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The equation characterization of Hermitian elements in a ring with involution

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Abstract. In this note, some new characterizations of Hermitian elements are considered by the solution to some related equations. Especially, the form of the general solution to these related equations is discussed, which has a pivotal role in the linear system theories and applications.

1. Introduction

Let R be a ring associate with an identity and $a \in R$. If there exists $b \in R$ such that

$$aba = a, \ bab = a, \ ab = ba,$$

then a is called a group invertible element of R and b is called a group inverse of a [3, 4, 14, 18, 19], and it is unique, usually we write it by $a^{\#}$. We write $R^{\#}$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a, \ (a+b)^* = a^* + b^*, \ (ab)^* = b^*a^* \ for \ a, \ b \in R,$$

then R is said to be an involution ring or a $*-$ring.

Let R be a $*$-ring and $a \in R$. If there exists $b \in R$ makes

$$a = aba, \ b = bab, \ (ab)^* = ab, \ (ba)^* = ba,$$

then a is called a Moore-Penrose invertible element, and b is called the Moore-Penrose inverse of a [5, 13, 16], and it is unique, usually we record it as a^+. Let R^+ denote the set of all Moore-Penrose invertible elements of R.

If $a \in R^{\#} \cap R^+$ and $a^{\#} = a^+$, then a is called an EP element. On the studies of EP, the readers can refer to [2, 11–13, 15, 17, 20–23].

If $a \in R$ and $a = a^*$, then a is called a Hermitian element [2, 6]. We write R^{Her} to denote the set of all Hermitian elements of R. Clearly, if $a \in R^+$, then a is Hermitian if and only if $a^+ = (a^+)^*$ [2].

Let R be a $*-$ring and $a \in R$. If there exists $b \in R$ such that

$$a = aba, \ Rb = Ra^*, \ aR = bR,$$

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then a is called a core invertible element, and b is called the core inverse of a, usually we write it by $a^{\tiny{\textcircled{\#}}}$.

Let R be a $*-$ring and $a \in R$. If there exists $b \in R$ such that

$$a = aba,\ bR = a^*R,\ Ra = Rb,$$

then a is called a dual core invertible element, and b is called the dual core inverse of a, usually we write it by $a_{\tiny{\textcircled{\#}}}$.

It is not difficult to find that $a^{\tiny{\textcircled{\#}}}$ and $a_{\tiny{\textcircled{\#}}}$ are all unique and $a^{\tiny{\textcircled{\#}}} = a^\# aa^+$, $a_{\tiny{\textcircled{\#}}} = a^+aa^\#$.

The research hotspots of Hermitian elements are mainly matrix directions, Hermitian matrix plays important roles in system theory, such as observer design, eigenstructure assignment, and detection. The study ways contain Lyapunov and Sylvester equations and their various forms of deformation equations [7–10]. In [2], many characterizations of Hermitian elements from the perspective of ring theory are given. In this paper, we gives some new portrayals of Hermitian elements from the perspective of ring theory and equations.

2. Characterizing Hermitian elements by Patrício-Puystjeus Theorem

In [1, Theorem 4], Patrício and Puystjens shows that a regular element a in a ring R is EP if and only if $a^* = v^{-1}au$, where

$$u = (aa^-)^*(aa^* - 1) + 1, v = (a^2 - 1)aa^- + 1$$

are invertible for some (any) inner inverse a^-.

Let $a \in R^\# \cap R^+$. By [2, Theorem 1.4.2], we know that $a \in R^{Her}$ if and only if $a^* = a^*a^*a^\#$. This inspires us to give the following theorem.

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = a^*a^*au$, where $u \in R^{-1}$ and $u^{-1} = a^+a^3 + 1 - a^+a$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{Her}$. Then

$$a^* = a^*a^*a^\# = a^*a^*aa^+a^\# = a^*a^*a(a^+a^\# + 1 - a^+a).$$

Take $u = a^+a^\# + 1 - a^+a$. Then

$$u(a^+a^3 + 1 - a^+a) = (a^+a^\# + 1 - a^+a)(a^+a^3 + 1 - a^+a) = 1,$$

and

$$(a^+a^3 + 1 - a^+a)u = (a^+a^3 + 1 - a^+a)(a^+a^\# + 1 - a^+a) = 1.$$

Consequently, $u \in R^{-1}$ and $a^* = a^*a^*au$, where $u^{-1} = a^+a^3 + 1 - a^+a$.

" $\Leftarrow$ " If there is $u \in R^{-1}$ that makes $a^* = a^*a^*au$ and $u^{-1} = a^+a^3 + 1 - a^+a$, then

$$a^*(a^+a^3 + 1 - a^+a) = a^*a^*au(a^+a^3 + 1 - a^+a) = a^*a^*a,$$

that is

$$a^* = a^*a^+a - a^*a^+a^3 + a^*a^*a.$$

Multiplying the equality on the right by a^+a, one has $a^* = a^*a^+a$. Hence, $a \in R^{EP}$. Now we obtain

$$a^* = a^*a^+a - a^*a^+a^3 + a^*a^*a = a^* - a^*a^2 + a^*a^*a,$$

i.e., $a^*a^2 = a^*a^*a$. Multiplying the equality on the left by $(a^+)^*$, one gets $a^2 = aa^+a^*a = a^*a$. So we infer that $a \in R^{Her}$ by [2, Theorem 1.4.1]. $\square$

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = a^*a^*au$, where $u \in R^{-1}$ and $a^+a = ua^3a^+$.*

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then, by Theorem 2.1, we have $a^* = a^*a^*au$, where $u \in R^{-1}$ and $u^{-1} = a^+a^3 + 1 - a^+a$. It follows that

$$u^{-1}a^+a = (a^+a^3 + 1 - a^+a)a^+a = a^+a^3.$$

Hence $a^+a = ua^+a^3 = ua^3a^+$.

$'' \Leftarrow ''$ If there exists $u \in R^{-1}$ that makes $a^* = a^*a^*au$ and $a^+a = ua^3a^+$. Then

$$a^*a^3a^+ = (a^*a^*au)a^3a^+ = a^*a^*a(ua^3a^+) = a^*a^*aa^+a = a^*a^*a.$$

Multiplying the equality on the left by $(a^+)^*$, one has

$$a^3a^+ = aa^+a^*a = (aa^+a^*a)a^+a = a^3a^+a^+a.$$

Again multiply the last equality on the left by $a^+(a^\#)^2$, one gets $a^+ = a^+a^+a$. Hence $a \in R^{EP}$. It follows that $a^*a^2 = a^*a^3a^+ = a^*a^*a$. Then

$$a^*a = a^*a^2a^\# = a^*a^*aa^\# = a^*a^*aa^+ = a^*a^*.$$

This infers $a^2 = a^*a$. Thus $a \in R^{Her}$. □

Theorem 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = a^*ua^+a^\#$, where $u \in R^{-1}$ and $ua^+(a^+)^* = aa^\#$.*

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, by [2, Theorem 1.4.2], we have

$$a^* = a^*a^*a^\# = a^*a^*aa^+a^\# = a^*(a^*a + 1 - aa^+)a^+a^\#.$$

Take $u = a^*a + 1 - aa^+$. Then $u^{-1} = a^+(a^+)^* + 1 - aa^+$ and

$$ua^+(a^+)^* = (a^*a + 1 - aa^+)a^+(a^+)^* = a^+a + (1 - aa^+)a^+(a^+)^*.$$

Since $a \in R^{EP}$, $(1 - aa^+)a^+ = (1 - a^+a)a^+ = 0$ and $a^+a = aa^\#$. Hence $ua^+(a^+)^* = aa^\#$ and $a^* = a^*ua^+a^\#$.

$'' \Leftarrow ''$ From the assumption, we have $a^* = a^*ua^+a^\#$ and $ua^+(a^+)^* = aa^\#$. Then

$$a^* = a^*u(a^+(a^+)^*a^*)a^\# = a^*(ua^+(a^+)^*)a^*a^\#$$

$$= a^*aa^\#a^*a^\# = (a^*aa^\#a^*a^\#)a^+a = a^*a^+a.$$

Hence $a \in R^{EP}$. This induces that

$$a^* = a^*aa^\#a^*a^\# = a^*aa^+a^*a^\# = a^*a^*a^\#.$$

Thus $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Theorem 2.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = a^+ua^*a^\#$, where $u \in R^{-1}$ and $u(a^+)^*a^+ = aa^\#$.*

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, by [2, Theorem 1.4.2], we have

$$a^* = a^*a^*a^\# = a^+aa^*a^*a^\# = a^+(aa^* + 1 - aa^+)a^*a^\#.$$

Take $u = aa^* + 1 - aa^+$. Then

$$u((a^+)^*a^+ + 1 - aa^+) = 1 = ((a^+)^*a^+ + 1 - aa^+)u.$$

This infers that $u \in R^{-1}$. Now we have

$$u(a^+)^*a^+ = (aa^* + 1 - aa^+)(a^+)^*a^+ = aa^*(a^+)^*a^+ = aa^+ = aa^\#$$

and $a^* = a^+ua^*a^\#$.

