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Generalization of parametric Baskakov operators based on the I-P-E distribution

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Abstract. This study explores the approximation properties of a non-negative real parametric generalization of the Baskakov operators using the inverse Pólya-Eggenberger (I-P-E) distribution. As a result of this study, we can obtain some approximation results, including the Voronovskaya type asymptotic formula, error estimate in terms of modulus of continuity and sense of k-functional, and weighted approximation.

1. Introduction

In literature, the inverse Pólya-Eggenberger (I-P-E) distribution is defined as:

$$P(X=\ddot{\varsigma}) = \binom{\dddot{m}+\ddot{\varsigma}-1}{\ddot{\varsigma}} \frac{\prod\limits_{\ddot{\varsigma}}^{\dddot{m}-1}(A+\ddot{\varsigma}S)\prod\limits_{\ddot{\varsigma}=0}^{\dddot{m}-1}(B+\ddot{\varsigma}S)}{\prod\limits_{\ddot{\varsigma}=0}^{\dddot{m}+\ddot{\varsigma}-1}(A+B+\ddot{\varsigma}S)}, \quad \ddot{\varsigma}=0,1,\ldots,\dddot{m}, \tag{1}$$

shows the probability that $\ddot{\varsigma}$ white balls are selected preceding the $\dddot{m}$-th black ball. The details have been given about this distributions (1) in [4, 8, 13].

In 1957, Baskakov [22] presented a sequence of positive linear operators, known as Baskakov operators on the unbounded interval $[0,\infty)$ for appropriate functions specified on the interval $[0,\infty)$. Afterwards, many mathematicians who studied the Baskakov operators came up with various modifications, including [3, 12, 15–17, 19, 21].

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Recall that for every $\mathrm{h} \in C_B[0,\infty)$ classical Baskakov operators are defined as:

$$\mathrm{I}_n(\mathrm{h}, x) = \sum_{k=0}^{\infty} v_{n,k}(x)\mathrm{h}\left(\frac{k}{n}\right), \tag{2}$$

where $n \geq 1$, $x \in [0,\infty)$ and

$$v_{n,k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{x^k}{(1+x)^{n+k}}.$$

Furthermore, using the inverse Pólya-Eggenberger distribution (1), Stancu [20] presented a special class of positive linear operators contextualising the Baskakov operators tied to a real-valued function bounded on $[0,\infty)$ as follows:

$$\begin{aligned}\mathrm{I}_n^{(\hat{\varrho})}(\mathrm{h}; x) &= \sum_{k=0}^{\infty} v_{n,k}^{(\hat{\varrho})}(x)\,\mathrm{h}\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{\infty} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\mathrm{h}\left(\frac{k}{n}\right)\end{aligned} \tag{3}$$

In the case when $\hat{\varrho} = 0$, operators (3) reduce to the classical Baskakov operators [22]. We came across a manuscript where in 2019, Ali and Hasan [1] introduced α-Baskakov operators, which are non-negative real parametric generalisations of Baskakov operators. The operators are reduced to classical Baskakov operators for $\alpha = 1$. Higher order derivatives are represented as α-Baskakov operators in this paper to obtain their new representation of independent variable x.

Now, for every $\mathrm{h} \in C_B[0,\infty)$, the parametric generalization of the Baskakov operators is defined as:

$$\mathcal{L}_n^{\alpha}(\mathrm{h}; x) = \sum_{k=0}^{\infty} \rho_{n,k}^{\alpha}(x)\mathrm{h}\left(\frac{k}{n}\right), \tag{4}$$

where $n \geq 1$, $x \in [0,\infty)$ and

$$\begin{aligned}\rho_{n,k}^{\alpha}(\mathrm{h}; x) = \sum_{k=0}^{\infty} \frac{x^{k-1}}{(1+x)^{n+k-1}} &\left\{\frac{\alpha x}{1+x}\begin{pmatrix} n+k-1 \\ k \end{pmatrix} - (1-\alpha)(1+x)\begin{pmatrix} n+k-1 \\ k-2 \end{pmatrix}\right. \\ &\left. + (1-\alpha)(1+x)\begin{pmatrix} n+k-1 \\ k-2 \end{pmatrix} + (1-\alpha)x\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\right\}\mathrm{h}\left(\frac{k}{n}\right),\end{aligned}$$

we call these operators α-Baskakov operators.

The α-Baskakov operators for $\mathrm{h}(x)$ can also expressed as:

$$\begin{aligned}\mathcal{L}_n^{\alpha}(\mathrm{h}; x) = (1-\alpha)&\sum_{k=0}^{\infty} \begin{pmatrix} n+k-2 \\ k \end{pmatrix} \frac{x^k}{(1+x)^{n+k-1,}} g_k \\ &+ \alpha \sum_{k=0}^{\infty} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{x^k}{(1+x)^{n+k,}}\mathrm{h}\left(\frac{k}{n}\right),\end{aligned}$$

where

$$g_k = \mathrm{h}\left(\frac{k}{n}\right)\left(1 + \frac{k}{n-1}\right) - \mathrm{h}\left(\frac{k+1}{n}\right)\frac{k}{n-1}.$$

