frac{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right)$$

$$\mathbb{I}_4(\lambda_1, \lambda_2, \vartheta) = \frac{1}{2}\left[\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 1; 3, 1 - \frac{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right),$$

and $0 < \vartheta \leq 1$.

Proof. Using Lemma 3.1 and Hölder's inequality, we exploit the harmonically convex property of $|\psi'|^q$ to deduce:

$$|I_{\psi_1}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2)| \leq \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2(\vartheta + 1)} \int_0^1 \frac{|1 - (\vartheta + 1)\ell^\vartheta|}{\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}\right)^2} \left|\psi'\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1}}\right)\right| d\ell.$$

Let $A_\ell = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{\ell}{\lambda_2} - \frac{(1-\ell)}{\lambda_1} = \ell\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}\right) + (1 - \ell)\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}\right)$.

$$|I_{\psi_1}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2)| \\ \leq \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2(\vartheta + 1)} \int_0^1 \left(|1 - (\vartheta + 1)\ell^\vartheta|^p d\ell\right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{A_\ell^{2q}} \left|\psi'\left(\frac{1}{A_\ell}\right)\right|^q d\ell\right)^{\frac{1}{q}} \\ \leq \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2(\vartheta + 1)} \int_0^1 \left(|1 - (\vartheta + 1)\ell^\vartheta|^p d\ell\right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{A_\ell^{2q}} \begin{bmatrix} |\psi'(\sigma_1)|^q + |\psi'(\sigma_2)|^q \\ -\ell|\psi'(\lambda_2)|^q - (1 - \ell)|\psi'(\lambda_1)|^q \end{bmatrix} d\ell\right)^{\frac{1}{q}} \\ \leq \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2(\vartheta + 1)} \int_0^1 \left(|1 - (\vartheta + 1)\ell^\vartheta|^p d\ell\right)^{\frac{1}{p}} \begin{pmatrix} |\psi'(\sigma_1)|^q \int_0^1 \frac{1}{A_\ell^{2q}} d\ell + \psi'(\sigma_2)|^q \int_0^1 \frac{1}{A_\ell^{2q}} d\ell \\ -\psi'(\lambda_2)|^q \int_0^1 \frac{\ell}{A_\ell^{2q}} d\ell - \psi'(\lambda_1)|^q \int_0^1 \frac{(1-\ell)}{A_\ell^{2q}} d\ell \end{pmatrix}^{\frac{1}{q}}. \tag{32}$$

After evaluating the integrals in (32), we obtain

$$\int_0^1 |1-(\vartheta+1)\ell^{\vartheta}|^p d\ell = \int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}} (1-(\vartheta+1)\ell^{\vartheta})^p d\ell + \int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1 ((\vartheta+1)\ell^{\vartheta}-1)^p d\ell$$
$$\leq \int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}} (1-(\sqrt[\vartheta]{\vartheta+1}\ell))^{\vartheta p} d\ell + \int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1 (\sqrt[\vartheta]{\vartheta+1}\ell-1)^{\vartheta p} d\ell$$
$$= \frac{(1-\sqrt[\vartheta]{\vartheta+1}\ell)^{\vartheta p+1}}{-\sqrt[\vartheta]{\vartheta+1}(\vartheta p+1)}\Bigg|_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}} + \frac{(\sqrt[\vartheta]{\vartheta+1}\ell-1)^{\vartheta p+1}}{\sqrt[\vartheta]{\vartheta+1}(\vartheta p+1)}\Bigg|_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1$$
$$= \frac{1}{\sqrt[\vartheta]{\vartheta p+1}(\vartheta+1)} + \frac{(\sqrt[\vartheta]{\vartheta+1}-1)^{\vartheta p+1}}{\sqrt[\vartheta]{\vartheta+1}(\vartheta p+1)}$$
$$= \mathbb{I}_1(\lambda_1,\lambda_2,\vartheta) \tag{33}$$

$$\int_0^1 \frac{1}{A_\ell^{2q}} d\ell = \int_0^1 \frac{1}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell$$
$$= \int_0^1 \frac{1}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell$$
$$= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 \left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell$$
$$= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$= \mathbb{I}_2(\lambda_1,\lambda_2,\vartheta) \tag{34}$$

$$\int_0^1 \frac{\ell}{A_\ell^{2q}} d\ell = \int_0^1 \frac{\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell$$
$$= \int_0^1 \frac{1-\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell$$
$$= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 (1-\ell)\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell$$
$$= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$= \mathbb{I}_3(\lambda_1,\lambda_2,\vartheta) \tag{35}$$

$$\int_0^1 \frac{1-\ell}{A_\ell^{2q}} d\ell = \int_0^1 \frac{1-\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell$$

$$= \int_0^1 \frac{\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell$$

$$= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 \ell\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell$$

$$= \frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 2; 3, 1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$

$$= \mathbb{I}_4(\lambda_1, \lambda_2, \vartheta). \tag{36}$$

Using 33, 34,35 and 36 in 32, we get 31. This completes the proof. □

Remark 4.2. *If we put $\vartheta = 1$ in Theorem 4.1, then we obtain*

$$\left|\frac{\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{2} - \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}} \frac{\psi(\epsilon)}{\epsilon^2} d\epsilon\right|$$

$$\leq \frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2}\mathbb{I}_1^{\frac{1}{p}}(\lambda_1,\lambda_2)\left[\begin{array}{c}\mathbb{I}_2(\lambda_1,\lambda_2)|\psi'(\sigma_1)|^q+\mathbb{I}_2(\lambda_1,\lambda_2)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_3(\lambda_1,\lambda_2)|\psi'(\lambda_2)|^q-\mathbb{I}_4(\lambda_1,\lambda_2)|\psi'(\lambda_1)|^q\end{array}\right]^{\frac{1}{q}},$$

where

$$\mathbb{I}_1(\lambda_1,\lambda_2) = \int_0^1 |1-2\ell|^p d\ell = \frac{1}{p+1},$$

and $\mathbb{I}_2, \mathbb{I}_3$ and $\mathbb{I}_4$ are same as in Theorem 4.1. Which is new in literature.

Remark 4.3. *If we put $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in Theorem 4.1, then we obtain an inequality appeared in [33].*

Remark 4.4. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Theorem 4.1, then we obtain an inequality appeared in [8].*

Example 4.5. ***Case 1***: *Let $\psi(\epsilon) = \frac{\epsilon^3}{3}$, $\epsilon > 0$. If we set $\vartheta = 1$, $\sigma_1 = \lambda_1$, $\sigma_1 = \lambda_2$, $\lambda_1 = 1$, $\lambda_2 = 5$ and $q \in [1.1, 100]$, then by Proposition 1.5 mapping, $|\psi'(\epsilon)|^q = \epsilon^{2q}$ is harmonic convex. So, we can say that the inequality 31 will deduce to*

$$-10\left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}} \times \left[\mu_3(1)^{2q}+\mu_4(5)^{2q}\right]^{\frac{1}{q}}$$

$$\leq 21-\frac{5}{4}\int_1^5 \frac{\epsilon}{3} d\epsilon \approx 16$$

$$\leq 10\left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}} \times \left[\mu_3(1)^{2q}+\mu_4(5)^{2q}\right]^{\frac{1}{q}}, \tag{37}$$

where

$$\mu_3 = \frac{\left[1^{2-2q}+5^{1-2q}[4(1-2q)-1]\right]}{2(4)^2(1-q)(1-2q)}$$

$$\mu_4 = \frac{\left[5^{2-2q}+1^{1-2q}[4(1-2q)-1]\right]}{2(4)^2(1-q)(1-2q)}.$$

Case 2: *Let* $\psi(\epsilon)=\frac{\epsilon^3}{3}$, $\epsilon>0$. *If we set* $\vartheta=1$, $\sigma_1=1$, $\sigma_2=5$, $q=2$ *and* $\lambda_1=[1,1.5]$, $\lambda_2=[4,5]$, *then we can say that the inequality 31 will deduce to*