$'' \Leftarrow ''$ Assume that $a^* = a^+ua^*a^\#$ and $u(a^+)^*a^+ = aa^\#$. Then

$$a^*a^+a = a^+ua^*a^\#a^+a = a^+ua^*a^\# = a^*.$$

Hence $a \in R^{EP}$. It follows that

$$a^* = a^+u(aa^+a^*)a^\# = a^+u(a^+)^*a^*a^*a^\# = a^+(u(a^+)^*a^+)aa^*a^*a^\#$$
$$= a^+aa^\#aa^*a^*a^\# = a^+aa^*a^*a^\# = a^*a^*a^\#.$$

Thus $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Theorem 2.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = uaa^*a^\#$, where $u \in R^{-1}$ and $uaa^+ = a^*a^+$.*

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then we have

$$a^* = a^*a^*a^\# = a^*a^+(aa^*a^\#) = (a^*a^+ + 1 - aa^+)aa^*a^\#.$$

Choose $u = a^*a^+ + 1 - aa^+$. Then by [2, Theorem 1.1.3], one gets

$$u = a^*a^+ + 1 - (aa^\#)^*.$$

Clearly

$$u^{-1} = (aa^\#)^*a(a^\#)^* + 1 - (aa^\#)^*.$$

Hence $a^* = uaa^*a^\#$ and $uaa^+ = (a^*a^+ + 1 - aa^+)aa^+ = a^*a^+$.

$'' \Leftarrow ''$ By hypothesis, we have $a^* = uaa^*a^\#$, where $u \in R^{-1}$ and $uaa^+ = a^*a^+$. Then

$$a^* = (uaa^+)aa^*a^\# = (a^*a^+)aa^*a^\# = a^*a^*a^\#.$$

Hence $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Remark 2.6. *In Theorem 2.5, if u is not required to be an invertible element, then we can choose $u = a^*a^+ + p(1 - aa^+)$, where $p \in R$.*

Example 2.7. *Choose $R = M_2(Z_5)$ with the transposition involution $*$.*

Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $A^\# = A^+ = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and $A^* = A$. Hence A is Hermitian. Then we can choose $U = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} + P\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, where $P \in R$.

3. Characterizing Hermitian elements by dual core invertible elements

Lemma 3.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^* = a^*a^*a_{\#}$.*

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then, by [2, Theorem 1.4.2], we have $a^* = a^*a^*a^\#$. Since $a \in R^{EP}$,

$$a_{\#} = a^+aa^\# = a^\#aa^\# = a^\#.$$

Hence $a^* = a^*a^*a_{\#}$.

$'' \Leftarrow ''$ Suppose that $a^* = a^*a^*a_{\#}$. Noting that $a_{\#}aa^\# = a_{\#}$. Then $a^* = a^*aa^\#$. It follows that $a \in R^{EP}$ by [2, Theorem 1.2.1]. This infers

$$a_{\#} = a^+aa^\# = a^+ = a^\#,$$

and then $a^* = a^*a^*a^\#$. Thus $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Theorem 3.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^* = a^* a_{\textcircled{\#}} a^* a a^{\#}$.*

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, $a \in R^{EP}$ and $a = a^*$, it follows that

$$a^* = a^* a^* a^{\#} = a^* a^+ a a^* a^{\#} = a^*(a^+ a a^{\#}) a a^* a^{\#} = a^* a_{\textcircled{\#}} a a^* a^{\#} = a^* a_{\textcircled{\#}} a^* a a^{\#}.$$

$'' \Leftarrow ''$ Assume that $a^* = a^* a_{\textcircled{\#}} a^* a a^{\#}$. Then $a^* = a^* a^+ a$ because $a^{\#} = a^{\#} a^+ a$. This implies $a \in R^{EP}$. Hence $a_{\textcircled{\#}} = a^+ a a^{\#} = a^{\#}$ and $a^* a a^{\#} = a^* a a^+ = a^*$. It follows that

$$a^* = a^* a_{\textcircled{\#}} a^* a a^{\#} = a^* a^{\#} a^*,$$

and

$$a a^+ = (a^+)^* a^* = (a^+)^* a^* a^{\#} a^* = a^{\#} a^*.$$

One gets $a = a^2 a^+ = a a^{\#} a^* = a^{\#} a a^* = a^+ a a^* = a^*$. Hence $a \in R^{Her}$. □

Let $a \in R^{\#} \cap R^{+}$. Then it is easy to show that $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $a^+ = (a^+)^*$. Hence we have the following theorem.

Theorem 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $(a^+)^* = a^+ a^2 (a^{\#})^* (a^+)^*$.*

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, one has $a^* = a^* a^* a_{\textcircled{\#}}$ by Lemma 3.1. Noting that $(a^* a^* a_{\textcircled{\#}})^+ = a^+ a^2 (a^{\#})^* (a^+)^*$. Then $(a^+)^* = a^+ a^2 (a^{\#})^* (a^+)^*$.

$'' \Leftarrow ''$ Assume that $(a^+)^* = a^+ a^2 (a^{\#})^* (a^+)^*$. Then

$$a a^+ = (a^+)^* a^* = a^+ a^2 (a^{\#})^* (a^+)^* a^* = a^+ a^2 (a^{\#})^* = a^+ a (a^+ a^2 (a^{\#})^*) = a^+ a^2 a^+.$$

This infers that $a = a a^+ a = a^+ a^2 a^+ a = a^+ a^2$. Hence $a \in R^{EP}$, this gives

$$a a^+ = a^+ a^2 (a^{\#})^* = a (a^{\#})^*.$$

Then $a^+ = a^+ a (a^{\#})^* = (a^{\#})^* = (a^+)^*$. Thus $a \in R^{Her}$. □

Let $a \in R^{\#} \cap R^{+}$. Then, clearly,

$$(a^* a^* a_{\textcircled{\#}})^{\#} = (a^* a^* a_{\textcircled{\#}})^+ = a^+ a^2 (a^{\#})^* (a^+)^*.$$

By Lemma 3.1, we have the following corollary which contrasts with Theorem 3.3.

Corollary 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $(a^{\#})^* = a^+ a^2 (a^{\#})^* (a^+)^*$.*

Theorem 3.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{\#} = a^+ a^{\#} a^* a_{\textcircled{\#}} a$.*

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, one has $(a^{\#})^* = a^+ a^2 (a^{\#})^* (a^+)^*$ by Corollary 3.4. Applying the involution on the equality, one gets $a^{\#} = a^+ a^{\#} a^* a^+ a = a^+ a^{\#} a^* a_{\textcircled{\#}} a$.

$'' \Leftarrow ''$ If $a^{\#} = a^+ a^{\#} a^* a_{\textcircled{\#}} a$, then

$$a_{\textcircled{\#}} = a^+ a a^{\#} = a^+ a (a^+ a^{\#} a^* a_{\textcircled{\#}} a) = a^+ a^{\#} a^* a_{\textcircled{\#}} a = a^{\#}.$$

Hence $a \in R^{EP}$. So we have

$$a^{\#} = a^+ a^{\#} a^* a^{\#} a = a^+ a^{\#} a^* a a^+ = a^+ a^{\#} a^* = a^{\#} a^{\#} a^*.$$

Hence $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Corollary 3.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{\textcircled{\#}} = a^+ a^{\#} a^* a_{\textcircled{\#}} a$.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then $a \in R^{EP}$ and $a^{\#} = a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a$ by Theorem 3.5. Since $a^2a^{+} = a$, we get $a^{\textcircled{\#}} = a^{\#}aa^{+} = (a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a)aa^{+} = a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a$.

" ⇐ " If $a^{\textcircled{\#}} = a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a$, then $a^{\#} = a^{\textcircled{\#}}aa^{\#} = a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a^2a^{\#} = a^{+}a^{\#}a^{*}a_{\textcircled{\#}}a$. Hence $a \in R^{Her}$ by Theorem 3.5. □

Noting that $a_{\textcircled{\#}}a^{\#} = a^{+}aa^{\#}a^{\#} = a^{+}a^{\#}$. Then we get the following corollary.

Corollary 3.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{\textcircled{\#}} = a_{\textcircled{\#}}a^{\#}a^{*}a_{\textcircled{\#}}a$.*

Remark 3.8. *In corollary 3.7. If only $a_{\textcircled{\#}} = a_{\textcircled{\#}}a^{\#}a^{*}a_{\textcircled{\#}}a$, then a need not be Hermitian.*

Example 3.9. *Choose $R = M_3(Z_2)$ with involution $*$, the transposition of a matrix in R.*

Take $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $A^{\#} = A$, $A^{+} = A^{*} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Clearly, $A_{\textcircled{\#}} = A^{+}AA^{\#} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $A_{\textcircled{\#}}A^{\#}A^{*}A_{\textcircled{\#}}A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = A_{\textcircled{\#}}$. *However $A \neq A^{*}$. Hence A is not Hermitian.*

4. Characterizing Hermitian elements by constructing invertible elements

Theorem 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{*}a^{*}a^{\#} + 1 - aa^{\#} \in R^{-1}$ and $(a^{*}a^{*}a^{\#} + 1 - aa^{\#})^{-1} = (a^{+})^{*} + 1 - aa^{\#}$.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then $a^{*}a^{*}a^{\#} = a^{*}$ by [2, Theorem 1.4.2] and $aa^{\#} = aa^{+} = a^{+}a$. This gives

$$(a^{*}a^{*}a^{\#} + 1 - aa^{\#})((a^{+})^{*} + 1 - aa^{\#}) = (a^{*} + 1 - aa^{\#})((a^{+})^{*} + 1 - aa^{\#})$$

$$= a^{+}a + a^{*}(1 - aa^{\#}) + (1 - aa^{\#})(a^{+})^{*} + 1 - aa^{\#} = a^{+}a + 1 - aa^{\#} = 1.$$

Similarly, we have

$$((a^{+})^{*} + 1 - aa^{\#})(a^{*}a^{*}a^{\#} + 1 - aa^{\#}) = 1.$$

Hence $(a^{*}a^{*}a^{\#} + 1 - aa^{\#})^{-1} = (a^{+})^{*} + 1 - aa^{\#}$.