In this manuscript, we consider generalised α-Baskakov operators (4) based on the inverse Pólya-Eggenberger distribution (1). This work was prompted by the Stancu [20], who introduced two classes of positive linear operators depending on a non-negative parameter α that may only depend on the natural number, and proceeded to establish some of their approximation properties to real-valued functions. In 2019, Deo and Dhamija [6] considered new modified Baskakov operators based on the inverse Pólya-Eggenberger distribution, and he also examined various modifications of the Baskakov operators in the context of the Lupaş operators based on the inverse Pólya-Eggenberger distribution. In 2016, Dhamija and Deo [8] provided Jain-Durrmeyer operators connected to the inverse Pólya-Eggenberger distribution and examined the approximation properties of the Jain-Durrmeyer operators based on the inverse Pólya-Eggenberger distribution, including uniform convergence and degree of approximation. Baskakov-Szász type operators on the inverse Pólya-Eggenberger distribution were presented by Kajla et al. [14, 15]. The rate of convergence for functions with bounded variation derivatives is established by using a Durrmeyer-type operator with basic functions in summation and integration based on Stancu [20] and Pltánea (2008). Numerous authors contributed to the "Baskakov operator and inverse Pólya-Eggenberger distribution" see [2, 4, 9].

Now we propose α-Pólya-Baskakov operator based on inverse Pólya-Eggenberger distribution (1) as follows:

$$\overset{\rightharpoonup}{Q}_{\mathscr{r}}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)=\sum_{\mathscr{j}=0}^{\infty} q_{\mathscr{r},\mathscr{j}}^{(\alpha,\hat{\varrho})}(x)\mathrm{h}\left(\frac{\mathscr{j}}{\mathscr{r}}\right), \tag{5}$$

where α is a non-negative parameter, which may depend only on the natural number $\mathscr{r}$, with $\alpha\to 0$ when $\mathscr{r}\to\infty$, $\mathscr{r}\geq 1$, $x\in[0,\infty)$

$$\begin{aligned} q_{\mathscr{r},\mathscr{j}}^{(\alpha,\hat{\varrho})}(x) &= \alpha\binom{\mathscr{r}+\mathscr{j}-1}{\mathscr{j}}\frac{1^{[\mathscr{r},-\hat{\varrho}]}x^{[\mathscr{j},-\hat{\varrho}]}}{(1+x)^{[\mathscr{r}+\mathscr{j},-\hat{\varrho}]}} \\ &\quad -(1-\alpha)\binom{\mathscr{r}+\mathscr{j}-3}{\mathscr{j}-2}\frac{1^{[\mathscr{r}-1,-\hat{\varrho}]}x^{[\mathscr{j}-1,-\hat{\varrho}]}}{(1+x)^{[\mathscr{r}+\mathscr{j}-2,-\hat{\varrho}]}} \\ &\quad +(1-\alpha)\binom{\mathscr{r}+\mathscr{j}-1}{\mathscr{j}}\frac{1^{[\mathscr{r}-1,-\hat{\varrho}]}x^{[\mathscr{j},-\hat{\varrho}]}}{(1+x)^{[\mathscr{r}+\mathscr{j}-1,-\hat{\varrho}]}}. \end{aligned}$$

Another way to express the above operators is as follows:

$$\begin{aligned} \overset{\rightharpoonup}{Q}_{\mathscr{r}}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) &= (1-\alpha)\sum_{\mathscr{j}=0}^{\infty}\binom{\mathscr{r}+\mathscr{j}-2}{\mathscr{j}}\frac{1^{[\mathscr{r}-1,-\hat{\varrho}]}x^{[\mathscr{j},-\hat{\varrho}]}}{(1+x)^{[\mathscr{r}+\mathscr{j}-1,-\hat{\varrho}]}}g_{\mathscr{j}} \\ &\quad +\alpha\sum_{\mathscr{j}=0}^{\infty}\binom{\mathscr{r}+\mathscr{j}-1}{\mathscr{j}}\frac{1^{[\mathscr{r},-\hat{\varrho}]}x^{[\mathscr{j},-\hat{\varrho}]}}{(1+x)^{[\mathscr{r}+\mathscr{j},-\hat{\varrho}]}}\mathrm{h}\left(\frac{\mathscr{j}}{\mathscr{r}}\right), \end{aligned}$$

where

$$g_{\mathscr{j}}=\mathrm{h}\left(\frac{\mathscr{j}}{\mathscr{r}}\right)\left(1+\frac{\mathscr{j}}{\mathscr{r}-1}\right)-\mathrm{h}\left(\frac{\mathscr{j}+1}{\mathscr{r}}\right)\frac{\mathscr{j}}{\mathscr{r}-1}.$$

The purpose of this note is to investigate the approximation behaviour of the proposed operators. The following is how the paper is structured: Section 2 discusses moments, central moments, and limits of the operators we've proposed. The next proceeding section of our article contains a general conclusion involving the Korovkin theorem for unbounded intervals, as well as definitions of moduli of continuity and K-functional. The last section is concerned with the approximation of functions in weighted space.