$$
\begin{aligned}
&-\frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2}Z_4{}^{\frac{1}{2}}\left[\begin{array}{c} Z_5(1)^4+Z_5(5)^4 \\ -Z_6(\lambda_2)^4-Z_7(\lambda_1)^4 \end{array}\right]^{\frac{1}{2}} \\
&\leq \frac{\left(\frac{1}{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}}\right)^3+\left(\frac{1}{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_1}}\right)^3}{6}-\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{6}{5}-\frac{1}{\lambda_1}}}^{\frac{1}{\frac{6}{5}-\frac{1}{\lambda_2}}}\frac{\epsilon}{3}d\epsilon \\
&\leq \frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2}Z_4{}^{\frac{1}{2}}\left[\begin{array}{c} Z_5(1)^4+Z_5(5)^4 \\ -Z_6(\lambda_2)^4-Z_7(\lambda_1)^4 \end{array}\right]^{\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
Z_4 &= \int_0^1 |1-2\ell|^2 d\ell = \frac{1}{3} \\
Z_5 &= \left[\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}\right]^{-4} {}_2F_1\left(4,1;2,1-\frac{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_1}}{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}}\right) \\
Z_6 &= \frac{1}{2}\left[\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}\right]^{-4} {}_2F_1\left(4,2;3,1-\frac{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_1}}{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}}\right) \\
Z_7 &= \frac{1}{2}\left[\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}\right]^{-4} {}_2F_1\left(4,1;3,1-\frac{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_1}}{\frac{1}{1}+\frac{1}{5}-\frac{1}{\lambda_2}}\right).
\end{aligned}
$$

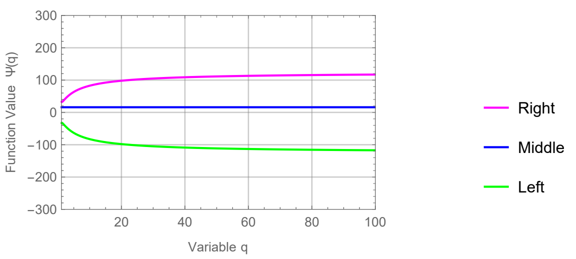

Figure 1: Case 1 visual illustration for $\vartheta=1$, $\sigma_1=\lambda_1$, $\sigma_1=\lambda_2$, $\lambda_1=1$, $\lambda_2=5$ and $q\in[1.1,100]$.

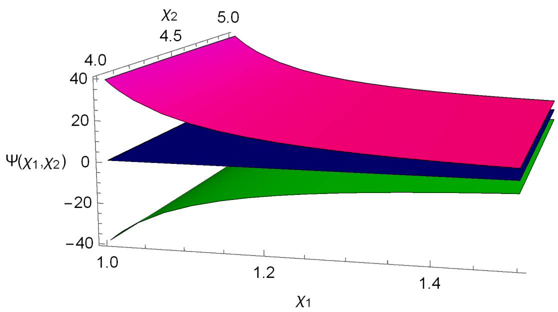

Figure 2: Case 2 visual illustration for $\vartheta=1$, $\sigma_1=1$, $\sigma_2=5$, $q=2$ and $\lambda_1=[1,1.5]$, $\lambda_2=[4,5]$.

Left, middle, and right mappings from the inequalities (31) *are plotted against* $q\in[1.1,100]$ *in Figure 1. Left, middle, and right mappings from the inequalities* (31) *are plotted against* $\lambda_1=[1,1.5]$, $\lambda_2=[4,5]$ *in Figure 2. The graphs of the functions prove the correctness of the Theorem 4.1 with* $\vartheta=1$.

Theorem 4.6. *If* $|\psi'|^q\in H_K(J)$ *for some fixed* $q\geq 1$ *and* $\psi'\in\mathcal{L}[\sigma_1,\sigma_2]$ *along with assumption* A_1, *then the following inequality for fractional integral holds:*

$$
\begin{aligned}
&|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)| \\
&\leq \frac{\lambda_2-\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}\mathbb{I}_5{}^{1-\frac{1}{q}}(\lambda_1,\lambda_2,\vartheta)\left[\begin{array}{c} \mathbb{I}_5(\lambda_1,\lambda_2,\vartheta)|\psi'(\sigma_1)|^q+\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_6(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_1)|^q-\mathbb{I}_7(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_2)|^q \end{array}\right]^{\frac{1}{q}}, \qquad (38)
\end{aligned}
$$

where

$$\mathbb{I}_5=\left[\begin{array}{r} 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ -\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\\ -\frac{2}{\vartheta+1}\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ +\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\end{array}\right]$$

$$\mathbb{I}_6=\left[\begin{array}{r} \left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ -\frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\\ -(\frac{1}{\vartheta+1})^{\frac{2}{\vartheta}}\frac{2(\vartheta+1)}{\vartheta+2}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ +\frac{(\vartheta+1)}{\vartheta+2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\end{array}\right]$$

$$\mathbb{I}_7(\lambda_1,\lambda_2,\vartheta)=\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta)-\mathbb{I}_6(\lambda_1,\lambda_2,\vartheta),$$

and $\vartheta>0$.

Proof. Using Lemma 3.1 and Power-mean inequality, we exploit the harmonically convex property of $|\psi'|^q$ to deduce:

$$|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)|\leq\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}\right)^2}\left|\psi'\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)\right|d\ell$$

Let $A_\ell=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}=\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)$.

$$\begin{aligned}&|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)|\\ &\leq\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\left(\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}d\ell\right)^{1-\frac{1}{q}}\left[\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}\left|\psi'\left(\frac{1}{A_\ell}\right)\right|^q d\ell\right]^{\frac{1}{q}}\\ &\leq\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\left(\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}d\ell\right)^{1-\frac{1}{q}}\\ &\left[\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}[|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q-\ell|\psi'(\lambda_2)|^q-(1-\ell)|\psi'(\lambda_1)|^q]d\ell\right]^{\frac{1}{q}}\\ &\leq\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\mathbb{I}_5{}^{1-\frac{1}{q}}(\lambda_1,\lambda_2,\vartheta)\left[\begin{array}{c}\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta)|\psi'(\sigma_1)|^q+\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta)|\psi'(\sigma_2)|^q\\ -\mathbb{I}_6(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_1)|^q-\mathbb{I}_7(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_2)|^q\end{array}\right],\end{aligned}\tag{39}$$

calculating $\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta),\mathbb{I}_6(\lambda_1,\lambda_2,\vartheta)$ and $\mathbb{I}_7(\lambda_1,\lambda_2,\vartheta)$, we have

$$\begin{aligned}\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}d\ell&=\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{1-(\vartheta+1)\ell^\vartheta}{(A_\ell)^2}d\ell+\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{(\vartheta+1)\ell^\vartheta-1}{(A_\ell)^2}d\ell\\ &=\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{1}{(A_\ell)^2}d\ell-\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{1}{(A_\ell)^2}d\ell-(\vartheta+1)\Big[\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{\ell^\vartheta}{(A_\ell)^2}d\ell-\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{\ell^\vartheta}{(A_\ell)^2}d\ell\Big]\end{aligned}$$

$$= 2\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}} \frac{1}{(A_\ell)^2}d\ell - \int_0^1 \frac{1}{(A_\ell)^2}d\ell - (\vartheta+1)\Big[2\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}} \frac{\ell^\vartheta}{(A_\ell)^2}d\ell - \int_0^1 \frac{\ell^\vartheta}{(A_\ell)^2}d\ell\Big]$$

$$= 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\int_0^1 \frac{1}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right))^2}d\mathfrak{u}$$

$$-\int_0^1 \frac{1}{(\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right))^2}d\mathfrak{u}$$

$$-\Big[2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\int_0^1 \frac{\mathfrak{u}^\vartheta}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right))^2}d\mathfrak{u}$$

$$-(\vartheta+1)\int_0^1 \frac{(1-\mathfrak{u})^\vartheta}{(\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right))^2}d\mathfrak{u}\Big]$$

$$= 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\int_0^1 \frac{1}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon))\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right))^2}d\epsilon$$