" ⇐ " From the assumption, we have $(a^{*}a^{*}a^{\#} + 1 - aa^{\#})((a^{+})^{*} + 1 - aa^{\#}) = 1$, this gives

$$a^{*}a^{*}a^{\#}(a^{+})^{*} + a^{*}a^{*}a^{\#}(1 - aa^{\#}) + (1 - aa^{\#})(a^{+})^{*} + 1 - aa^{\#} = 1,$$

e.g.,

$$a^{*}a^{*}a^{\#}(a^{+})^{*} = aa^{\#}.$$

Multiplying the equality on the left by $a^{+}a$, one yields $aa^{\#} = a^{+}a$. Hence $a \in R^{EP}$. This induces

$$(a^{+})^{*} = (a^{+})^{*}aa^{\#} = (a^{+})^{*}a^{*}a^{*}a^{\#}(a^{+})^{*} = aa^{+}a^{*}a^{\#}(a^{+})^{*} = a^{*}a^{\#}(a^{+})^{*}$$

and

$$aa^{+} = (a^{+})^{*}a^{*} = a^{*}a^{\#}(a^{+})^{*}a^{*} = a^{*}a^{\#}aa^{+} = a^{*}a^{+}.$$

It follows from [2, Theorem 1.4.1] that $a \in R^{Her}$. □

Corollary 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{*}a^{*}a^{\#} + 1 - aa^{+} \in R^{-1}$ and $(a^{*}a^{*}a^{\#} + 1 - aa^{+})^{-1} = (a^{+})^{*} + 1 - aa^{+}$.*

Proof. " ⇒ " It follows from Theorem 4.1 because $aa^{\#} = aa^{+}$.

" ⇐ " From the assumption, one has

$$1 = (a^*a^*a^{\#} + 1 - aa^{+})((a^{+})^* + 1 - aa^{+}) = a^*a^*a^{\#}(a^{+})^* + a^*a^*a^{\#}(1 - aa^{+}) + 1 - aa^{+},$$

e.g.,

$$aa^{+} = a^*a^*a^{\#}(a^{+})^* + a^*a^*a^{\#}(1 - aa^{+}).$$

Multiplying the equality on the left by $a^{+}a$, one has $aa^{+} = a^{+}a^2a^{+}$. Hence $a \in R^{EP}$ and so $aa^{\#} = aa^{+}$. Now we have $(a^*a^*a^{\#} + 1 - aa^{\#})^{-1} = (a^{+})^* + 1 - aa^{\#}$. Thus $a \in R^{Her}$ by Theorem 4.1. □

Corollary 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^*a^*a^{\#} + 1 - a^{+}a \in R^{-1}$ and $(a^*a^*a^{\#} + 1 - a^{+}a)^{-1} = (a^{+})^* + 1 - a^{+}a$.*

Proof. " ⇒ " It follows from Theorem 4.1 because $aa^{\#} = a^{+}a$.

" ⇐ " From the assumption, one yields

$$1 = (a^*a^*a^{\#} + 1 - a^{+}a)((a^{+})^* + 1 - a^{+}a) = a^*a^*a^{\#}(a^{+})^* + (1 - a^{+}a)(a^{+})^* + 1 - a^{+}a,$$

e.g.,

$$a^{+}a + a^{+}a(a^{+})^* = a^*a^*a^{\#}(a^{+})^* + (a^{+})^*.$$

Multiplying the equality on the left by $a^{+}a$, one gets $(a^{+})^* = a^{+}a(a^{+})^*$. Applying the involution on the equality, one has $a^{+} = a^{+}a^{+}a$. Hence $a \in R^{EP}$ and so $aa^{+} = a^{+}a$. Now we have $(a^*a^*a^{\#} + 1 - aa^{+})^{-1} = (a^{+})^* + 1 - aa^{+}$. Thus $a \in R^{Her}$ by Corollary 4.2. □

Lemma 4.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a^*a^*a^{\#} + 1 - a^{+}a \in R^{-1}$ and $(a^*a^*a^{\#} + 1 - a^{+}a)^{-1} = a^{+}a^3a^{+}(a^{\#})^*(a^{+})^* + 1 - a^{+}a$.*

Proof. Since

$$(a^*a^*a^{\#} + 1 - a^{+}a)(a^{+}a^3a^{+}(a^{\#})^*(a^{+})^* + 1 - a^{+}a) = a^{+}a + a^*a^*a^{\#}(1 - a^{+}a) +$$

$$(1 - a^{+}a)a^{+}a^3a^{+}(a^{\#})^*(a^{+})^* + 1 - a^{+}a = a^{+}a + 1 - a^{+}a = 1$$

and

$$(a^{+}a^3a^{+}(a^{\#})^*(a^{+})^* + 1 - a^{+}a)(a^*a^*a^{\#} + 1 - a^{+}a)$$

$$= a^{+}a^3a^{+}(a^{\#})^*(a^{+})^*a^*a^*a^{\#} + 1 - a^{+}a = a^{+}a + 1 - a^{+}a = 1.$$

Hence $a^*a^*a^{\#} + 1 - a^{+}a \in R^{-1}$ and $(a^*a^*a^{\#} + 1 - a^{+}a)^{-1} = a^{+}a^3a^{+}(a^{\#})^*(a^{+})^* + 1 - a^{+}a$. □

Theorem 4.5. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*

(1) $a \in R^{Her}$;

(2) $(a^{+})^* = a^{+}a^3a^{+}(a^{\#})^*(a^{+})^*$;

(3) $a^{+} = a^{+}a^{\#}aa^{+}a^*a^{+}a = a^{+}a^{\textcircled{\#}}a^*a^{+}a$.

Proof. It is an immediate result of Corollary 4.3 and Lemma 4.4. □

5. Solutions to related equations in a given set.

In this section, we characterize Hermitian elements by the form of solutions to related equations. However, we need the following lemma.

Lemma 5.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^{+}(a^{+})^* = aa^{+}(a^{\#})^*(a^{+})^*$.*

Proof. $''\Rightarrow''$ Assume that $a \in R^{Her}$. Then $a^\# = a^+$ and $(a^+)^* = a^+a^3a^+(a^\#)^*(a^+)^*$ by Theorem 4.5. This gives

$$a^+(a^+)^* = a^\#(a^+)^* = a^\#a^+a^3a^+(a^\#)^*(a^+)^* = aa^+(a^\#)^*(a^+)^*.$$

$''\Leftarrow''$ Suppose that $a^+(a^+)^* = aa^+(a^\#)^*(a^+)^*$. Then

$$a^+ = a^+(a^+)^*a^* = aa^+(a^\#)^*(a^+)^*a^* = aa^+(a^\#)^*$$

and

$$a^+a^* = aa^+(a^\#)^*a^* = aa^+.$$

This infers that $a^+a^*a = aa^+a = a$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Observing Lemma 5.1, we can construct the following equation.

$$x(a^+)^* = ax(a^\#)^*(a^\#)^*. \tag{5.1}$$

Theorem 5.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(5.1) has at least one solution in $\chi_a =: \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$.*

Proof. $''\Rightarrow''$ Assume that $a \in R^{Her}$. Then $a \in R^{EP}$ and $(a^+)^* = (a^\#)^*$. It follows from Lemma 5.1 that $x = a^+$ is a solution.

$''\Leftarrow''$ (1) If $x = a$, then $a(a^+)^* = aa(a^\#)^*(a^\#)^*$. Multiplying the equality on the right by aa^+, one has

$$a(a^+)^*aa^+ = a(a^+)^*.$$

Multiplying the last equality on the left by $a^\#$, one obtains

$$(a^+)^*aa^+ = (a^+)^*.$$

This induces $aa^+a^+ = a^+$. Hence $a \in R^{EP}$. It follows that

$$(a^\#)^* = (a^+)^* = aa^+(a^+)^* = a^+a(a^+)^* = a^\#aa(a^\#)^*(a^\#)^* = a(a^\#)^*(a^\#)^*.$$

Applying the involution on the equality, one has $a^\# = a^\#a^\#a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

(2) If $x = a^\#$, then $a^\#(a^+)^* = aa^\#(a^\#)^*(a^\#)^*$. Multiplying the equality on the left by a^2, one has $a(a^+)^* = aa(a^\#)^*(a^\#)^*$. Hence $a \in R^{Her}$ by (1).

(3) If $x = a^+$, then $a^+(a^+)^* = aa^+(a^\#)^*(a^\#)^*$. Multiplying the equality on the left by aa^+, one has

$$aa^+a^+(a^+)^* = a^+(a^+)^*.$$

Multiplying the last equality on the right by a^*, one gets

$$aa^+a^+ = a^+.$$

Hence $a \in R^{EP}$. This infers $x = a^+ = a^\#$. Hence $a \in R^{Her}$ by (2).