2. Premiinary

Lemma 2.1. *Deo and Dhamija [5] use the Vandermonde convolution formula and the substitutions $\lambda = 0,\ p = 0$ to derive the moments of I-P-E distribution based operators* (3). *Stancu [20] however, already calculated these moments using hypergeometric series.*

$$
\begin{aligned}
\mathrm{I}_{n}^{\hat{\varrho}}(1;x) =& 1\\
\mathrm{I}_{n}^{\hat{\varrho}}(\mathrm{v};x) =& \frac{x}{1-\hat{\varrho}}\\
\mathrm{I}_{n}^{\hat{\varrho}}\left(\mathrm{v}^2;x\right) =& \frac{x(\hat{\varrho}(n-1)+nx+x+1)}{(1-\hat{\varrho})(1-2\hat{\varrho})n}.\\
\mathrm{I}_{n}^{\hat{\varrho}}\left(\mathrm{v}^3;x\right) =& \frac{(n+1)x((n+2)x+3)-x\left(\hat{\varrho}^2((3-2n)n-1)-\hat{\varrho}(-3n(nx+1)+3x+2)-1\right)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})n^2}\\
\mathrm{I}_{n}^{\hat{\varrho}}\left(\mathrm{v}^4;x\right) =& \frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^3}\left[(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})\right.\\
&-(n+1)(\hat{\varrho}+x)\left\{6\hat{\varrho}^2((n-3)n+4)+\hat{\varrho}((n+2)(5n-9)x+12n-25)\right.\\
&\left.\left.+(n+2)x((n+3)x+6)+7\right\}\right]
\end{aligned}
$$

Lemma 2.2. *The moments of* $\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(f;x)$ *are given as follows:*

$$
\begin{aligned}
\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(1;x) =& (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\\
&+\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\\
=&1.\\
\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(\mathrm{v};x) =& (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k(n-2)}{n(n-1)}\right]\\
&+\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k}{n}\right]\\
=&(1-\alpha)\frac{(n-2)}{n}\mathrm{I}_{n-1}^{\hat{\varrho}}(t;x)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}(t;x)\\
=&(1-\alpha)\frac{(n-2)}{n}\frac{x}{1-\hat{\varrho}}+\frac{\alpha x}{1-\hat{\varrho}}\\
=&\frac{2\alpha+n-2}{(1-\hat{\varrho})n}x.
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{Q}_{n}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^2;x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^2\,(n-3)}{n^2\,(n-1)}\right] \\
&\quad -(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k}{n^2\,(n-1)}\right] \\
&\quad +\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k^2}{n^2}\right] \\
&= \frac{(1-\alpha)\,(n-3)\,(n-1)}{n^2}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^2;x\right)-\frac{(1-\alpha)}{n^2}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t;x\right)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}\left(t^2;x\right) \\
&= \frac{(4\alpha n+(n-3)\,n)}{(1-\hat{\varrho})(1-2\hat{\varrho})\,n^2}x^2+\frac{4\alpha\hat{\varrho}(n-2)+4\alpha+\hat{\varrho}((n-5)\,n+8)+n-4}{(1-\hat{\varrho})(1-2\hat{\varrho})\,n^2}x.
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{Q}_{n}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^3;x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^3\,(n-4)}{n^3\,(n-1)}\right] \\
&= -3\,(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^2}{n^3\,(n-1)}\right] \\
&\quad -(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k}{n^3\,(n-1)}\right] \\
&\quad +\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k^3}{n^3}\right] \\
&= \frac{(1-\alpha)\,(n-4)\,(n-1)^2}{n^3}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^3;x\right)-\frac{(1-\alpha)\,(n-1)}{n^3}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^2;x\right) \\
&\quad -\frac{1}{n^3}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t;x\right)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}\left(t^3;x\right) \\
&= \frac{(n+1)(6\alpha+n-4)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})\,n^2}x^3 \\
&\quad +\frac{3(6\alpha(\hat{\varrho}(n-2)+1)+\hat{\varrho}((n-6)\,n+11)+n-5)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})\,n^2}x^2 \\
&\quad -\frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})\,n^3}\left[\begin{array}{c}(-\hat{\varrho}(n-3)-1)(\hat{\varrho}(n(2n-9)+16)+n-8)\\ -2\alpha(\hat{\varrho}(3\hat{\varrho}(n(2n-7)+8)+9n-20)+4)\end{array}\right]x.
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{Q}_{n}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^4;x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^4\,(n-5)}{n^4\,(n-1)}\right] \\
&\quad -6\,(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^3}{n^4\,(n-1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-4(1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^{2}}{n^{4}(n-1)}\right] \\
&-(1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k}{n^{4}(n-1)}\right] \\
&+\alpha\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k^{4}}{n^{4}}\right] \\
=&\frac{(1-\alpha)(n-5)(n-1)^{3}}{n^{4}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{4};x\right)-\frac{6(1-\alpha)(n-1)^{2}}{n^{4}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{3};x\right) \\
&-\frac{4(1-\alpha)(n-1)}{n^{4}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{2};x\right)-\frac{1}{n^{4}}\mathrm{I}_{n-1}^{\hat{\varrho}}(t;x)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}\left(t^{4};x\right) \\
=&\frac{(8\alpha n(n+1)(n+2)+(n-5)n(n+1)(n+2))}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^{4}}x^{4} \\
&+\frac{(48\alpha n(n+1)(\hat{\varrho}(n-2)+1)+6n(n+1)(\hat{\varrho}((n-7)n+14)+n-6))}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^{4}}x^{3} \\
&+\frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^{4}}\left[\begin{array}{c} 8\alpha n(\hat{\varrho}(\hat{\varrho}(n(11n-39)+46) \\ +18n-38)+8)+n(\hat{\varrho}(\hat{\varrho}(n(n \\ (11n-94)+301)-362) \\ +nn(18n-139)+291) \\ +7n-57) \end{array}\right]x^{2} \\
&+\frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^{4}}\left[\begin{array}{c} (16\alpha(\hat{\varrho}(n-2)+1)(\hat{\varrho}(3\hat{\varrho}((\bar{m}-3 \\ n+4)+3n-7)+1)) \\ +\hat{\varrho}(6\hat{\varrho}^{3}((n-5)n+8)^{2} \\ +\hat{\varrho}^{2}n(n(12n-109)+359)-416\hat{\varrho} \\ +\hat{\varrho}(n-6)(7n-24))+n-16 \end{array}\right]x.
\end{aligned}
$$