$$-\int_0^1 \frac{1}{(\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right))^2}d\mathfrak{u}$$

$$-\Big[2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\int_0^1 \frac{(1-\epsilon)^\vartheta}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon))\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right))^2}d\epsilon$$

$$-(\vartheta+1)\int_0^1 \frac{(1-\mathfrak{u})^\vartheta}{\left(\mathfrak{u}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\mathfrak{u})\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right)^2}d\mathfrak{u}\Big]$$

$$= 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}\int_0^1\left[1-\epsilon\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\right]^{-2}d\epsilon$$

$$-\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}\int_0^1\left(1-\mathfrak{u}\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2}d\mathfrak{u}-\Big[\frac{2}{\vartheta+1}\sqrt[\vartheta]{\frac{1}{\vartheta+1}}$$

$$\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}\int_0^1(1-\epsilon)^\vartheta\left(1-\epsilon\left[1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right]\right)^{-2}d\epsilon$$

$$-(\vartheta+1)\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2}\int_0^1(1-\mathfrak{u})^\vartheta\left(1-\mathfrak{u}\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2}d\mathfrak{u}\Big]$$

$$
=\left[\begin{array}{r} 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ -\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \\ -\frac{2}{\vartheta+1}\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ +\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{array}\right]
$$

$$
=\mathbb{I}_5(\lambda_1,\lambda_2,\vartheta) \tag{40}
$$

$$
\int_0^1\frac{|1-(\vartheta+1)\ell^\vartheta|}{(A_\ell)^2}\ell d\ell=\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{1-(\vartheta+1)\ell^\vartheta}{(A_\ell)^2}\ell d\ell+\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{(\vartheta+1)\ell^\vartheta-1}{(A_\ell)^2}\ell d\ell
$$

$$
=\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{\ell}{(A_\ell)^2}d\ell-\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{\ell}{(A_\ell)^2}d\ell-(\vartheta+1)\Big[\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{\ell^{\vartheta+1}}{(A_\ell)^2}d\ell-\int_{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}^1\frac{\ell^{\vartheta+1}}{(A_\ell)^2}d\ell\Big]
$$

$$
=2\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{\ell}{(A_\ell)^2}d\ell-\int_0^1\frac{\ell}{(A_\ell)^2}d\ell-(\vartheta+1)\Big[2\int_0^{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}}\frac{\ell^{\vartheta+1}}{(A_\ell)^2}d\ell-\int_0^1\frac{\ell^{\vartheta+1}}{(A_\ell)^2}d\ell\Big]
$$

$$
=2\Big(\frac{1}{\vartheta+1}\Big)^{\frac{2}{\vartheta}}\int_0^1\frac{\mathfrak{u}}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u}\Big(\frac{}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big))^2}d\mathfrak{u}
$$

$$
-\int_0^1\frac{1-\mathfrak{u}}{(\mathfrak{u}\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)+(1-\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big))^2}d\mathfrak{u}
$$

$$
-\Big[2\Big(\frac{1}{\vartheta+1}\Big)^{\frac{2}{\vartheta}}\int_0^1\frac{\mathfrak{u}^{\vartheta+1}}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u}\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big))^2}d\mathfrak{u}
$$

$$
-(\vartheta+1)\int_0^1\frac{(1-\mathfrak{u})^{\vartheta+1}}{(\mathfrak{u}\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)+(1-\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big))^2}d\mathfrak{u}\Big]
$$

$$
=2\Big(\frac{1}{\vartheta+1}\Big)^{\frac{2}{\vartheta}}\int_0^1\frac{(1-\epsilon)}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon)\Big(\frac{}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon))\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big))^2}d\mathfrak{u}
$$

$$
-\int_0^1\frac{1-\mathfrak{u}}{(\mathfrak{u}\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)+(1-\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big))^2}d\mathfrak{u}
$$

$$
-\Big[2\Big(\frac{1}{\vartheta+1}\Big)^{\frac{2}{\vartheta}}\int_0^1\frac{(1-\epsilon)^{\vartheta+1}}{(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon)\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big)+(1-\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(1-\epsilon))\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big))^2}d\mathfrak{u}
$$

$$
-(\vartheta+1)\int_0^1\frac{(1-\mathfrak{u})^{\vartheta+1}}{(\mathfrak{u}\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)+(1-\mathfrak{u})\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big))^2}d\mathfrak{u}\Big]
$$

$$= 2\left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}\int_0^1\left[1-\epsilon\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\right]^{-2}$$

$$(1-\epsilon)d\epsilon-\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}\int_0^1\left(1-u\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2}(1-u)du$$

$$-\left[\begin{array}{r} 2\left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}\int_0^1(1-\epsilon)^{\vartheta+1}\left(1-\epsilon\left[1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right]\right)^{-2}d\epsilon \\ -(\vartheta+1)\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}\int_0^1(1-u)^{\vartheta+1}\left(1-u\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2}du \end{array}\right]$$

$$=\left[\begin{array}{r} \left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ -\frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \\ -(\frac{1}{\vartheta+1})^{\frac{2}{\vartheta}}\frac{2(\vartheta+1)}{\vartheta+2}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ +\frac{(\vartheta+1)}{\vartheta+2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{array}\right]$$

$$= \mathbb{I}_6(\lambda_1,\lambda_2,\vartheta) \tag{41}$$

$$\mathbb{I}_7(\lambda_1,\lambda_2,\vartheta) = \mathbb{I}_5(\lambda_1,\lambda_2,\vartheta) - \mathbb{I}_6(\lambda_1,\lambda_2,\vartheta). \tag{42}$$

Using 40, 41 and 42 in 39, we get the inequality of 38. This completes the proof. □

Remark 4.7. *If we put $\vartheta = 1$ in Theorem 4.6, then we obtain*

$$\left|\frac{\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{2}-\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon\right|$$
$$\leq \frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2}\mathbb{I}_5{}^{1-\frac{1}{q}}(\lambda_1,\lambda_2)\left[\begin{array}{c} \mathbb{I}_5(\lambda_1,\lambda_2)|\psi'(\sigma_1)|^q+\mathbb{I}_5(\lambda_1,\lambda_2)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_6(\lambda_1,\lambda_2)|\psi'(\lambda_1)|^q-\mathbb{I}_7(\lambda_1,\lambda_2)|\psi'(\lambda_2)|^q \end{array}\right],$$

where

$$\mathbb{I}_5(\lambda_1,\lambda_2)=\left[\begin{array}{r} \left(\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ -\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \\ -\frac{1}{2}\left(\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ +\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{array}\right]$$

$$\mathbb{I}_6(\lambda_1,\lambda_2) = \begin{bmatrix} \left(\frac{1}{4}\right)\left(\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}) + (\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2} {}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ -\frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2} {}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \\ -(\frac{1}{4})\frac{4}{3}\left(\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}) + (\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2} {}_2F_1\left(2,1;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{2}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right) \\ +\frac{2}{3}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2} {}_2F_1\left(2,1;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{bmatrix}$$

$$\mathbb{I}_7(\lambda_1,\lambda_2) = \mathbb{I}_5(\lambda_1,\lambda_2) - \mathbb{I}_6(\lambda_1,\lambda_2),$$

which is new in literature.