(4) If $x = a^*$, then $a^*(a^+)^* = aa^*(a^\#)^*(a^\#)^*$. Applying the involution on the equality, one obtains $a^+a = a^\#a^\#aa^* = a^\#a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

(5) If $x = (a^\#)^*$, then $(a^\#)^*(a^+)^* = a(a^\#)^*(a^\#)^*(a^\#)^*$. Applying the involution on the equality, one yields

$$a^+a^\# = a^\#a^\#a^\#a^*.$$

Multiplying the equality on the left by a, one has $a^\# = a^\#a^\#a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

(6) If $x = (a^+)^*$, then $(a^+)^*(a^+)^* = a(a^+)^*(a^\#)^*(a^\#)^*$. Applying the involution on the equality, one obtains

$$a^+a^+ = a^\#a^\#a^+a^*.$$

Multiplying the equality on the left by a^+a, one has

$$a^+a^\#a^+a^* = a^\#a^\#a^+a^*.$$

Multiplying the last equality on the right by $(a^\#)^*a^3$, one gets

$$a^+a = a^\#a.$$

Hence $a \in R^{EP}$. This infers $x = (a^+)^* = (a^\#)^*$. Thus $a \in R^{Her}$ by (5). □

Multiplying Eq.(5.1) on the right by a^*, one gets

$$xaa^+ = ax(a^\#)^*. \tag{5.2}$$

Now we construct the following two equations:

$$xaa^+ + a^\# = ax(a^\#)^* + a^+. \tag{5.3}$$

and

$$xa^+a = ax(a^\#)^*. \tag{5.4}$$

Theorem 5.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(5.3) has at least one solution in χ_a.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then $a^+ = a^\#$ and $a^+ = (a^*)^+ = (a^+)^* = (a^\#)^*$. It follows that $x = a$ is a solution.

" ⇐ " Multiplying Eq.(5.3) on the right by aa^+, one has $a^\#aa^+ = a^\#$. Hence $a \in R^{EP}$. This infers $a^\# = a^+$. Now we have $\chi_a = \{a, a^\#, a^*, (a^+)^*\}$ and Eq.(5.3) is equivalent to Eq.(5.2).

(1) If $x = a$, then $aaa^+ = aa(a^\#)^*$. Multiplying the equality on the left by $a^+a^\#$, one gets $a^+ = (a^\#)^*$. It follows that $(a^\#)^* = a^\#$. Hence $a \in R^{Her}$.

(2) If $x = a^\#$, then $a^\#aa^+ = aa^\#(a^\#)^*$. Multiplying the equality on the left by a^2, one has $aaa^+ = aa(a^\#)^*$. Hence $a \in R^{Her}$ by (1).

(3) If $x = a^*$, then $a^*aa^+ = aa^*(a^\#)^*$, e.g., $a^* = aa^*(a^\#)^*$. Noting that $a \in R^{EP}$. Then

$$a = aa^+a = aa^*(a^+)^* = aa^*(a^\#)^* = a^*.$$

Hence $a \in R^{Her}$.

(4) If $x = (a^+)^*$, then $(a^+)^*aa^+ = a(a^+)^*(a^\#)^*$. Since $a^\# = a^+$, one gets

$$(a^\#)^* = (a^+)^* = (aa^+a^+)^* = (a^+)^*aa^+ = a(a^+)^*(a^\#)^*.$$

Applying the involution on the equality, one obtains $a^\# = a^\#a^+a^*$, e.g., $a^\# = a^\#a^\#a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Theorem 5.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(5.4) has at least one solution in χ_a.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then $a^\# = a^+$ and $aa^+ = a^+a$. Hence the Eq.(5.4) has the same solution as Eq.(5.3). Then by Theorem 5.3, we are done.

" ⇐ " (1) If $x = a$, then $a = aa(a^\#)^*$. Applying the involution on the equality, one yields $a^* = a^\#a^*a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

(2) If $x = a^\#$, then $a^\# = aa^\#(a^\#)^*$. Multiplying the equality on the left by a^2, one gets $a = a^2(a^\#)^*$. Thus $a \in R^{Her}$ by (1).

(3) If $x = a^+$, then $a^+a^+a = aa^+(a^\#)^*$. Multiplying the equality on the right by a^*, one has $a^+a^* = aa^+$. Hence $a \in R^{Her}$ by the proof of Lemma 5.1.

(4) If $x = a^*$, then $a^*a^+a = aa^*(a^\#)^*$. Applying the involution on the equality, one yields

$$a^+aa = a^\#aa^*.$$

Multiplying the last equality on the left by a, one has

$$aa = aa^*.$$

Hence $a \in R^{Her}$ by [2, Theorem 1.4.1].

(5) If $x = (a^\#)^*$, then $(a^\#)^* a^+ a = a(a^\#)^*(a^\#)^*$. Applying the involution on the equality, one gets

$$a^+ aa^\# = a^\# a^\# a^*.$$

Multiplying the last equality on the left by a, one obtains $aa^\# = a^\# a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

(6) If $x = (a^+)^*$, then $(a^+)^* a^+ a = a(a^+)^*(a^\#)^*$. Applying the involution on the equality, one yields $a^+ = a^\# a^+ a^*$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2]. □

Multiplying Eq.(5.4) on the right by a^*, one has $xa^* = ax(aa^\#)^*$. We revised it as follows:

$$xa^* = axaa^\#. \tag{5.5}$$

The following theorem gives a new characterization of Hermitian element which proof is routine.

Theorem 5.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(5.5) has at least one solution in χ_a.*

Now we change Eq.(5.5) as follows:

$$xa^* = aa^\# xa. \tag{5.6}$$

Theorem 5.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(5.6) has at least one solution in χ_a.*

Example 5.7. *Choose $R = M_2(Z_8)$ and $*$ be the transposition of a matrix. Take $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Then $a^* = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $a^+ = a^\# = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$. Hence $\chi_a = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \right\}$. Clearly, $aa^\# = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence Eq.(5.6) changes $x\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = x\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, that is $x\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix} = 0$. Clearly, this equation has not a solution in χ_a. Hence, by Theorem 5.6, a is not Hermitian. In fact, $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = a^*$.*

6. Solutions to constructed bivariate equations

Change Eq.(5.2) as follows:

$$xaa^\# = ax(a^\#)^*. \tag{6.1}$$

This infers the following equation:

$$xya^\# = yx(a^\#)^*. \tag{6.2}$$

Theorem 6.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $aa^* a^\# = a$ and Eq.(6.2) has at least one solution in $\chi_a^2 =: \{(x, y) \mid x, y \in \chi_a\}$.*

Proof. " ⇒ " Since $a \in R^{Her}$, then $a^\# = (a^\#)^*$. It follows that $(x, y) = (a, a)$ is a solution.

" ⇐ " From the assumption, there exists $(x_0, y_0) \in \chi_a^2$ such that $x_0 y_0 a^\# = y_0 x_0 (a^\#)^*$.

Noting that

$$x_0^+ aa^\# x_0 = \begin{cases} a^+ a, & x_0 \in \{a, a^\#, (a^+)^*\} =: \tau_a \\ aa^+, & x_0 \in \{a^+, a^*, (a^\#)^*\} =: \gamma_a \end{cases}.$$

Then in the case of $x_0 \in \tau_a$, one gets

$$a^+ay_0a^\# = (x_0^+aa^\#x_0)y_0a^\# = x_0^+aa^\#y_0x_0(a^\#)^*$$
$$= (x_0^+aa^\#y_0x_0(a^\#)^*)aa^+ = a^+ay_0a^\#aa^+.$$

This infers $y_0^\#a^+ay_0a^\# = y_0^\#a^+ay_0a^\#aa^+$.

In the case of $x_0 \in \gamma_a$, one has

$$aa^+y_0a^\# = (x_0^+aa^\#x_0)y_0a^\# = x_0^+aa^\#y_0x_0(a^\#)^*$$
$$= (x_0^+aa^\#y_0x_0(a^\#)^*)aa^+ = aa^+y_0a^\#aa^+.$$

This induces $y_0^\#aa^+y_0a^\# = y_0^\#aa^+y_0a^\#aa^+$.

Since

$$y_0^\#aa^+y_0 = y_0^\#a^+ay_0 = \begin{cases} a^\#a, & y_0 \in \tau_a \\ (aa^\#)^*, & y_0 \in \gamma_a \end{cases}.$$

Then we have

(a) if $x_0 \in \tau_a,\ y_0 \in \tau_a$, then we get

$$a^\# = a^\#aa^\# = y_0^\#a^+ay_0a^\# = y_0^\#a^+ay_0a^\#aa^+ = a^\#aa^+.$$

(b) if $x_0 \in \tau_a,\ y_0 \in \gamma_a$, then we get

$$(aa^\#)^*a^\# = (y_0^\#a^+ay_0)a^\# = y_0^\#a^+ay_0a^\#aa^+ = (aa^\#)^*a^\#aa^+,$$

and then

$$a^\# = aa^+a^\# = aa^+(aa^\#)^*a^\# = aa^+(aa^\#)^*a^\#aa^+ = a^\#aa^+.$$

(c) if $x_0 \in \gamma_a,\ y_0 \in \tau_a$, then we get

$$a^\# = a^\#aa^\# = y_0^\#aa^+y_0a^\# = y_0^\#aa^+y_0a^\#aa^+ = a^\#aa^+.$$

(d) if $x_0 \in \gamma_a,\ y_0 \in \gamma_a$, then we get

$$(aa^\#)^*a^\# = (y_0^\#aa^+y_0)a^\# = y_0^\#aa^+y_0a^\#aa^+ = (aa^\#)^*a^\#aa^+,$$

and then

$$a^\# = aa^+(aa^\#)^*a^\# = aa^+(aa^\#)^*a^\#aa^+ = a^\#aa^+.$$

To sum up, we have $a^\# = a^\#aa^+$ in any case. Thus $a \in R^{EP}$. Now we have $\chi_a = \{a, a^\#, a^*, (a^+)^*\}$.

(1) If $y_0 = a$, then we have $x_0aa^\# = ax_0(a^\#)^*$, which has the same solution as Eq.(5.4). By Theorem 5.4, we have $a \in R^{Her}$.

(2) If $y_0 = a^\#$, then $x_0a^\#a^\# = a^\#x_0(a^\#)^*$.