Lemma 2.3. *We establish the following limits of central moments by using Lemma 2.2 and,* $\lim_{n\to\infty} n\hat{\varrho}=l$*:*

(i) $\lim_{n\to\infty} n\overrightarrow{\mathcal{Q}}_{n}^{(\alpha,\hat{\varrho})}(\vartheta;x)=(l+1)x(1+x).$

(ii) $\lim_{n\to\infty} n^{2}\overrightarrow{\mathcal{Q}}_{n}^{(\alpha,\hat{\varrho})}(\vartheta;x)=3x^{2}(1+l)^{2}(1+x)^{2}.$

where $\vartheta(\mathrm{v})=(\mathrm{v}-x)^{i}$ *and,* $i=2,4.$.

3. Direct Results

The renowned Bohman-Korovkin-Popoviciu theorem is used to obtain the uniform convergence of the α-Pólya-Baskakov operator (5).

Theorem 3.1. *Let* $h\in C[0,\infty)$ *and* α *being a non-negative parameter, which may depend only on the natural number* n*, with* $\hat{\varrho}\to 0$ *when* $n\to\infty$*, then we have*

$$\lim_{n\to\infty}\overrightarrow{\mathcal{Q}}_{n}^{(\alpha,\hat{\varrho})}(h;x)=h(x)$$

uniformly on each compact subset of $[0,\infty)$*, where* $C[0,\infty)$ *is the space of all real-valued functions continuous on* $[0,\infty)$.

Proof. Taking Lemma 2.2 into consideration, it follows that:

$$\lim_{\wp\to\infty} \overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^{\mathrm{i}};x\right)=x^{i},\ \ i=0,1,2$$

uniformly on each compact subset of $[0,\infty)$. Thus, we arrive at the desired conclusion by applying the well-known Bohman-Korovkin-Popoviciu theorem. □

α-Pólya-Baskakov operator (5) asymptotic behaviour is now presented.

Theorem 3.2. *Let* h *be a bounded and integrable function on* $[0,\infty)$. *In the case where there are first and second derivatives of* h *at a fixed point* $x\in[0,\infty)$, *then the function could be written as follows.*

$$\lim_{\wp\to\infty} \wp\left(\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}\,(x)\right)=(l+1)\,x\,(1+x)\,\mathrm{h}'\,(x)+3x^{2}(1+l)^{2}(1+x)^{2}\mathrm{h}''(x).$$

Proof. We can express the function h, using Taylor's expansion by writing,

$$\mathrm{h}(\mathrm{v})=\mathrm{h}(x)+(\mathrm{v}-x)\mathrm{h}'(x)+\frac{1}{2!}(\mathrm{v}-x)^{2}\mathrm{h}''(x)+\varepsilon(\mathrm{v},x)(\mathrm{v}-x)^{2},$$

where $\varepsilon(\mathrm{v},x)$ is a bounded function and $\lim_{\mathrm{v}\to x}\varepsilon(\mathrm{v},x)=0$. By linearity of α-Pólya-Baskakov operator (5), it follows

$$\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)=\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\mathrm{v}-x;x)\,\mathrm{h}'(x)+\frac{1}{2}\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^{2};x\right)\mathrm{h}''(x)+$$
$$+\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^{2};x\right).$$

Taking Lemma 2.3 into account, we get

$$\lim_{\wp\to\infty}\wp\left(\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}\,(x)\right)=(-2+l)\,x\mathrm{h}'\,(x)+(l+1)\,x\,(1+x)\,\mathrm{h}''(x)$$
$$+\lim_{\wp\to\infty}\wp\left(\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^{2};x\right)\right). \tag{6}$$

As a result of Cauchy-Schwarz inequality, we have

$$\wp\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\varepsilon(\mathrm{v},x)(\mathrm{v}-x)^{2};x\right)\leq\sqrt{\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\varepsilon^{2}(\mathrm{v},x);x)}\sqrt{\wp^{2}\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^{4};x\right)}. \tag{7}$$