Remark 4.8. *If we put $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in Theorem 4.6, then we obtain an inequality appeared in [33].*

Remark 4.9. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Theorem 4.6, then we obtain an inequality appeared in [8].*

Example 4.10. ***Case 1***: *Let $\psi(\epsilon) = \frac{\epsilon^3}{3}$, $\epsilon > 0$. If we set $\vartheta = 1$, $\sigma_1 = \lambda_1$, $\sigma_1 = \lambda_2$, $\lambda_1 = 2$, $\lambda_2 = 5$ and $q \in [1.1, 100]$, then by Proposition 1.5 mapping, $|\psi'(\epsilon)|^q = \epsilon^{2q}$ is harmonic convex.*

Case 2: *Let $\psi(\epsilon) = \frac{\epsilon^3}{3}$, $\epsilon > 0$. Suppose we set $\vartheta = 1$, $\sigma_1 = 2$, $\sigma_2 = 5$, $q = 2$ and $\lambda_1 = [1, 1.5]$, $\lambda_2 = [4, 5]$.*

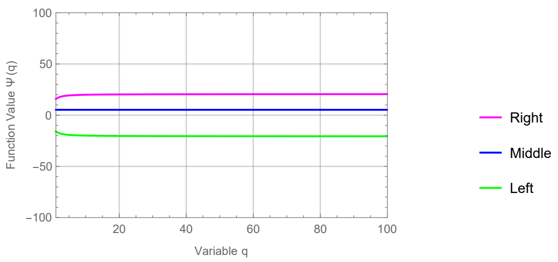

Figure 3: Case 1 visual illustration for $\vartheta = 1$, $\sigma_1 = \lambda_1$, $\sigma_1 = \lambda_2$, $\lambda_1 = 2$, $\lambda_2 = 5$ and $q \in [1.1, 100]$.

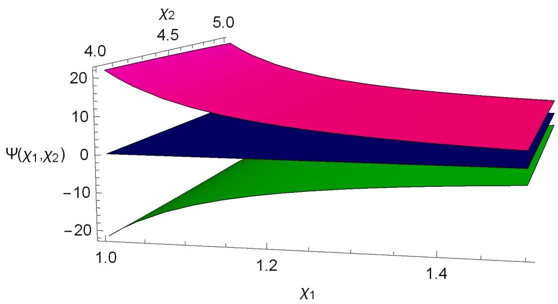

Figure 4: Case 2 visual illustration for $\vartheta = 1$, $\sigma_1 = 2$, $\sigma_2 = 5$, $q = 2$ and $\lambda_1 = [1, 1.5]$, $\lambda_2 = [4, 5]$.

Left, middle, and right mappings from the inequalities (38) *are plotted against $q \in [1.1, 100]$ in Figure 3. Left, middle, and right mappings from the inequalities* (38) *are plotted against $\lambda_1 = [1, 1.5]$, $\lambda_2 = [4, 5]$ in Figure 4. The graphs of the functions prove the correctness of the Theorem 4.6 with $\vartheta = 1$.*

Theorem 4.11. *Under the same assumptions as in Theorem 4.1, following inequality for fractional integral holds:*

$$|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)| \leq \frac{\lambda_2-\lambda_1}{(\vartheta+1)\lambda_1\lambda_2}$$
$$\left\{ \Upsilon_1^{\frac{1}{p}} \begin{bmatrix} \Lambda_1|\psi'(\sigma_1)|^q + \Lambda_1|\psi'(\sigma_2)|^q \\ -\Lambda_2|\psi'(\lambda_2)|^q - \Lambda_3|\psi'(\lambda_1)|^q \end{bmatrix}^{\frac{1}{q}} + \Upsilon_2^{\frac{1}{p}} \begin{bmatrix} \Lambda_4|\psi'(\sigma_1)|^q + \Lambda_4|\psi'(\sigma_2)|^q \\ -\Lambda_5|\psi'(\lambda_2)|^q - \Lambda_2|\psi'(\lambda_1)|^q \end{bmatrix}^{\frac{1}{q}} \right\},$$

where

$$\Lambda_1=\frac{1}{2}\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2q}{}_2F_1\left(2q,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$\Lambda_2=\frac{1}{6}\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2q}{}_2F_1\left(2q,2;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$\Lambda_3=\frac{1}{3}\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2q}{}_2F_1\left(2q,1;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$\Lambda_4=\frac{1}{2}\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2q}{}_2F_1\left(2q,2;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$\Lambda_5=\frac{1}{3}\Big[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big]^{-2q}{}_2F_1\left(2q,3;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)$$
$$\Upsilon_1=\int_0^1\Big((1-\ell)|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}$$
$$\Upsilon_2=\int_0^1\Big(\ell|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}},$$

and $0<\vartheta\le 1$.

Proof. Using Lemma 3.1 and improved Hölder's inequality, we exploit the harmonically convex property of $|\psi'|^q$ to deduce:

$$|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)|\le\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\int_0^1\frac{|1-(\vartheta+1)\ell^{\vartheta}|}{\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}\Big)^2}\left|\psi'\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}}\right)\right|d\ell.$$

Let $A_\ell=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{\ell}{\lambda_2}-\frac{(1-\ell)}{\lambda_1}=\ell\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\Big)+(1-\ell)\Big(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\Big)$.

$$\begin{aligned}
&|I_{\psi_1}(\mathfrak{h};\vartheta,\lambda_1,\lambda_2)|\\
&\le\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\Big\{\int_0^1\Big((1-\ell)|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\left(\int_0^1\frac{(1-\ell)}{A_\ell^{2q}}\left|\psi'\Big(\frac{1}{A_\ell}\Big)\right|^q d\ell\right)^{\frac{1}{q}}\\
&+\int_0^1\Big(\ell|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\left(\int_0^1\frac{\ell}{A_\ell^{2q}}\left|\psi'\Big(\frac{1}{A_\ell}\Big)\right|^q d\ell\right)^{\frac{1}{q}}\Big\}\\
&\le\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\Big\{\int_0^1\Big((1-\ell)|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{(1-\ell)}{A_\ell^{2q}}\begin{bmatrix}|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\\-\ell|\psi'(\lambda_2)|^q-(1-\ell)|\psi'(\lambda_1)|^q\end{bmatrix}d\ell\Big)^{\frac{1}{q}}\\
&+\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\int_0^1\Big(\ell|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{\ell}{A_\ell^{2q}}\begin{bmatrix}|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\\-\ell|\psi'(\lambda_2)|^q-(1-\ell)|\psi'(\lambda_1)|^q\end{bmatrix}d\ell\Big)^{\frac{1}{q}}\Big\}\\
&=\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\Big\{\int_0^1\Big((1-\ell)|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\left(\begin{matrix}|\psi'(\sigma_1)|^q\int_0^1\frac{1-\ell}{A_\ell^{2q}}+\psi'(\sigma_2)|^q\int_0^1\frac{1-\ell}{A_\ell^{2q}}\\-\psi'(\lambda_2)|^q\int_0^1\frac{\ell(1-\ell)}{A_\ell^{2q}}-\psi'(\lambda_1)|^q\int_0^1\frac{(1-\ell)^2}{A_\ell^{2q}}\end{matrix}\right)^{\frac{1}{q}}\\
&+\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(\vartheta+1)}\int_0^1\Big(\ell|1-(\vartheta+1)\ell^{\vartheta}|^p d\ell\Big)^{\frac{1}{p}}\left(\begin{matrix}|\psi'(\sigma_1)|^q\int_0^1\frac{\ell}{A_\ell^{2q}}+\psi'(\sigma_2)|^q\int_0^1\frac{\ell}{A_\ell^{2q}}\\-\psi'(\lambda_2)|^q\int_0^1\frac{\ell^2}{A_\ell^{2q}}-\psi'(\lambda_1)|^q\int_0^1\frac{\ell(1-\ell)}{A_\ell^{2q}}\end{matrix}\right)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

After evaluating the appearing integrals, we obtain

$$\begin{aligned}
&\int_0^1 \frac{1-\ell}{A_\ell^{2q}} d\ell \\
&= \int_0^1 \frac{1-\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell \\
&= \int_0^1 \frac{1-\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell \\
&= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 (1-\ell)\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell \\
&= \frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{\ell(1-\ell)}{A_\ell^{2q}} d\ell \\
&= \int_0^1 \frac{\ell(1-\ell)}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell \\
&= \int_0^1 \frac{\ell(1-\ell)}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell \\
&= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 \ell(1-\ell)\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell \\
&= \frac{1}{6}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q,2;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{(1-\ell)^2}{A_\ell^{2q}} d\ell \\
&= \int_0^1 \frac{(1-\ell)^2}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell \\
&= \int_0^1 \frac{(1-\ell)^2}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell \\
&= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 (1-\ell)^2\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell \\
&= \frac{1}{3}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q,1;4,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{\ell}{A_\ell^{2q}} d\ell \\
&= \int_0^1 \frac{\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell \\
&= \int_0^1 \frac{\ell}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell \\
&= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 \ell\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell \\
&= \frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 2; 3, 1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{\ell^2}{A_\ell^{2q}} d\ell \\
&= \int_0^1 \frac{\ell^2}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)\right]^{2q}} d\ell \\
&= \int_0^1 \frac{\ell^2}{\left[\ell\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}\right)+(1-\ell)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right)\right]^{2q}} d\ell \\
&= \left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} \int_0^1 \ell^2\left(1-\ell\left(1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\right)^{-2q} d\ell \\
&= \frac{1}{3}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2q} {}_2F_1\left(2q, 3; 4, 1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right).
\end{aligned}$$