① If $x_0 = a$, then $a^\# = aa^\#a^\# = a^\#a(a^\#)^* = a^\#a(a^+)^* = (a^+)^* = (a^\#)^*$. Hence $a \in R^{Her}$.

② If $x_0 = a^\#$, then $a^\#a^\#a^\# = a^\#a^\#(a^\#)^*$. It follows that

$$a^\# = a^2(a^\#)^3 = a^2a^\#a^\#(a^\#)^* = aa^\#(a^+)^* = (a^+)^* = (a^\#)^*.$$

Hence $a \in R^{Her}$.

③ If $x_0 = a^*$, then $a^*a^\#a^\# = a^\#a^*(a^\#)^* = a^+(aa^\#)^* = a^+ = a^\#$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.2].

④ If $x_0 = (a^+)^*$, then $(a^+)^* = (a^+)^*a^\#a^\#a^2 = a^\#(a^+)^*(a^\#)^*a^2$. Multiplying the equality on the left by aa^*, one gets

$$a = aa^*a^\#(a^+)^*(a^\#)^*a^2 = a(a^+)^*(a^\#)^*a^2,$$

and then

$$a^{\#}a = a^{\#}a(a^{+})^{*}(a^{\#})^{*}a^{2} = (a^{+})^{*}(a^{\#})^{*}a^{2}.$$

This gives

$$a^{*} = a^{*}aa^{+} = a^{*}aa^{\#} = a^{*}((a^{+})^{*}(a^{\#})^{*}a^{2}) = (a^{\#})^{*}a^{2}$$

and

$$a = aa^{*}a^{\#} = a(a^{\#})^{*}a^{2}a^{\#} = a(a^{\#})^{*}a.$$

So $a^{\#} = a^{+} = a^{+}aa^{+} = a^{+}(a(a^{\#})^{*}a)a^{+} = (a^{\#})^{*}$. Hence $a \in R^{Her}$.

(3) If $y_0 = a^{*}$, then we have $x_0a^{*}a^{\#} = a^{*}x_0(a^{\#})^{*}$.

⑤ If $x_0 = a$, then $a = aa^{*}a^{\#} = a^{*}a(a^{\#})^{*}$ and

$$a^{2} = aa^{*}a(a^{\#})^{*} = aa^{*}a^{\#}a^{2}(a^{\#})^{*} = a^{3}(a^{\#})^{*}.$$

This implies

$$a^{\#} = (a^{\#})^{3}a^{2} = (a^{\#})^{3}a^{3}(a^{\#})^{*} = a^{\#}a(a^{\#})^{*} = a^{+}a(a^{\#})^{*} = (a^{\#})^{*}.$$

Hence $a \in R^{Her}$.

⑥ If $x_0 = a^{\#}$, then $a^{\#}a^{*}a^{\#} = a^{*}a^{\#}(a^{\#})^{*}$. It follows that

$$a^{*}a^{\#} = a^{+}aa^{*}a^{\#} = a^{\#}aa^{*}a^{\#} = a(a^{*}a^{\#}(a^{\#})^{*}) = (aa^{*}a^{\#})(a^{\#})^{*} = a(a^{\#})^{*},$$

and then

$$a^{2} = a(aa^{*}a^{\#}) = a^{2}(a^{*}a^{\#}) = a^{3}(a^{\#})^{*}.$$

Hence $a \in R^{Her}$ by ⑤.

⑦ If $x_0 = a^{*}$, then $a^{*}a^{*}a^{\#} = a^{*}a^{*}(a^{\#})^{*} = a^{*}$. It follows from [2, Theorem 1.4.2] that $a \in R^{Her}$.

⑧ If $x_0 = (a^{+})^{*}$, then $a^{\#} = (a^{+})^{*}a^{*}a^{\#} = a^{*}(a^{+})^{*}(a^{\#})^{*} = (a^{\#})^{*}$. Hence $a \in R^{Her}$.

(4) If $y_0 = (a^{+})^{*}$, then $x_0(a^{+})^{*}a^{\#} = (a^{+})^{*}x_0(a^{\#})^{*}$.

⑨ If $x_0 = a$, then $a(a^{+})^{*}a^{\#} = (a^{+})^{*}a(a^{\#})^{*}$. This infers

$$a^{*}a(a^{+})^{*}a^{\#} = a^{*}(a^{+})^{*}a(a^{\#})^{*} = a^{+}a^{2}(a^{\#})^{*} = a(a^{\#})^{*}$$

and

$$a^{*}a(a^{+})^{*}a^{\#} = a^{+}(aa^{*}a^{\#})a^{2}(a^{+})^{*}a^{\#} = a^{+}a^{3}(a^{+})^{*}a^{\#} = a^{2}(a^{+})^{*}a^{\#}.$$

Hence $a^{2}(a^{+})^{*}a^{\#} = a(a^{\#})^{*}$. It follows that

$$a(a^{+})^{*}a^{\#} = a^{\#}(a^{2}(a^{+})^{*}a^{\#}) = a^{\#}a(a^{\#})^{*} = a^{+}a(a^{\#})^{*} = (a^{\#})^{*}$$

and

$$(a^{+})^{*}a^{\#} = (a^{\#}a(a^{+})^{*})a^{\#} = a^{\#}(a^{\#})^{*}.$$

Then

$$a^{\#} = aa^{+}a^{\#} = a^{+}aa^{\#} = a^{*}(a^{+})^{*}a^{\#} = a^{*}a^{\#}(a^{\#})^{*}$$

and

$$aa^{\#} = aa^{*}a^{\#}(a^{\#})^{*} = a(a^{\#})^{*}.$$

Finally, we have $a^{\#} = a^{+}aa^{\#} = a^{+}a(a^{\#})^{*} = (a^{\#})^{*}$. Hence $a \in R^{Her}$.

⑩ If $x_0 = a^{\#}$, then $a^{\#}(a^{+})^{*}a^{\#} = (a^{+})^{*}a^{\#}(a^{\#})^{*}$. This gives

$$a(a^{+})^{*}a^{\#} = (aa^{*}a^{\#})(a^{+})^{*}a^{\#} = aa^{*}((a^{+})^{*}a^{\#}(a^{\#})^{*}) = aa^{\#}(a^{\#})^{*},$$

and then

$$(a^{+})^{*}a^{\#} = a^{\#}a(a^{+})^{*}a^{\#} = a^{\#}(aa^{\#}(a^{\#})^{*}) = a^{\#}(a^{\#})^{*}.$$

Thus $a \in R^{Her}$ by the proof of ⑨.

① If $x_0 = a^*$, then $a^\# = a^+aa^\# = a^*(a^+)^*a^\# = (a^+)^*a^*(a^\#)^* = aa^+(a^\#)^* = a^+a(a^\#)^* = (a^\#)^*$. Hence $a \in R^{Her}$.
② If $x_0 = (a^+)^*$, then $(a^+)^*(a^+)^*a^\# = (a^+)^*(a^+)^*(a^\#)^*$. Multiplying the equality on the left by a^*, one has

$$a^+a(a^+)^*a^\# = a^+a(a^+)^*(a^\#)^*.$$

Noting that $a^+a = aa^+$. Then $(a^+)^*a^\# = (a^+)^*(a^\#)^*$. This infers that

$$a^\# = a^+aa^\# = a^*(a^+)^*a^\# = a^*(a^+)^*(a^\#)^* = a^+a(a^\#)^* = (a^\#)^*.$$

Hence $a \in R^{Her}$. □

Noting that $aa^*a^\# = a$ if and only if $a^*a^\# = a^+a$ if and only if $a^*a^\#a^* = a^*$. Hence we obtain the following corollaries.

Corollary 6.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^*a^\# = a^+a$ and Eq.(6.2) has at least one solution in χ_a^2.*

Corollary 6.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^*a^\#a^* = a^*$ and Eq.(6.2) has at least one solution in χ_a^2.*

7. The influence of the form of the general solution to related equation on Hermitian elements

Now we generalize Eq.(5.6) as follows:

$$xa^* = aa^\#ya. \tag{7.1}$$

Theorem 7.1. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = aa^\#pa^+(a^+)^* + u - ua^+a \\ y = pa^+a^+ + v - a^+avaa^+ \end{cases}, \text{where } p, u, v \in R \text{ satisfying } pa^+ = pa^+a^+a. \tag{7.2}$$

Proof. First, we claim that the formula (7.2) is the solution of Eq.(7.1). In fact,

$$(aa^\#pa^+(a^+)^* + u - ua^+a)a^* = aa^\#pa^+$$
$$= aa^\#pa^+a^+a = aa^\#(pa^+a^+ + v - a^+avaa^+)a.$$

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.1). Then

$$x_0a^* = aa^\#y_0a.$$

Choose $p = a^+ax_0a^*a$, $u = x_0 - aa^\#pa^+(a^+)^*$, $v = y_0$. Then we have

$$pa^+ = a^+ax_0a^*aa^+ = a^+ax_0a^* = a^+a(aa^\#y_0a) = a^+a(aa^\#y_0a)a^+a = pa^+a^+a,$$
$$ua^+a = (x_0 - aa^\#pa^+(a^+)^*)a^+a = x_0a^+a - aa^\#(pa^+)(a^+)^* =$$
$$x_0a^+a - aa^\#(a^+ax_0a^*)(a^+)^* = x_0a^+a - aa^\#x_0a^*(a^+)^* =$$
$$x_0a^+a - aa^\#(aa^\#y_0a)(a^+)^* = x_0a^+a - aa^\#y_0a(a^+)^* = x_0a^+a - x_0a^*(a^+)^* = 0.$$

This induces that

$$x_0 = aa^\#pa^+(a^+)^* + (x_0 - aa^\#pa^+(a^+)^*)$$
$$= aa^\#pa^+(a^+)^* + u = aa^\#pa^+(a^+)^* + u - ua^+a$$

and

$$a^+avaa^+ = a^+ay_0aa^+ = a^+a(aa^\#y_0a)a^+$$
$$= a^+ax_0a^*a^+ = (a^+ax_0a^*a)a^+a^+ = pa^+a^+.$$

It follows that

$$y_0 = pa^+a^+ + y_0 - a^+avaa^+ = pa^+a^+ + v - a^+avaa^+.$$

Hence the general solution of Eq.(7.1) is given by the formula (7.2). □

Theorem 7.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if the general solution of Eq.*(7.1) *is given by*

$$\begin{cases} x = a^*a^{\#}pa^+(a^+)^* + u - ua^+a \\ y = pa^+a^+ + v - a^+avaa^+ \end{cases}, \text{where } p, u, v \in R. \tag{7.3}$$

Proof. $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then $a \in R^{EP}$ and $a = a^*$, this gives $pa^+a^+a = pa^{\#}a^{\#}a = pa^{\#} = pa^+$. Hence the formula (7.3) is the same as the formula (7.2). By Theorem 7.1, we are done.