Because $\varepsilon^{2}(x,x)=0$ and $\varepsilon^{2}(\cdot,x)\in C[0,\infty)$, using uniform convergence from Theorem 3.1, we find

$$\lim_{\wp\to\infty}\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\varepsilon^{2}(\mathrm{v},x);x\right)=\varepsilon^{2}(x,x)=0. \tag{8}$$

Therefore, from Lemma 2.3 and (8) yields

$$\lim_{\wp\to\infty}\wp\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^{2};x\right)=0$$

and using (6) we obtain the asymptotic behavior of the α-Pólya-Baskakov operator (5). □

Lemma 3.3. *The following inequality is true for positive linear operators* (5).

$$\left|\overrightarrow{Q}_{\wp}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)\right|\leqslant\|\mathrm{h}\|\,.$$

Proof. From operators (5), we obtain

$$\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)\right| = \left|\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\mathrm{h}\left(\frac{\jmath}{\imath}\right)\right| \leqslant \sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right)\right|$$
$$\leqslant \sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\sup|\mathrm{h}(x)| = \|\mathrm{h}\|.$$

□

Let $C_B[0,\infty)$ denotes the space containing the real-valued continuous and bounded functions $\mathrm{h} \in C_B[0,\infty)$, equipped with the norm.

$$\|\mathrm{h}\| = \sup_{x\in[0,\infty)} |\mathrm{h}(x)|.$$

The Peetre's K-functional for h has been characterized as follows:

$$K_2(h;\delta) := \inf_{x\in C_B^2[0,\infty)} \left\{\left\|\mathrm{h}-g\right\| + \delta\left\|g''\right\|\right\},\ \delta > 0,$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz([7], p.177. Theorem 2.4) there exists $C > 0$, such that

$$K_2(\mathrm{h};\delta) \leqslant C\omega_2\left(\mathrm{h};\sqrt{\delta}\right), \tag{9}$$

The modulus of continuity of second order $\omega_2\left(\mathrm{h};\sqrt{\delta}\right)$ is specified by

$$\omega_2\left(\mathrm{h};\sqrt{\delta}\right) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x\in I} |\mathrm{h}(x+2h) - 2\mathrm{h}(x+h) + \mathrm{h}(x)|.$$

Furthermore, the modulus of smoothness of the first order is provided by

$$\omega\left(\mathrm{h};\sqrt{\delta}\right) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x\in I} |\mathrm{h}(x+h) - \mathrm{h}(x)|.$$

Theorem 3.4. *For* $\mathrm{h} \in C_B[0,\infty)$, *we conclude*

$$\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant \omega\left(\mathrm{h}, \frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right) + C\omega_2\left(\mathrm{h}, \frac{\sqrt{\psi_{\imath,\lambda}^{(\alpha)}(x)}}{2}\right),$$

where $C > 0$ *is a constant and*

$$\psi_{\imath,\lambda}^{(\alpha)}(x) = \overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right) + \left\{\frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right\}^2.$$

Proof. We proceed with auxiliary operators

$$\mathcal{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) = \overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(h;x) + \mathrm{h}(x) - \mathrm{h}\left(\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x\right). \tag{10}$$

We found that $\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)$ are linear for all $x \in [0,\infty)$, hence

$$\mathcal{Q}_{\imath}^{(\alpha,\hat{\varrho})}(1;x) = 1 \ \ and \ \ \mathcal{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{v};x) = x,$$

i.e., $\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}$ preserves linear functions intact as a result.

$$\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}(\mathrm{v}-x;x)=0. \tag{11}$$

Let $g\in C_B^2[0,\infty)$ and $\mathrm{v},x\in[0,\infty)$, then according to Taylor's theorem we can imply that,

$$g(v)=g(x)+(\mathrm{v}-x)g'(x)+\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi,$$

Applying the operator $\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}$ on both sides of above inequality,we can write

$$\begin{aligned}\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}(g;x)-g(x)&=g'(x)\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}((\mathrm{v}-x);x)+\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\&=\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\&=\overrightarrow{\mathcal{Q}}_{\wr}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\&\quad-\int_x^{\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x}\left(\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x-\varpi\right)g''(\varpi)d\varpi.\end{aligned}$$

Furthermore, we acquire

$$\begin{aligned}\left|\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}(g;x)-g(x)\right|&\leqslant\overrightarrow{\mathcal{Q}}_{\wr}^{(\alpha,\hat{\varrho})}\left(\left|\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi\right|;x\right)\\&\quad+\left|\int_x^{\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x}\left(\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x-\varpi\right)g''(\varpi)d\varpi\right|.\end{aligned} \tag{12}$$

Since $\left|\int_x^t(\mathrm{v}-\varpi)g''(\varpi)d\varpi\right|\leqslant(\mathrm{v}-x)^2\left\|g''\right\|$ and

$$\left|\int_x^{\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x}\left(\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x-\varpi\right)g''(\varpi)d\varpi\right|\leqslant\left\{\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x-x\right\}^2\|g''\|.$$