This completes the proof. □

Remark 4.12. *If we put $\vartheta = 1$ in Theorem 4.11, then we obtain*

$$\begin{aligned}
&\left|\frac{\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)+\psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)}{2} - \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}} \frac{\psi(\epsilon)}{\epsilon^2} d\epsilon\right| \\
&\leq \frac{\lambda_2-\lambda_1}{2\lambda_1\lambda_2} \\
&\left\{\Upsilon_1^{\frac{1}{p}}\left[\begin{array}{c}\Lambda_1|\psi'(\sigma_1)|^q+\Lambda_1|\psi'(\sigma_2)|^q \\ -\Lambda_2|\psi'(\lambda_2)|^q-\Lambda_3|\psi'(\lambda_1)|^q\end{array}\right]^{\frac{1}{q}}\right. \\
&\left.+\Upsilon_2^{\frac{1}{p}}\left[\begin{array}{c}\Lambda_4|\psi'(\sigma_1)|^q+\Lambda_4|\psi'(\sigma_2)|^q \\ -\Lambda_5|\psi'(\lambda_2)|^q-\Lambda_2|\psi'(\lambda_1)|^q\end{array}\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\Upsilon_1 = \int_0^1 |1-2\ell|^p d\ell = \frac{1}{2(p+1)} = \Upsilon_2,$$

and $\Lambda_1 - \Lambda_5$ are same as in Theorem 4.11. Which is new in literature.

Remark 4.13. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Theorem 4.11, then we obtain*

$$\left|\frac{\psi(\lambda_1)+\psi(\lambda_2)}{2} - \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{\psi(\epsilon)}{\epsilon^2}d\epsilon\right| \leq \frac{\lambda_1\lambda_2(\lambda_2-\lambda_1)}{2}\left[\frac{1}{2(p+1)}\right]^{\frac{1}{p}}$$
$$\left\{\left[\Lambda_2|\psi'(\lambda_1)|^q + \Lambda_3|\psi'(\lambda_2)|^q\right]^{\frac{1}{q}} + \left[\Lambda_2|\psi'(\lambda_2)|^q + \Lambda_5|\psi'(\lambda_1)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$\Lambda_2 = \int_0^1 \frac{\ell(1-\ell)}{(\ell\lambda_2+(1-\ell)\lambda_1)^{2q}}d\ell$$
$$= \frac{1}{(\lambda_2-\lambda_1)^2}\left[\frac{\lambda_2\left({\lambda_2}^{1-2q}-{\lambda_1}^{1-2q}\right)}{(1-2q)} - \frac{\left({\lambda_2}^{2-2q}-{\lambda_1}^{2-2q}\right)}{(2-2q)}\right]$$
$$- \frac{1}{(\lambda_2-\lambda_1)^3}\left[\frac{{\lambda_2}^2\left({\lambda_2}^{1-2q}-{\lambda_1}^{1-2q}\right)}{(1-2q)} + \frac{\left({\lambda_2}^{3-2q}-{\lambda_1}^{3-2q}\right)}{(3-2q)} - \frac{2\lambda_2\left({\lambda_2}^{2-2q}-{\lambda_1}^{2-2q}\right)}{(2-2q)}\right]$$
$$\Lambda_3 = \int_0^1 \frac{(1-\ell)^2}{(\ell\lambda_2+(1-\ell)\lambda_1)^{2q}}d\ell$$
$$= \frac{1}{(\lambda_2-\lambda_1)^3}\left[\frac{{\lambda_2}^2\left({\lambda_2}^{1-2q}-{\lambda_1}^{1-2q}\right)}{(1-2q)} + \frac{\left({\lambda_2}^{3-2q}-{\lambda_1}^{3-2q}\right)}{(3-2q)} - \frac{2\lambda_2\left({\lambda_2}^{2-2q}-{\lambda_1}^{2-2q}\right)}{(2-2q)}\right],$$

and Λ_5 is calculated above. Which is new in literature.

4.1. Comparison Between Hölder and Improved Hölder Fractional Integral Inequalities

Here, we give comparative analysis of Hölder's and improved Hölder's integral inequalities. In Theorem 4.11, by utilising the improved form of the Hölder's inequality, we obtain the lower upper bound better than that of the original form in Theorem 4.1.

Example 4.14. *If one chooses $\psi(\epsilon) = \frac{1}{2}\epsilon^2$, $\epsilon > 0$, then by Proposition 1.5, mapping $|\psi'(\epsilon)|^q = \epsilon^q$ for $q > 1$ and $\epsilon > 0$ is $\psi \in H_K(J)$. In the instance of $\lambda_1 = 2$, $\lambda_2 = 3$ and $q = 2$, let us find the right side of the inequalities in Theorem 4.1 and Theorem 4.11 with $\vartheta = 1$ and $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ i.e. Remark 4.4 and Remark 4.13, respectively. After reducing common factor $\frac{\lambda_1\lambda_2(\lambda_2-\lambda_1)}{2}$, the right side of Remark 4.4 is*

$$= \left[\frac{1}{p+1}\right]^{\frac{1}{p}}\left[\mu_1|\psi'(\lambda_1)|^q + \mu_2|\psi'(\lambda_2)|^q\right]^{\frac{1}{q}}$$
$$= \left[\frac{1}{3}\right]^{\frac{1}{2}} \times \left[\frac{7}{648}(4) + \frac{1}{54}(9)\right]^{\frac{1}{2}}$$
$$\approx 0.2645.$$

The right side of Remark 4.13, is

$$= \left[\frac{1}{2(p+1)}\right]^{\frac{1}{p}} \times \left\{\left[\Lambda_2|\psi'(\lambda_1)|^q + \Lambda_3|\psi'(\lambda_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[\Lambda_2|\psi'(\lambda_2)|^q + \Lambda_5|\psi'(\lambda_1)|^q\right]^{\frac{1}{q}}\right\}$$
$$= \left[\frac{1}{6}\right]^{\frac{1}{2}} \times \left\{\left[\frac{1}{216}(4) + \frac{1}{72}(9)\right]^{\frac{1}{2}} + \left[\frac{1}{162}(4) + \frac{1}{216}(9)\right]^{\frac{1}{2}}\right\}$$
$$\approx 0.2598.$$

Clearly,

$$0.2645 > 0.2598,$$

which verifies that the Remark 4.13 error estimate is more accurate than the one in the Remark 4.4. We obtain the lower upper bound better than that of the original form of Hölder inequality.
***Case 1**: Furthermore, let us set $\lambda_1 = 1$, $\lambda_2 = 3$ and $q \in [1.1, 100]$ as the unknown. The right-hand sides of Remark 4.4 and Remark 4.13 are denoted by $\psi_1(q)$ and $\psi_2(q)$, respectively.*

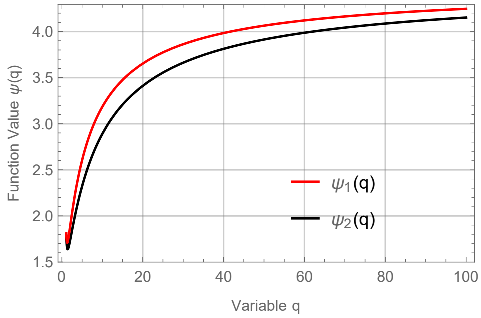

Figure 5: For the case $\lambda_1 = 1$, $\lambda_2 = 2$ and $q \in [1.1, 100]$ the diagram for Example 4.14.