$'' \Leftarrow ''$ From the assumption, we have

$$(a^*a^{\#}pa^+(a^+)^* + u - ua^+a)a^* = aa^{\#}(pa^+a^+ + v - a^+avaa^+)a,$$

e.g., for each $p \in R$, we have

$$a^*a^{\#}pa^+ = aa^{\#}pa^+a^+a.$$

Choose $p = a^2$, one gets

$$a^* = a^2a^+a^+a = aa^+(a^2a^+a^+a) = aa^+a^*.$$

This implies $a = a^2a^+$, it follows that $a^* = a^2a^+a^+a = aa^+a = a$. Hence $a \in R^{Her}$. □

Consider the general solution of which equation is given by the formula (7.3). For this we construct the following equation:

$$xa^* = a^*a^{\#}ya(aa^{\#})^*. \tag{7.4}$$

Theorem 7.3. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.*(7.4) *is given by the formula* (7.3).

Proof. First, we claim that the formula (7.3) is the solution of Eq.(7.4). In fact,

$$(a^*a^{\#}pa^+(a^+)^* + u - ua^+a)a^* = a^*a^{\#}pa^+ = a^*a^{\#}pa^+(aa^{\#})^* =$$
$$a^*a^{\#}pa^+a^+a(aa^{\#})^* = a^*a^{\#}(pa^+a^+ + v - a^+avaa^+)a(aa^{\#})^*.$$

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.4). Then

$$x_0a^* = a^*a^{\#}y_0a(aa^{\#})^*.$$

Choose $p = a^+a^2(a^+)^*x_0a^*a$, $u = x_0 - a^*a^{\#}pa^+(a^+)^*$, $v = y_0$. Then

$$ua^+a = (x_0 - a^*a^{\#}pa^+(a^+)^*)a^+a = x_0a^+a - a^*a^{\#}(a^+a^2(a^+)^*x_0a^*a)a^+(a^+)^* =$$
$$x_0a^+a - a^*a^{\#}a(a^+)^*x_0a^*(a^+)^* = x_0a^+a - a^*a^{\#}a(a^+)^*(a^*a^{\#}y_0a(aa^{\#})^*)(a^+)^* =$$
$$x_0a^+a - a^*a^{\#}y_0a(aa^{\#})^*(a^+)^* = x_0a^+a - x_0a^*(a^+)^* = 0$$

and

$$pa^+a^+ = a^+a^2(a^+)^*x_0a^*aa^+a^+ = a^+a^2(a^+)^*x_0a^*a^+$$
$$= a^+a^2(a^+)^*(a^*a^{\#}y_0a(aa^{\#})^*)a^+ = a^+a^2aa^+a^{\#}y_0aa^+ = a^+ay_0aa^+ = a^+avaa^+.$$

It follows that

$$x_0 = a^*a^{\#}pa^+(a^+)^* + (x_0 - a^*a^{\#}pa^+(a^+)^*)$$
$$= a^*a^{\#}pa^+(a^+)^* + u = a^*a^{\#}pa^+(a^+)^* + u - ua^+a$$

and

$$y_0 = pa^+a^+ + y_0 - a^+avaa^+ = pa^+a^+ + v - a^+avaa^+.$$

Hence the general solution of Eq.(7.4) is given by the formula (7.3). □

Theorem 7.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if Eq.*(7.1) *has the same solution as Eq.*(7.4).

Proof. $'' \Rightarrow ''$ Since $a \in R^{Her}$, by Theorem 7.2, the general solution of Eq.(7.1) is given by the formula (7.3) . Hence Eq.(7.1) has the same solution as Eq.(7.4) by Theorem 7.3.

$'' \Leftarrow ''$ Assume that Eq.(7.1) has the same solution as Eq.(7.4). Then the general solution of Eq.(7.1) is given by the formula (7.3) by Theorem 7.3. Hence $a \in R^{Her}$ by Theorem 7.2. □

8. Generalized inverse representation of elements with parameter

Theorem 8.1. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(xa^*)^+ = (a^+)^*(aa^{\#})^*x^+$, for each $x \in \chi_a$.
(2) $(xa^*)^{\#} = (a^+)^*(aa^{\#})^*x^+$, for each $x \in \tau_a$.
(3) $(xa^*)^{\#} = (aa^{\#})^*(a^+)^*x^{\#}$, for each $x \in \gamma_a$.

Proof. (1) Noting that

$$x(aa^{\#})^*x^+ = \begin{cases} aa^+, x \in \tau_a \\ a^+a, x \in \gamma_a \end{cases}, x^+x = \begin{cases} a^+a, x \in \tau_a \\ aa^+, x \in \gamma_a \end{cases}$$

and $(xa^*)((a^+)^*(aa^{\#})^*x^+) = xa^+a(aa^{\#})^*x^+ = x(aa^{\#})^*x^+$.

Then we have

$$((xa^*)((a^+)^*(aa^{\#})^*x^+))^* = (x(aa^{\#})^*x^+)^* = x(aa^{\#})^*x^+ = (xa^*)((a^+)^*(aa^{\#})^*x^+),$$

$$(xa^*)((a^+)^*(aa^{\#})^*x^+)(xa^*) = x(aa^{\#})^*x^+xa^* = x(aa^{\#})^*a^* = xa^*,$$

$$(((a^+)^*(aa^{\#})^*x^+)(xa^*))^* = ((a^+)^*(aa^{\#})^*a^*)^* = aa^+ = ((a^+)^*(aa^{\#})^*x^+)(xa^*)$$

and

$$((a^+)^*(aa^{\#})^*x^+)(xa^*)((a^+)^*(aa^{\#})^*x^+) = aa^+(a^+)^*(aa^{\#})^*x^+ = (a^+)^*(aa^{\#})^*x^+.$$

Hence $(xa^*)^+ = (a^+)^*(aa^{\#})^*x^+$ for each $x \in \chi_a$.

(2) It is an immediate result of (1) in case of $x \in \tau_a$.

(3) Noting that $xa^+ax^{\#} = (aa^{\#})^* = x^{\#}x$ and $(aa^{\#})^*x = x$. Then

$$(xa^*)((aa^{\#})^*(a^+)^*x^{\#}) = xa^+ax^{\#} = (aa^{\#})^*,$$

$$(xa^*)((aa^{\#})^*(a^+)^*x^{\#})(xa^*) = (aa^{\#})^*xa^* = xa^*,$$

$$((aa^{\#})^*(a^+)^*x^{\#})(xa^*) = (aa^{\#})^*(a^+)^*(aa^{\#})^*a^* = (aa^{\#})^*$$

and

$$((aa^{\#})^*(a^+)^*x^{\#})xa^*((aa^{\#})^*(a^+)^*x^{\#}) = (aa^{\#})^*(aa^{\#})^*(a^+)^*x^{\#} = (aa^{\#})^*(a^+)^*x^{\#}.$$

Thus $(xa^*)^{\#} = (aa^{\#})^*(a^+)^*x^{\#}$ for each $x \in \gamma_a$. □

Corollary 8.2. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $xa^* \in R^{EP}$ for each $x \in \tau_a$.
(2) $a \in R^{EP}$ if and only if $xa^* \in R^{EP}$ for some $x \in \gamma_a$.

Proof. (1) It follows from (1) and (2) of Theorem 8.1.

(2) " ⇒ " Since $x \in \gamma_a$, $(xa^*)^{\#} = (aa^{\#})^*(a^+)^*x^{\#}$ by (3) of Theorem 8.1. Noting that $a \in R^{EP}$. Then

$$(aa^{\#})^*(a^+)^* = aa^{\#}(a^+)^* = (a^+)^* = (a^+)^*aa^{\#} = (a^+)^*(aa^{\#})^*$$

by [2, Theorem 1.1.3]. This gives $(xa^*)^{\#} = (a^+)^*(aa^{\#})^*x^{\#}$. Since

$$(a^+)^+ = a = (a^{\#})^{\#} = (a^+)^{\#},$$

$$(a^*)^+ = (a^+)^* = (a^{\#})^* = (a^*)^{\#}$$

and

$$((a^{\#})^*)^+ = ((a^{\#})^+)^* = ((a^+)^+)^* = a^* = ((a^{\#})^{\#})^* = ((a^{\#})^*)^{\#},$$

one yields $x^+ = x^{\#}$ for any $x \in \gamma_a$. Hence $(xa^*)^{\#} = (a^+)^*(aa^{\#})^*x^+ = (xa^*)^+$ for each $x \in \gamma_a$ by Theorem 8.1. Therefore $xa^* \in R^{EP}$ for each $x \in \gamma_a$.