As a equation, (12) indicates

$$\begin{aligned}\left|\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}(g;x)-g(x)\right|&\leqslant\left[\overrightarrow{\mathcal{Q}}_{\wr}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)+\left\{\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x-x\right\}^2\right]\left\|g''\right\|\\&\leqslant\left[\overrightarrow{\mathcal{Q}}_{\wr}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)+\left\{\frac{(2\alpha+\hat{\varrho}\wr-2)}{(1-\hat{\varrho})\wr}x\right\}^2\right]\left\|g''\right\|\\&=\psi_{\wr,\lambda}^{(\alpha)}(x)\left\|g''\right\|,\end{aligned} \tag{13}$$

In accordance with Lemma 3.3 and auxiliary operators,we have

$$\begin{aligned}\left|\overrightarrow{\mathcal{Q}}_{\wr}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right|&\leqslant\left|\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}((\mathrm{h}-g);x)\right|+\left|g(x)-\mathrm{h}(x)\right|+\left|\mathcal{Q}_{\wr}^{(\alpha,\hat{\varrho})}(g;x)-g(x)\right|\\&\quad+\left|\mathrm{h}\left(\frac{2\alpha+\wr-2}{(1-\hat{\varrho})\wr}x\right)-\mathrm{h}(x)\right|\\&\leqslant4\left\|\mathrm{h}-g\right\|+\psi_{\wr,\lambda}^{(\alpha)}(x)\left\|g''\right\|+\omega\left(\mathrm{h};\frac{(2\alpha+\hat{\varrho}\wr-2)}{(1-\hat{\varrho})\wr}x\right).\end{aligned}$$

taking infimum on both side over $g \in C_B^2[0,\infty)$,

$$\left|\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant 4K_2\left(\mathrm{h};\frac{\psi_{\imath,\lambda}^{(\alpha)}(x)}{4}\right) + \omega\left(\mathrm{h};\frac{(2\alpha+\varrho\imath-2)}{(1-\varrho)\imath}x\right).$$

using equation (9), we get

$$\left|\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant C\omega_2\left(\mathrm{h};\frac{\sqrt{\psi_{\imath,\lambda}^{(\alpha)}(x)}}{2}\right) + \omega\left(\mathrm{h};\frac{(2\alpha+\varrho\imath-2)}{(1-\varrho)\imath}x\right).$$

□

For $\beta \in (0,1]$, the Lipschitz-type space is defined as: (see [18])

$$Lip_{\mathcal{D}}^*(\beta) := \left\{\mathrm{h} \in C_B[0,\infty) : |\mathrm{h}(v) - \mathrm{h}(x)| \leqslant \mathcal{D}\frac{|v-x|^\beta}{(x+v)^{\beta/2}}; x, v \in [0,\infty)\right\},$$

Here, $\mathcal{D} > 0$ is constant and $\beta \in (0,1]$.

Theorem 3.5. *For all $x \in [0,\infty)$ and $\mathrm{h} \in Lip_{\mathcal{D}}^*(\beta)$, $0 < \beta \in (0,1]$ we get*

$$\left|\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| \mathcal{D}\left(\frac{\Phi_{\imath}^{(\alpha,\varrho)}(x)}{x}\right)^{\beta/2}, \tag{14}$$

where $\Phi_{\imath}^{(\alpha,\varrho)}(x) = \overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^2;x\right)$.

Proof. Let us suppose that $\beta = 1$. Then, for $\mathrm{h} \in Lip_{\mathcal{D}}^*(1)$ and $x \in [0,\infty)$, we conclude

$$\begin{aligned}\left|\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| &\leqslant \left|\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\mathrm{h}\left(\frac{\jmath}{\imath}\right) - \mathrm{h}(x)\right| \\ &\leqslant \sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right) - f(x)\right| \leqslant \mathcal{D}\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\frac{\left|\frac{\jmath}{\imath}-x\right|}{\left(\frac{\jmath}{\imath}+x\right)^{1/2}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality for sum and $\frac{1}{\sqrt{\frac{k}{\imath}+x}} \leqslant \frac{1}{\sqrt{x}}$, we obtain

$$\begin{aligned}\left|\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left\{\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{1/2} \\ &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\left\{\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\right\}^{1/2}\left\{\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{1/2} \\ &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\left\{\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}(1;x)\right\}^{1/2}\left\{\overrightarrow{Q}_{\imath}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^2;x\right)\right\}^{1/2} = \mathcal{D}\left(\frac{\Phi_{\imath}^{(\alpha,\varrho)}(x)}{x}\right)^{1/2}.\end{aligned}$$

As a result, the outcome for $\beta = 1$ is correct.