Plotting $\psi_1(q)$ and $\psi_2(q)$ in Figure 6, which illustrates the upper bound derived in Remark 4.13 is better than that of Remark 4.4.

***Case 2**: Now, we set $\vartheta = 1$, $\sigma_1 = 1$, $\sigma_2 = 4$, $q = 2$ and $\lambda_1 = [1, 1.1]$, $\lambda_2 = [3, 4]$ as the unknown. The right-hand sides of Theorem 4.1 and Theorem 4.11 are denoted by $\psi_3(\lambda_1, \lambda_2)$ and $\psi_4(\lambda_1, \lambda_2)$, respectively. Plotting $\psi_3(\lambda_1, \lambda_2)$*

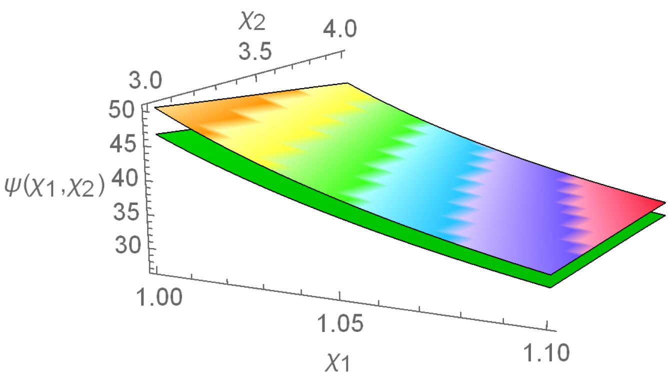

Figure 6: For the case $\sigma_1 = 1$, $\sigma_2 = 4$, $q = 2$ and $\lambda_1 = [1, 1.1]$, $\lambda_2 = [3, 4]$ the diagram for Example 4.14.

and $\psi_4(\lambda_1, \lambda_2)$ in Figure 6, which illustrates the upper bound derived in Theorem 4.11 is better than that of Theorem 4.1.

Theorem 4.15. *Under the same assumptions as in Theorem 4.1, then following inequality for fractional integral holds:*

$$\begin{aligned}&|I_{\psi_2}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2)| \\ &\leq \frac{\lambda_2 - \lambda_1}{(\vartheta + 1)\lambda_1\lambda_2} \mathbb{I}_8^{\,1-\frac{1}{q}}(\lambda_1, \lambda_2, \vartheta) \left[\begin{array}{c} \mathbb{I}_8(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_1)|^q + \mathbb{I}_8(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_9(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_1)|^q - \mathbb{I}_{10}(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_2)|^q \end{array} \right], \end{aligned} \tag{43}$$

where

$$\mathbb{I}_8=\left[\begin{array}{r} 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ -\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\\ -\frac{2}{\vartheta+1}\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ +\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{array}\right]$$

$$\mathbb{I}_9=\left[\begin{array}{r} \left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ -\frac{1}{2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right)\\ -(\frac{1}{\vartheta+1})^{\frac{2}{\vartheta}}\frac{2(\vartheta+1)}{\vartheta+2}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1})\right)^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2})+\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}\right)\\ +\frac{(\vartheta+1)}{\vartheta+2}\left[\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}\right]^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}\right) \end{array}\right]$$

$$\mathbb{I}_{10}(\lambda_1,\lambda_2,\vartheta)=\mathbb{I}_8(\lambda_1,\lambda_2,\vartheta)-\mathbb{I}_9(\lambda_1,\lambda_2,\vartheta),$$

and $\vartheta>0$.

Proof. Similarly to the proof of Theorem 4.6, using Lemma 3.5, Power-mean inequality, and the harmonic convexity of $|\psi'|^q$, we obtain (43). □

Remark 4.16. *If we put* $\sigma_1=\lambda_1$ *and* $\sigma_2=\lambda_2$ *in Theorem 4.15, then we obtain*

$$\left|\frac{\vartheta\psi(\lambda_2)+\psi(\lambda_1)}{\vartheta+1}-\Gamma(\vartheta+1)\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^{\vartheta}J^{\vartheta}_{\frac{1}{\lambda_2}+}(\psi\circ\mathfrak{h})(\frac{1}{\lambda_2})\right|$$
$$\leq\frac{\lambda_1\lambda_2(\lambda_2-\lambda_1)}{(\vartheta+1)}{Z_7}^{1-\frac{1}{q}}(\lambda_1,\lambda_2,\vartheta)\left[Z_8(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_2)|^q+Z_9(\lambda_1,\lambda_2,\vartheta)|\psi'(\lambda_1)|^q\right],$$

where

$$Z_7=\left[\begin{array}{r} 2\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\lambda_1}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1}\right)\\ -{\lambda_2}^{-2}{}_2F_1\left(2,1;2,1-\frac{\lambda_1}{\lambda_2}\right)\\ -\frac{2}{\vartheta+1}\sqrt[\vartheta]{\frac{1}{\vartheta+1}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1\right)^{-2}{}_2F_1\left(2,1;2,1-\frac{\lambda_1}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1}\right)\\ +{\lambda_2}^{-2}{}_2F_1\left(2,1;\vartheta+2,1-\frac{\lambda_1}{\lambda_2}\right) \end{array}\right]$$

$$Z_8=\left[\begin{array}{r} \left(\frac{1}{\vartheta+1}\right)^{\frac{2}{\vartheta}}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1\right)^{-2}{}_2F_1\left(2,1;3,1-\frac{\lambda_1}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1}\right)\\ -\frac{1}{2}{\lambda_2}^{-2}{}_2F_1\left(2,1;3,1-\frac{\lambda_1}{\lambda_2}\right)\\ -(\frac{1}{\vartheta+1})^{\frac{2}{\vartheta}}\frac{2(\vartheta+1)}{\vartheta+2}\left(\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1\right)^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\lambda_1}{\sqrt[\vartheta]{\frac{1}{\vartheta+1}}(\lambda_2-\lambda_1)+\lambda_1}\right)\\ +\frac{(\vartheta+1)}{\vartheta+2}{\lambda_2}^{-2}{}_2F_1\left(2,1;\vartheta+3,1-\frac{\lambda_1}{\lambda_2}\right) \end{array}\right]$$

$$Z_9(\lambda_1, \lambda_2, \vartheta) = Z_7(\lambda_1, \lambda_2, \vartheta) - Z_8(\lambda_1, \lambda_2, \vartheta),$$

which is appeared in [33].

Remark 4.17. *If we put $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\vartheta = 1$ in Theorem 4.15, then we derive an inequality appeared in [8].*

Theorem 4.18. *Let ψ be defined as in Theorem 4.1, then following inequality for fractional integral holds*

$$\begin{aligned} &|I_{\psi_2}(\mathfrak{h}; \vartheta, \lambda_1, \lambda_2)| \\ &\leq \frac{\lambda_2 - \lambda_1}{(\vartheta+1)\lambda_1\lambda_2} \mathbb{I}_1^{\frac{1}{p}}(\lambda_1, \lambda_2, \vartheta) \begin{bmatrix} \mathbb{I}_1(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_1)|^q + \mathbb{I}_2(\lambda_1, \lambda_2, \vartheta)|\psi'(\sigma_2)|^q \\ -\mathbb{I}_3(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_2)|^q - \mathbb{I}_4(\lambda_1, \lambda_2, \vartheta)|\psi'(\lambda_1)|^q \end{bmatrix}^{\frac{1}{q}}, \end{aligned} \tag{44}$$

where $\mathbb{I}_1(\lambda_1, \lambda_2, \vartheta) - \mathbb{I}_4(\lambda_1, \lambda_2)$ are same as in Theorem 4.1 and $0 < \vartheta \leq 1$.