$'' \Leftarrow ''$ Since $xa^* \in R^{EP}$ for some $x \in \gamma_a$, one has $(xa^*)^\# = (xa^*)^+$. By Theorem 8.1, we have

$$(aa^\#)^*(a^+)^*x^\# = (a^+)^*(aa^\#)^*x^+.$$

Multiplying the equality on the left by $aa^\#$, one has

$$(aa^\#)^*(a^+)^*x^\# = aa^\#(aa^\#)^*(a^+)^*x^\#,$$

e.g.,

$$(a^\#)^*a^+ax^\# = aa^\#(a^\#)^*a^+ax^\#.$$

Then

$$\begin{aligned}(a^\#)^* &= (a^\#)^*(aa^\#)^* = (a^\#)^*a^+a(aa^\#)^* = (a^\#)^*a^+ax^\#x\\ &= aa^\#(a^\#)^*a^+ax^\#x = aa^\#(a^\#)^*a^+a(aa^\#)^* = aa^\#(a^\#)^*.\end{aligned}$$

Hence $a \in R^{EP}$ by [2, Theorem 1.1.3]. □

Theorem 8.3. *Let $a \in R^\# \cap R^+$. Then*
(1) $a \in R^{Her}$ if and only if $(xa^*)^+ = a^+x^+$ for some $x \in \chi_a$.
(2) $a \in R^{Her}$ if and only if $(xa^*)^\# = a^\#x^\#$ for some $x \in \gamma_a$.

Proof. (1) $'' \Rightarrow ''$ Since $a \in R^{Her}$, $a^+ = a^\#$ and $(a^\#)^* = a^\#$. Hence, by Theorem 8.1,

$$(xa^*)^+ = (a^+)^*(aa^\#)^*x^+ = (a^\#)^*(aa^\#)^*x^+ = (a^\#)^*x^+ = a^\#x^+ = a^+x^+$$

for each $x \in \chi_a$.

$'' \Leftarrow ''$ From the assumption and Theorem 8.1, there exists some $x \in \chi_a$, such that

$$a^+x^+ = (a^+)^*(aa^\#)^*x^+ = aa^+(a^\#)^*x^+.$$

Noting that

$$x^+x = \begin{cases} a^+a, & x \in \tau_a\\ aa^+, & x \in \gamma_a \end{cases}.$$

Then when $x \in \tau_a$, we have

$$\begin{aligned}a^+a^+a &= a^+x^+x = aa^+(a^\#)^*x^+x\\ &= aa^+(a^\#)^*a^+a = (a^+)^* = aa^\#(a^+)^* = aa^\#a^+a^+a.\end{aligned}$$

Multiplying the equality on the right by $(aa^\#)^*$, one gets $a^+ = aa^\#a^+$. Hence $a \in R^{EP}$, it follows that $a^\# = a^+ = a^+a^+a = (a^+)^* = (a^\#)^*$. Thus $a \in R^{Her}$.

When $x \in \gamma_a$, we have

$$a^+ = a^+aa^+ = a^+x^+x = aa^+(a^\#)^*x^+x = aa^+(a^\#)^*aa^+ = aa^+(a^\#)^*,$$

and then $a^+a^* = aa^+(a^\#)^*a^* = aa^+$. Hence $a \in R^{Her}$ by [2, Theorem 1.4.1].

(2) $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then $(xa^*)^\# = (aa^\#)^*(a^+)^*x^\# = (a^\#)^*a^+ax^\#$ for each $x \in \gamma_a$ by Theorem 8.1. Since $(a^\#)^* = a^\#$, then $(xa^*)^\# = a^\#a^+ax^\# = a^\#x^\#$ for each $x \in \gamma_a$.

$'' \Leftarrow ''$ By the hypothesis and Theorem 8.1, we have $a^\#x^\# = (aa^\#)^*(a^+)^*x^\#$ for some $x \in \gamma_a$. It follows that

$$a^\#(aa^\#)^* = a^\#x^\#x = (aa^\#)^*(a^+)^*x^\#x = (aa^\#)^*(a^+)^*(aa^\#)^*.$$

This gives $aa^\#(a^\#)^* = aa^\#a^+aa^\# = a^\#$. Hence $a \in R^{EP}$ by [2, Theorem 1.1.3]. Now we obtain

$$a^\# = a^\#aa^\# = a^\#(aa^\#)^* = (aa^\#)^*(a^+)^*(aa^\#)^* = aa^\#(a^+)^*aa^\# = (a^+)^* = (a^\#)^*.$$

Thus $a \in R^{Her}$. □

Theorem 8.4. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(aa^{\#}xa)^{+} = a^{+}x^{\#}a^{+}a^{2}a^{+}$ for each $x \in \chi_a$.
(2) $(aa^{\#}xa)^{\#} = aa^{\#}a^{+}x^{\#}a^{+}a$ for each $x \in \chi_a$.

Proof. (1) Noting that

$$xaa^{+}x^{\#} = x^{\#}a^{+}ax = \begin{cases} aa^{\#}, & x \in \tau_a \\ (aa^{\#})^{*}, & x \in \gamma_a \end{cases}.$$

Then

$$(aa^{\#}xa)(a^{+}x^{\#}a^{+}a^{2}a^{+}) = aa^{\#}a^{+}a^{2}a^{+} = aa^{+},$$
$$(aa^{\#}xa)(a^{+}x^{\#}a^{+}a^{2}a^{+})(aa^{\#}xa) = aa^{+}aa^{\#}xa = aa^{\#}xa,$$
$$(a^{+}x^{\#}a^{+}a^{2}a^{+})(aa^{\#}xa) = a^{+}x^{\#}a^{+}axa = a^{+}a$$

and

$$(a^{+}x^{\#}a^{+}a^{2}a^{+})(aa^{\#}xa)(a^{+}x^{\#}a^{+}a^{2}a^{+}) = a^{+}a(a^{+}x^{\#}a^{+}a^{2}a^{+}) = a^{+}x^{\#}a^{+}a^{2}a^{+}.$$

Hence $(aa^{\#}xa)^{+} = a^{+}x^{\#}a^{+}a^{2}a^{+}$ for each $x \in \chi_a$.

(2) Since

$$(aa^{\#}xa)(aa^{\#}a^{+}x^{\#}a^{+}a) = aa^{\#}xaa^{+}x^{\#}a^{+}a = aa^{\#}a^{+}a = aa^{\#},$$
$$(aa^{\#}xa)(aa^{\#}a^{+}x^{\#}a^{+}a)(aa^{\#}xa) = aa^{\#}aa^{\#}xa = aa^{\#}xa,$$
$$(aa^{\#}a^{+}x^{\#}a^{+}a)(aa^{\#}xa) = aa^{\#}a^{+}x^{\#}a^{+}axa = aa^{\#}a^{+}a = aa^{\#}$$

and

$$(aa^{\#}a^{+}x^{\#}a^{+}a)(aa^{\#}xa)(aa^{\#}a^{+}x^{\#}a^{+}a) = aa^{\#}aa^{\#}a^{+}x^{\#}a^{+}a = aa^{\#}a^{+}x^{\#}a^{+}a.$$

Thus $(aa^{\#}xa)^{\#} = aa^{\#}a^{+}x^{\#}a^{+}a$ for each $x \in \chi_a$. □

Theorem 8.5. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $a \in R^{Her}$ if and only if $(aa^{\#}xa)^{+} = a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*}$ for some $x \in \chi_a$.
(2) $a \in R^{Her}$ if and only if $(aa^{\#}xa)^{\#} = (a^{\#})^{*}x^{\#}a^{+}a$ for some $x \in \chi_a$.

Proof. (1) $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then we get $(a^{+})^{*} = (a^{*})^{+} = a^{+}$ and

$$a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*} = a^{+}x^{\#}a^{+}a^{2}a^{+}.$$

It follows from Theorem 8.4 that $(aa^{\#}xa)^{+} = a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*}$ for each $x \in \chi_a$.

$'' \Leftarrow ''$ By the hypothesis and Theorem 8.4, we have $a^{+}x^{\#}a^{+}a^{2}a^{+} = a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*}$ for some $x \in \chi_a$.

When $x \in \tau_a$, we have

$$aa^{+} = aa^{\#}a^{+}a^{2}a^{+} = (xaa^{+}x^{\#})a^{+}a^{2}a^{+} = (xa)(a^{+}x^{\#}a^{+}a^{2}a^{+})$$
$$= (xa)a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*} = (xaa^{+}x^{\#})a^{+}a^{2}(a^{+})^{*} = aa^{\#}a^{+}a^{2}(a^{+})^{*} = a(a^{+})^{*}.$$

Hence $a \in R^{Her}$.

When $x \in \gamma_a$, we have

$$a^{+}a^{2}a^{+} = (aa^{\#})^{*}a^{+}a^{2}a^{+} = (xaa^{+}x^{\#})a^{+}a^{2}a^{+} = (xa)(a^{+}x^{\#}a^{+}a^{2}a^{+})$$
$$= (xa)a^{+}x^{\#}a^{+}a^{2}(a^{+})^{*} = (xaa^{+}x^{\#})a^{+}a^{2}(a^{+})^{*} = (aa^{\#})^{*}a^{+}a^{2}(a^{+})^{*} = a^{+}a^{2}(a^{+})^{*}$$

This gives

$$aa^{+} = aa^{\#}a^{+}a^{2}a^{+} = aa^{\#}a^{+}a^{2}(a^{+})^{*} = a(a^{+})^{*}.$$

Hence $a \in R^{Her}$.