As we continue, we will demonstrate the necessary outcome for $0 < \beta < 1$ and Consider $\mathrm{h} \in Lip^{*}_{\mathcal{D}}(\beta)$.

$$\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant \sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right) - \mathrm{h}(x)\right| \leqslant \mathcal{D}\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\frac{\left|\frac{\jmath}{\imath} - x\right|^{\beta}}{\left(\frac{\jmath}{\imath} + x\right)^{\beta/2}}.$$

making use of inequality $\frac{1}{\sqrt{\frac{\jmath}{\imath}+x}} \leqslant \frac{1}{\sqrt{x}}$ and using Holder's inequality to the sum with $p = 2/\beta, q = 2/(2-\beta)$

$$\begin{aligned}
\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right| &\leqslant \frac{\mathcal{D}}{x^{\beta/2}}\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left\{\left(\frac{\jmath}{\imath} - x\right)^{2}\right\}^{\beta/2} \\
&\leqslant \frac{\mathcal{D}}{x^{\beta/2}}\left\{\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\left(\frac{\jmath}{\imath} - x\right)^{2}\right\}^{\beta/2}\left\{\sum_{\jmath=0}^{\infty} q_{\imath,\jmath}^{(\alpha,\varrho)}(x)\right\}^{\frac{2-\beta}{2}} \\
&\leqslant \mathcal{D}\left\{\frac{\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^{2};x\right)}{x}\right\}^{\beta/2} = \mathcal{D}\left(\frac{\Phi_{\imath}^{(\alpha,\varrho)}(x)}{x}\right)^{\beta/2}.
\end{aligned}$$

□

4. Weighted Approximation

Gadjiev [10, 11] investigated the weight spaces $C_{\Phi}[0,\infty)$ and $B_{\Phi}[0,\infty)$ of real-valued functions defined on $[0,\infty)$ with $\Phi(x) = 1 + x^{2}$, to demonstrate Korovkin's theorem, which does not commonly hold to all of these spaces.

$$B_{\Phi}[0,\infty) := \{\mathrm{h} : |\mathrm{h}(x)| \leqslant \mathcal{D}_{\mathrm{h}}\Phi(x)\},$$

with

$$\|\mathrm{h}\|_{\Phi} = \sup_{x\in[0,\infty)}\frac{|\mathrm{h}(x)|}{\Phi(x)},$$

and

$$C_{\Phi}[0,\infty) := \{\mathrm{h} : \mathrm{h} \in B_{\Phi}[0,\infty)\},$$

i.e. $C_{\Phi}[0,\infty) = C[0,\infty) \cap B_{\Phi}[0,\infty)$ is the subspace of $B_{\Phi}[0,\infty)$ containing continuous functions and

$$C^{*}_{\Phi}[0,\infty) := \left\{\mathrm{h} \in C_{\Phi}[0,\infty) : \lim_{x\to\infty}\frac{|\mathrm{h}(x)|}{\Phi(x)} < \infty\right\}.$$

Korovkin's theorem, however, is valid throughout the range $C^{*}_{\Phi}[0,\infty)$.

The usual modulus of continuity of h on $[0,b]$ is defined as follows:

$$\omega_{b}(\mathrm{h},\delta) = \sup_{|\mathrm{v}-x|\leqslant\delta}\ \sup_{x,\mathrm{v}\in[0,b]} |\mathrm{h}(\mathrm{v}) - \mathrm{h}(x)|.$$

Theorem 4.1. *Let* $\mathrm{h} \in C^{*}_{\Phi}[0,\infty)$ *then for operators* $\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x)$ *we conculde*

$$\left\|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right\|_{C[a,b]} \leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^{2}\right)\Phi_{\imath}^{(\alpha,\varrho)}(x) + 2\omega\left(\mathrm{h}, \sqrt{\Phi_{\imath}^{(\alpha,\varrho)}(x)}\right), \tag{15}$$

where $\Phi_{\imath}^{(\alpha,\varrho)}(x) = \overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^{2};x\right)$.

Proof. suppose $x \in [0,b]$, $\mathrm{v} \in (b+1,\infty)$, and $\mathrm{v}-x>1$ we conclude

$$\begin{aligned}|\mathrm{h}(\mathrm{v})-\mathrm{h}(x)| &\leqslant \mathcal{D}_{\mathrm{h}}\Phi(\mathrm{v}-x)=\mathcal{D}_{\mathrm{h}}\left\{1+(\mathrm{v}-x)^2\right\}=\mathcal{D}_{\mathrm{h}}\left(\mathrm{v}^2-2x\mathrm{v}+x^2+1\right)\\ &\leqslant \mathcal{D}_{\mathrm{h}}\left(\mathrm{v}^2+x^2+2\right)=\mathcal{D}_{\mathrm{h}}\left\{(\mathrm{v}-x)^2+2x(\mathrm{v}-x)+2+2x^2\right\}\\ &\leqslant \mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left\{2x^2+2x+3\right\}\leqslant 4\mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left(1+x^2\right)\\ &\leqslant 4\mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left(1+b^2\right).\end{aligned} \tag{16}$$

For $x \in [0,b]$ and $\mathrm{v} \in [0,b+1]$, we conclude

$$|\mathrm{h}(\mathrm{v})-\mathrm{h}(x)| \leqslant \omega_{b+1}(|\mathrm{v}-x|) \leqslant \left(1+\frac{|\mathrm{v}-x|}{\delta}\right)\omega_{b+1}(\mathrm{h},\delta), \delta>0. \tag{17}$$