Proof. *Similarly to the proof of Theorem 4.1, using Lemma 3.5, Hölder's inequality, and the harmonic convexity of $|\psi'|^q$, we obtain* (44). □

Remark 4.19. *If we put $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$ in Theorem 4.18, then we derive an inequality appeared in [33].*

Remark 4.20. *If we put $\vartheta = 1$ and $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ in Theorem 4.18, then we derive an inequality appeared in [8].*

5. Applications

5.1. Special Means

Now, for numbers $0 < \lambda_1 < \lambda_2$, we consider special means and j-logarithmic means:

$$A(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2},$$
$$G(\lambda_1, \lambda_2) = (\lambda_1\lambda_2)^{\frac{1}{2}},$$

and

$$L_j(\lambda_1, \lambda_2) = \left[\frac{\lambda_2^{(j+1)} - \lambda_1^{(j+1)}}{(j+1)(\lambda_2 - \lambda_1)}\right]^{\frac{1}{j}}, \quad j \in Z\{-1, 0\}.$$

Proposition 5.1. *Using the same assumptions as in Theorem 4.1, if we take $\psi(\sigma) = \frac{\sigma^{(k+1)}}{k+1}$ with $\sigma > 0$, $k \geq 1$ and $\vartheta = 1$, then we get*

$$\begin{aligned} &\Bigg|A\left(\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right)^{k+1}, \left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right)^{k+1}\right) \\ &- (k+1)G^2(\lambda_1, \lambda_2)L_{(k-1)}^{k-1}\left(\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right), \left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right)\right)\Bigg| \\ &\leq \frac{(1+k)(\lambda_2 - \lambda_1)}{2G^2(\lambda_1, \lambda_2)} \times \mathbb{I}_1^{\frac{1}{p}}(\lambda_1, \lambda_2) \\ &\begin{bmatrix} \mathbb{I}_2(\lambda_1, \lambda_2)(\lambda_1)^{kq} + \mathbb{I}_2(\lambda_1, \lambda_2)(\lambda_2)^{kq} \\ -\mathbb{I}_3(\lambda_1, \lambda_2)(\lambda_2)^{kq} - \mathbb{I}_4(\lambda_1, \lambda_2)(\lambda_1)^{kq} \end{bmatrix}^{\frac{1}{q}}, \end{aligned}$$

where $\mathbb{I}_1 - \mathbb{I}_4$ are same as in Remark 4.2.

Corollary 5.2. *Under the same assumptions as in Proposition 5.1, if we take $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$, then we get*

$$\left|A({\lambda_1}^{k+1}, {\lambda_2}^{k+1}) - G^2(\lambda_1, \lambda_2)L_{(k-1)}^{k-1}(\lambda_1, \lambda_2)\right|$$
$$\leq (1+k)\frac{G^2(\lambda_1, \lambda_2)(\lambda_2 - \lambda_1)}{2}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}\left[\mu_1(\lambda_1)^{kq} + \mu_2(\lambda_2)^{kq}\right]^{\frac{1}{q}},$$

where μ_1 and μ_2 are same as in Remark 4.4.

Proposition 5.3. *Under the same assumptions as in Theorem 4.6, if we take $\psi(\sigma) = \frac{\sigma^{(k+1)}}{k+1}$ with $\sigma > 0$, $k \geq 1$ and $\vartheta = 1$, then we get*

$$\left|A\left(\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right)^{k+1}, \left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right)^{k+1}\right)\right.$$
$$\left. - (k+1)G^2(\lambda_1, \lambda_2)L_{(k-1)}^{k-1}\left(\left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_2}}\right), \left(\frac{1}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{1}{\lambda_1}}\right)\right)\right|$$
$$\leq \frac{(1+k)(\lambda_2 - \lambda_1)}{2G^2(\lambda_1, \lambda_2)} \times {\mathbb{I}_5}^{1-\frac{1}{q}}(\lambda_1, \lambda_2)$$
$$\left[\begin{array}{c} \mathbb{I}_5(\lambda_1, \lambda_2)(\lambda_1)^{kq} + \mathbb{I}_5(\lambda_1, \lambda_2)(\lambda_2)^{kq} \\ -\mathbb{I}_6(\lambda_1, \lambda_2)(\lambda_1)^{kq} - \mathbb{I}_7(\lambda_1, \lambda_2)(\lambda_2)^{kq} \end{array}\right]^{\frac{1}{q}},$$

where $\mathbb{I}_5 - \mathbb{I}_7$ are same as in Remark 4.7.

Corollary 5.4. *Under the same assumptions as in Proposition 5.1, if we take $\sigma_1 = \lambda_1$ and $\sigma_2 = \lambda_2$, then we get*

$$\left|A({\lambda_1}^{k+1}, {\lambda_2}^{k+1}) - G^2(\lambda_1, \lambda_2)L_{(k-1)}^{k-1}(\lambda_1, \lambda_2)\right|$$
$$\leq (1+k)\frac{G^2(\lambda_1, \lambda_2)(\lambda_2 - \lambda_1)}{2}[\lambda_1]^{\frac{1}{p}}\left[\lambda_2(\lambda_1)^{kq} + \lambda_3(\lambda_2)^{kq}\right]^{\frac{1}{q}},$$

where $\lambda_1 - \lambda_3$ are same as in Remark 4.9.

5.2. Applications to Numerical Quadrature Rule

In this section, we analyse how the integral inequalities engaging fractional integral proposed in the previous section can be used to yield an estimate of composite quadrature rules that, it turns out have a significantly smaller error than those that may be acquired by the classical results. We will extend the idea by giving applications to numerical quadrature rule for harmonic convex inequality.

Theorem 5.5. *Under the assumption of Theorem 4.6 for $\vartheta = 1$, let $J_{\mathfrak{n}} : \sigma_1 = \epsilon_0 < \epsilon_1 < \epsilon_2 ... < \epsilon_{\mathfrak{n}-1} < \epsilon_{\mathfrak{n}} = \sigma_2$ is a partition of $[\sigma_1, \sigma_2]$, $\lambda_1, \lambda_2 \in [\epsilon_{\mathfrak{i}}, \epsilon_{\mathfrak{i}+1}]$ and $\mathfrak{h}_{\mathfrak{i}} = (\epsilon_{\mathfrak{i}+1} - \epsilon_{\mathfrak{i}})$, $\mathfrak{i} = 0, 1, ..., \mathfrak{n}-1$, then we have:*

$$\int_{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_1}}}^{\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_2}-\frac{1}{\lambda_2}}} \frac{\psi(\epsilon)}{\epsilon^2} d\epsilon = B(J_{\mathfrak{n}}, \psi) + R(J_{\mathfrak{n}}, \psi),$$

where

$$B(J_{\mathfrak{n}}, \psi) = \sum_{\mathfrak{i}=0}^{\mathfrak{n}-1} \frac{\psi\left(\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},}}}\right) + \psi\left(\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},g}}}\right)}{2\epsilon_{\mathfrak{i},g}\epsilon_{\mathfrak{i},}} \mathfrak{h}_{\mathfrak{i}}, \tag{45}$$

and the remainder term satisfy the estimation:

$$|R(J_{\mathfrak{n}},\psi)| \leq \sum_{\mathfrak{i}=0}^{\mathfrak{n}-1} \frac{\mathfrak{h}_{\mathfrak{i}}^{2}}{2(\epsilon_{\mathfrak{i},g}\epsilon_{\mathfrak{i},})^{2}} \mathbb{I}_{5}^{1-\frac{1}{q}} \begin{bmatrix} \mathbb{I}_{5}|\psi'(\epsilon_{\mathfrak{i}})|^{q} + \mathbb{I}_{5}|\psi'(\epsilon_{\mathfrak{i}+1})|^{q} \\ -\mathbb{I}_{6}|\psi'(\epsilon_{\mathfrak{i},g})|^{q} - \mathbb{I}_{7}|\psi'(\epsilon_{\mathfrak{i},})|^{q} \end{bmatrix}^{\frac{1}{q}},$$

where $\mathbb{I}_5, \mathbb{I}_6, \mathbb{I}_7$ *are same as defined in Remark 4.7.*