(2) $'' \Rightarrow ''$ Assume that $a \in R^{Her}$. Then $aa^{\#}a^{+} = (aa^{\#})^{*}a^{+} = a^{+} = (a^{*})^{+} = (a^{+})^{*} = (a^{\#})^{*}$. It follows from Theorem 8.4 that $(aa^{\#}xa)^{\#} = (a^{\#})^{*}x^{\#}a^{+}a$ for each $x \in \chi_a$.

" $\Leftarrow$ " From the assumption and Theorem 8.4, we get $aa^{\#}a^{+}x^{\#}a^{+}a = (a^{\#})^{*}x^{\#}a^{+}a$ for some $x \in \chi_a$.
When $x \in \tau_a$, we have

$$a^{\#} = aa^{\#}a^{+}aa^{\#} = aa^{\#}a^{+}(x^{\#}a^{+}ax) = (a^{\#})^{*}x^{\#}a^{+}ax = (a^{\#})^{*}aa^{\#}.$$

Hence $a \in R^{Her}$.
When $x \in \gamma_a$, we have

$$\begin{aligned} aa^{\#}a^{+} &= aa^{\#}a^{+}(aa^{\#})^{*} = aa^{\#}a^{+}(x^{\#}a^{+}ax) \\ &= (a^{\#})^{*}x^{\#}a^{+}ax = (a^{\#})^{*}(aa^{\#})^{*} = (a^{\#})^{*}. \end{aligned}$$

This gives

$$a^{\#} = aa^{\#}a^{+}aa^{\#} = (a^{\#})^{*}aa^{\#}.$$

Hence $a \in R^{Her}$. □

By Theorem 5.6, Theorem 8.1 and Theorem 8.4, we have

Theorem 8.6. *Let $a \in R^{\#} \cap R^{+}$. Then the followings are equavilent:*
(1) $a \in R^{Her}$;
(2) $(a^{+})^{*}(aa^{\#})^{*}x^{+} = a^{+}x^{\#}a^{+}a^{2}a^{+}$ for some $x \in \chi_a$;
(3) $(a^{+})^{*}(aa^{\#})^{*}x^{+} = aa^{\#}a^{+}x^{\#}a^{+}a$ for some $x \in \tau_a$;
(4) $(aa^{\#})^{*}(a^{+})^{*}x^{\#} = aa^{\#}a^{+}x^{\#}a^{+}a$ for some $x \in \gamma_a$.

9. Deformed Desouza-Bhattacharyya equation

Observing Eq.(7.1), we can construct the following equation.

$$xa^{*} - aa^{\#}ya = a^{+}. \tag{9.1}$$

Theorem 9.1. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.*(9.1) *is given by*

$$\begin{cases} x = a^{+}(a^{+})^{*} + u - ua^{+}a + a^{\#}awa(a^{+})^{*} \\ y = v - a^{+}avaa^{+} + a^{+}awaa^{+} \end{cases}, \text{where } u, v, w \in R \text{ with } wa = wa^{2}a^{+}. \tag{9.2}$$

Proof. First, we show that the formula (9.2) is exactly the solution of Eq.(9.1). In fact,

$$\begin{aligned} &(a^{+}(a^{+})^{*} + u - ua^{+}a + a^{\#}awa(a^{+})^{*})a^{*} - aa^{\#}(v - a^{+}avaa^{+} + a^{+}awaa^{+})a \\ &= a^{+} + a^{\#}awa^{2}a^{+} - a^{\#}awa = a^{+} + a^{\#}awa - a^{\#}awa = a^{+}. \end{aligned}$$

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(9.1). Then

$$x_0a^{*} - aa^{\#}y_0a = a^{+}.$$

Choose $w = aa^{\#}y_0aa^{+}$, $u = x_0 - a^{+}(a^{+})^{*}$, $v = y_0$. Then

$$\begin{aligned} wa^{2}a^{+} &= aa^{\#}y_0a^{2}a^{+} = (aa^{\#}y_0a)aa^{+} = (x_0a^{*} - a^{+})aa^{+} = x_0a^{*} - a^{+} = aa^{\#}y_0a = wa, \\ ua^{+}a &= (x_0 - a^{+}(a^{+})^{*})a^{+}a = x_0a^{+}a - a^{+}(a^{+})^{*} = x_0a^{*}(a^{+})^{*} - a^{+}(a^{+})^{*} = aa^{\#}y_0a(a^{+})^{*} \\ &= aa^{\#}y_0aa^{+}a(a^{+})^{*} = aa^{\#}(aa^{\#}y_0aa^{+})a(a^{+})^{*} = aa^{\#}wa(a^{+})^{*} \end{aligned}$$

and

$$a^{+}avaa^{+} = a^{+}ay_0aa^{+} = a^{+}a(aa^{\#}y_0a)a^{+} = a^{+}a(wa)a^{+} = a^{+}awaa^{+}.$$

It follows that

$$x_0 = a^{+}(a^{+})^{*} + x_0 - a^{+}(a^{+})^{*} = a^{+}(a^{+})^{*} + u = a^{+}(a^{+})^{*} + u - ua^{+}a + a^{\#}awa(a^{+})^{*}$$

and

$$y_0 = y_0 - a^{+}ay_0aa^{+} + a^{+}ay_0aa^{+} = v - a^{+}avaa^{+} + a^{+}awaa^{+}.$$

Hence the general solution of Eq.(9.1) is given by the formula (9.2). □

Theorem 9.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if the general solution of Eq.*(9.1) *is given by*

$$\begin{cases} x = (a^+)^*(a^+)^* + u - ua^+a + a^{\#}awa(a^+)^* \\ y = v - a^+avaa^+ + a^+awaa^+ \end{cases}, \text{where } u, v, w \in R. \tag{9.3}$$

Proof. " $\Rightarrow$ " Since $a \in R^{Her}$, $a^+ = (a^+)^*$ and $a \in R^{EP}$, it follows that $wa^2a^+ = wa$. Hence the formula (9.3) is the same as the formula (9.2). By Theorem 9.1, we are finished.

" $\Leftarrow$ " From the assumption, we have

$$((a^+)^*(a^+)^* + u - ua^+a + a^{\#}awa(a^+)^*)a^* - aa^{\#}(v - a^+avaa^+ + a^+awaa^+)a = a^+,$$

e.g.,

$$(a^+)^*aa^+ + a^{\#}awa^2a^+ - aa^{\#}wa = a^+, \ for\ all\ w \in R.$$

Especially, choose $w = aa^{\#}$, we obtain

$$(a^+)^*aa^+ + a^2a^+ - a = a^+.$$

Multiplying the equality on the right by aa^+, we get $a = a^2a^+$. Hence $a \in R^{EP}$, it follows that

$$a^+ = (a^+)^*aa^+ + a^2a^+ - a = (a^+)^*aa^+ = (a^+)^*aa^{\#} = (a^+)^*.$$

Thus $a \in R^{Her}$. □

Now, we construct the following equation

$$xa^* - a^{\#}aya^2a^+ = (a^+)^*aa^+. \tag{9.4}$$

Lemma 9.3. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.*(9.4) *is given by the formula* (9.3).

Proof. It is routine. □

Theorem 9.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if Eq.*(9.1) *has the same solution as Eq.*(9.4).

Proof. It follows from Theorem 9.1, Theorem 9.2 and Lemma 9.3. □

Also, we can establish the following equation.

$$xa^*aa^{\#} - a^{\#}aya = (a^+)^*. \tag{9.5}$$

Theorem 9.5. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.*(9.5) *is given by the formula* (9.3).

Proof. First, we show that the formula (9.3) is exactly the solution of Eq.(9.5). In fact,

$$\begin{aligned} &((a^+)^*(a^+)^* + u - ua^+a + a^{\#}awa(a^+)^*)a^*aa^{\#} - a^{\#}a(v - a^+avaa^+ + a^+awaa^+)a \\ &= (a^+)^* + a^{\#}awa - a^{\#}ava + a^{\#}ava - a^{\#}awa = (a^+)^*. \end{aligned}$$

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(9.5). Then

$$x_0a^*aa^{\#} - a^{\#}ay_0a = (a^+)^*.$$

Choose $u = x_0$, $v = y_0$, $w = a^{\#}ay_0aa^+$. Then

$$\begin{aligned} a^{\#}awa(a^+)^* &= a^{\#}a(a^{\#}ay_0aa^+)a(a^+)^* = a^{\#}ay_0a(a^+)^* = (x_0a^*aa^{\#} - (a^+)^*)(a^+)^* \\ &= x_0a^+a - (a^+)^*(a^+)^* = ua^+a - (a^+)^*(a^+)^* \end{aligned}$$

and

$$a^+awaa^+ = a^+aa^\# ay_0aa^+ = a^+ay_0aa^+ = a^+avaa^+.$$

It follows that

$$x_0 = (a^+)^*(a^+)^* + x_0 - ua^+a + (ua^+a - (a^+)^*(a^+)^*)$$
$$= (a^+)^*(a^+)^* + u - ua^+a + a^\# awa(a^+)^*$$

and

$$y_0 = y_0 - a^+avaa^+ + a^+avaa^+ = v - a^+avaa^+ + a^+awaa^+.$$

Hence the general solution of Eq.(9.5) is given by the formula (9.3). □

Theorem 9.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.*(9.1) *has the same solution as Eq.*(9.5).

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