We find by solving (16) and (17)

$$|\mathrm{h}(\mathrm{v})-\mathrm{h}(x)| \leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^2\right)(\mathrm{v}-x)^2+\left(1+\frac{|\mathrm{v}-x|}{\delta}\right)\omega_{b+1}(\mathrm{h},\delta).$$

making use of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\vec{Q}_{\ell}^{(\alpha,\varrho)}\mathrm{h}(\mathrm{v};x)-\mathrm{h}(x)\right| &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^2\right)\vec{Q}_{\ell}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^2;x\right)\\ &\quad+\left(1+\frac{1}{\delta}\vec{Q}_{\ell}^{(\alpha,\varrho)}(|\mathrm{v}-x|;x)\right)\omega_{b+1}(\mathrm{h},\delta)\\ &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^2\right)\vec{Q}_{\ell}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^2;x\right)\\ &\quad+\left[1+\frac{1}{\delta}\left\{\vec{Q}_{\ell}^{(\alpha,\varrho)}\left((\mathrm{v}-x)^2;x\right)\right\}^{1/2}\right]\omega_{b+1}(\mathrm{h},\delta)\\ &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^2\right)\Phi_{\ell}^{(\alpha,\varrho)}(x)+\left[1+\frac{1}{\delta}\sqrt{\Phi_{\ell}^{(\alpha,\varrho)}(x)}\right]\omega_{b+1}(\delta)\end{aligned}$$

choosing $\delta=\sqrt{\Phi_{\ell}^{(\alpha,\varrho)}(x)}$, we receive the desired outcome.
□

Theorem 4.2. *suppose* $\mathrm{h} \in C^*_{\Phi}[0,\infty)$ *then, we obtain*

$$\lim_{\ell\to\infty}\left\|\vec{Q}_{\ell}^{(\alpha,\varrho)}(\mathrm{h};x)-\mathrm{h}(x)\right\|_{\Phi}=0. \tag{18}$$

Proof. From [11], This is sufficient to validate the three equations shown below.

$$\lim_{\ell\to\infty}\left\|\vec{Q}_{\ell}^{(\alpha,\varrho)}\left(\mathrm{v}^i;x\right)-\mathrm{v}^i\right\|_{\Phi}=0,\ i=0,1,2. \tag{19}$$

Clearly equation (19) holds for $i=0$ as $\vec{Q}_{\ell}^{(\alpha,\varrho)}(1;x)=1$. Now using Lemma 2.2 we have

$$\begin{aligned}\left\|\vec{Q}_{\ell}^{(\alpha,\varrho)}(\mathrm{v};x)-x\right\|_{\Phi} &= \sup_{x\in[0,\infty)}\frac{1}{\Phi(x)}\left|\frac{2\alpha+\ell-2}{(1-\varrho)\ell}x-x\right|\\ &= \sup_{x\in[0,\infty)}\frac{x}{x^2+1}\left|\frac{(2\alpha+\varrho\ell-2)}{(1-\varrho)\ell}\right| \leqslant \frac{(2\alpha+\varrho\ell-2)}{(1-\varrho)\ell},\end{aligned}$$

This implies that $\lim\limits_{\mathfrak{z}\to\infty}\left\|\overrightarrow{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\mathrm{v};x)-x\right\|_{\Phi}=0.$ similarly, we have

$$\begin{aligned}\left\|\overrightarrow{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^2;x\right)-x^2\right\|_{\Phi} &= \sup_{x\in[0,\infty)}\frac{1}{\Phi(x)}\left|\begin{array}{c}\frac{(4\alpha\mathfrak{z}+(\mathfrak{z}-3)n)}{(1-\hat{\varrho})(1-2\hat{\varrho})\mathfrak{z}^2}x^2\\ +\frac{4\alpha\hat{\varrho}(\mathfrak{z}-2)+4\alpha+\hat{\varrho}((\mathfrak{z}-5)\mathfrak{z}+8)+\mathfrak{z}-4}{(1-\hat{\varrho})(1-2\hat{\varrho})\mathfrak{z}^2}x-x^2\end{array}\right|\\ &\leqslant \sup_{x\in[0,\infty)}\frac{x^2}{\Phi(x)}\left|\frac{(4\alpha\mathfrak{z}+(\mathfrak{z}-3)\mathfrak{z})}{(1-\hat{\varrho})(1-2\hat{\varrho})\mathfrak{z}^2}-1\right|\\ &\quad+\sup_{x\in[0,\infty)}\frac{x}{\Phi(x)}\left|\frac{x(4\alpha\hat{\varrho}(\mathfrak{z}-2)+4\alpha+\hat{\varrho}((\mathfrak{z}-5)\mathfrak{z}+8)+\mathfrak{z}-4)}{(1-\hat{\varrho})(1-2\hat{\varrho})\mathfrak{z}^2}\right|.\end{aligned}$$

Thus $\lim\limits_{\mathfrak{z}\to\infty}\left\|\overrightarrow{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^2;x\right)-x^2\right\|_{\Phi}=0.$
The anticipated outcome is attained. □

Significance statement

This work is carried out to find the rate of convergence of modified α-Pólya-Baskakov operators in terms of K-functionality, error estimate in terms of modulus of continuity, and weighted approximation. We have also proved Voronovskaya's theorems to establish the degree of approximation.

Conflict of interest

This work does not have any conflict of interest.

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