Proof. Applying Theorem 4.6 with $\vartheta = 1$ on interval $[\epsilon_{\mathfrak{i}}, \epsilon_{\mathfrak{i}+1}]$, $\mathfrak{i} = 0, 1, ..., \mathfrak{n}-1$, we get

$$\left| \frac{\psi\left(\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},}}}\right) + \psi\left(\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},g}}}\right)}{2\epsilon_{\mathfrak{i},g}\epsilon_{\mathfrak{i},}} \mathfrak{h}_{\mathfrak{i}} - \int_{\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},g}}}}^{\frac{1}{\frac{1}{\epsilon_{\mathfrak{i}}}+\frac{1}{\epsilon_{\mathfrak{i}+1}}-\frac{1}{\epsilon_{\mathfrak{i},}}}} \frac{\psi(\epsilon)}{\epsilon^{2}} d\epsilon \right|$$
$$\leq \frac{\mathfrak{h}_{\mathfrak{i}}^{2}}{2(\epsilon_{\mathfrak{i},g}\epsilon_{\mathfrak{i},})^{2}} \mathbb{I}_{5}^{1-\frac{1}{q}} \begin{bmatrix} \mathbb{I}_{5}|\psi'(\epsilon_{\mathfrak{i}})|^{q} + \mathbb{I}_{5}|\psi'(\epsilon_{\mathfrak{i}+1})|^{q} \\ -\mathbb{I}_{6}|\psi'(\epsilon_{\mathfrak{i},g})|^{q} - \mathbb{I}_{7}|\psi'(\epsilon_{\mathfrak{i},})|^{q} \end{bmatrix}^{\frac{1}{q}},$$

for all $\mathfrak{i} = 0, 1, ..., \mathfrak{n}-1$. Summing over 0 to $\mathfrak{n}-1$ and using the triangular inequality we obtain the above estimation. □

6. Conclusion

The study dealt with the analysis of novel (**H-H-M**) type inequalities in fractional calculus for harmonic convex functions. We presented two new Riemann-Liouville fractional trapezoidal type auxiliary equalities in Mercer sense. We enhanced the study of Mercer type integral inequalities using Power-mean, Hölder's and modified Hölder integral inequalities using the novel approach. Similarly, smaller upper bounds can be deduced by using modified Power-mean integral inequality. Which is left for reader interest. The remarkable techniques and ideas presented in this article may be developed to coordinates and fractional integral calculus. Our goal for the future is to continue and expand our research in this regard.

References

[1] A. W. Roberts, D. E. Varberg, *Convex functions academic press*, New York, 62 (1973).
[2] C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, Second edition, Springer, (2018).
[3] T. Rasheed, S. I. Butt, D. Pečarić, J. Pečarić, *Generalized cyclic Jensen and information inequalities*, Chaos, Solitons & Fractals, 163, 112602, (2022).
[4] P. Xu, Y. Sayyari, S. I. Butt, *Logical entropy of information sources*, Entropy. 24(9) (2022), 1174.
[5] J. E. Nápoles Valdés, F. Rabossi, A. D. Samaniego, *Convex functions: Ariadne's thread or charlotte's spiderweb?*, Advanced Mathematical Models & Applications, 5(2) (2020), 176–191.
[6] B. Bayraktar, J. E. Nápoles Valdés, F. Rabossi, *Some refinements of the Hermite–Hadamard inequality with the help of weighted integrals*, Ukrains'kyi Matematychnyi Zhurnal, 75(6) (2023), 736–752.
[7] B. Bayraktar, P. Kórus, J. E. Nápoles Valdés, *Some new Jensen-Mercer type integral inequalities via fractional operators*, Axioms, 12(6) (2021), 517.
[8] I. Işcan, *Hermite-Hadamard inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics, 43(6) (2014), 935–942.
[9] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for HA-convex functions*, Moroccan Journal of Pure and Applied Analysis, 3(1) (2017), 143–153.
[10] S. I. Butt, P. Agarwal, S. Yousaf, J.L.G. Guirao, *Generalized fractal Jensen and Jensen-Mercer inequalities for harmonic convex function with applications*, Journal of Inequalities and Applications. 2022 (1), 1–18.
[11] S. I. Butt, M. Tariq, A. Aslam, H. Ahmad, T. A. Nofal, *Hermite-Hadamard type Inequalities via generalized harmonic exponential convexity and applications*, J. Funct. Spaces, (2021).
[12] M. U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, Y. M. Chu,*ew Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions*, Journal of Inequalities and Applications, 2020(1), 1-12.

[13] S. I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, W. Gao, *Hermite-Hadamard-type inequalities via n-polynomial exponential-type convexity and their applications*, Adv. Differ. Equ. (2020), 1–25.

[14] S. S. Dragomir, *Inequalities of Jensen type for HA-convex cunctions*, Fasciculi Mathematici, (2015), 35–52.

[15] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, M. D. L. Sen, *A varient of Jensen- type inequality and related results for harmonic convex functions*, AIMS Mathematics, 5(6) (2020), 6404–6418.

[16] I. Işcan, *New refinements for integral and sum forms of Hölder inequality*, Journal of Inequalities and Applications, 2019, 1–11.

[17] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam (2006).

[18] T. Abdeljawad, D. Baleanu, *On fractional derivatives with exponential kernel and their discrete versions*, Reports on Mathematical Physics, 80(1) (2017), 11–27.

[19] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. , 264 (2014), 65–70.

[20] D. Baleanu, R. P. Agarwal, *Fractional calculus in the sky*, Advances in Difference Equations, 1 (2021).

[21] P. Z. Tan, T. S. Du, *On the multi-parameterized inequalities involving the tempered fractional integral operators*, Filomat, 37 (15)(2023), 4919-4941.

[22] T. S. Du, C. Y. Luo, Z. J. Cao, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals, 29 (7) (2021), Article ID 2150188, 1–20.

[23] T. S. Du, T. C. Zhou, *On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings*, Chaos, Solitons and Fractals, 156 (2022), Article ID 111846, 19 pages.

[24] T. C. Zhou, Z. R. Yuan, T. S. Du, *On the fractional integral inclusions having exponential kernels for interval-valued convex functions*, Math. Sci., 17 (2) (2023), 107-120.

[25] M. Kunt, I. Işcan, S. Turhan, D. Karapinar, *Improvement of frational Hermite-Hadamard type inequality for convex functions*, Miskolc Mathematical Notes, 19(2) (2018), 1007-17.

[26] E. Set, S. I. Butt, A. O. Akdemir, A. Karaoglan, T. Abdeljawad, *New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators*, Chaos, Solitons and Fractals, 143 (2021), 110554.

[27] S. I. Butt, J. Nasir, M. A. Dokuyucu, A. O. Akdemir, E. Set, *Some Ostrowski-Mercer type inequalities for differentiable convex functions via fractional integral operators with strong kernels*, Applied and computational mathematics, 21(3) (2022), 329–348.

[28] P. Xu, S. I. Butt, S. Yousaf, A. Aslam, T. J. Zia, *Generalized fractal Jensen-Mercer and Hermite-Mercer type inequalities via h-convex functions involving Mittag-Leffler kernel*, Alexandria Engineering Journal. 61(6) (2022), 4837-4846.

[29] I. Işcan, S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals* , Applied Mathematics and Computation, 238 (2014), 237–244.

[30] M. Tunc, U. Sanal, E. Gov, *Some Hermite-Hadamard inequalities for Beta-convex and its fractional applications*, New Trends in Mathematical Sciences, 3(4) (2015), 18–33.

[31] A. P. Prudnikov, Y. A. Brychkov, O. J. Marichev, *Integral and series of elementry functions*, Nauka, Moacow, 1981.

[32] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, M. D. L. Sen, *Improvement and generalization of some results related to the class of harmonically convex functions and applications*, Journal of Mathematics and Computer Science, 22(2021), 282–294.

[33] Z. Sanli, M. Kunt, T. *Köroğlu*, *New Riemann-Liouville fractional Hermite-Hadamard type inequalities for harmonically convex functions*, Arabian Journal of Mathematics, 9(2) (2020), 431–441.