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Stone spaces I

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Abstract. The aim of this paper is to propose a categorical definition of extremally disconnected and Stone objects (spaces) in a topological category and examine the relationship between them. Moreover, we characterize each of T_k, $k = 0, 1, 2$, compact, extremally disconnected, and Stone (transitive) spatial graphs and prebornological spaces as well as investigate some properties of them. Finally, we compare our results.

1. Introduction

Recall from [20], p.70 that a topological space Y is a Stone space if and only if Y is Hausdorff, compact, and totally disconnected (i.e., the only connected subspaces of Y are the empty set and the singletons [20], p.69); totally disconnected spaces are referred to as hereditarily disconnected in [16], p.360. One of the application of Stone spaces in topology and functional analysis is Stone's construction of the compactification of a completely regular space [32] which contains a universal property that has a connection with the notion of adjoint functor in category theory. Another application of Stone spaces in topology and functional analysis is Stone's generalization of the Weierstrass approximation theorem [32].

Stone [33], also, introduced the notion of extremal disconnectedness in order to find out the topological equivalent of the condition of completeness for Boolean algebras. The extremally disconnected spaces have a connection with projective topological spaces which have applications in functional analysis, sheaf theory, topos theory, and logic [18–20, 31–33].

Compact Hausdorff spaces have many useful properties which one can use in proving theorems and making constructions. Since such spaces are defined in terms of clossedness, several authors studied them in a category. Hausdorffness and compactness with respect to a factorization structure were defined in [21, 25] for a general category, with respect to closure operators was done in [14] for abstract categories, and with respect to the notion of closedness was defined in [3, 6] for a topological category [30].

If one wishes to extend a particular concept in general topology to topological categories, it is necessary first to describe and formulate it in terms of initial lifts, final lifts, products, pushouts, copruducts, discreteness, and indiscreteness which are available. Extensions of concepts may have several equivalent descriptions for topological spaces and when interpreted in other topological categories they may give rise to distinct concepts, one of these distinct forms may be more useful than another. There are several generalizations of usual topological T_2-axiom to the topological categories [3, 5], with no reference to points and neighbourhoods since the point (resp. neighbourhood) notion may not be available for non set-based

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topological categories (resp. for topological categories not related to topological spaces) (see Definition 2.1, below). Also, T_2-axiom can be generalized to a topological category by using that the diagonal map embeds as a closed subspace of its product with itself [3, 15]. One way to define the notion of Hausdorff locale would be to say that the diagonal map embeds the locale as a closed sublocale of its product with itself [20].

The notion of closedness is a basic concept of general topology that is used to define, for example, the T_2-axiom, compactness, extremal disconnectedness; this notion was extended in [3] (see Definitions 3.2 and 4.1, below).

In view of this, it will be useful to be able to not only extend these notions to an arbitrary topological category but also to have the characterization of each of them and present important theorems in general topology such as the Tietze Extension Theorem, the Tychonoff Theorem, the Baire Theorem, the Urysohn Lemma among others in certain topological categories of interest. Recently, the presentation of the Urysohn lemma and Tietze extension theorem were studied in [17, 23, 24, 29].

In this paper, we introduce various forms of Stone objects (spaces) in a topological category and investigate the relationship between them. Also,

(i) we characterize Stone (transitive) spatial graphs and Stone prebornological spaces, and investigate some of their properties.

(ii) we characterize each of T_k, $k = 0, 1, 2$, (transitive) spatial graphs as well as examine the relationship among them.

(iii) we show that, in the presence of $PreT_2$ topological spaces, all T_0, T_1, T_2, and sober spaces are equivalent.

(iv) we introduce extremally disconnected objects in a topological category and give the characterization of each of extremally disconnected and compact (transitive) spatial graphs and prebornological spaces.

(v) we compare our results.

2. Premilinaries

Let $\mathcal{E}$ be a category and **Set** be the category of sets.

A functor $U : \mathcal{E} \to$ **Set** is said to be topological or $\mathcal{E}$ is a topological category over **Set** if and only if the following conditions hold:

(1) U is concrete, i.e., faithful (U is mono on hom sets) and amnestic (if $U(f) = id$ and f is an isomorphism, then $f = id$) [28], p.278.

(2) U has small fibers, i.e., $U^{-1}(b)$ is a set for all b in **Set** [28], p.278.

(3) For every U-source, i.e., family $g_i : b \to U(X_i)$ of maps in **Set**, there exists a family $f_i : X \to X_i$ in $\mathcal{E}$ such that $U(f_i) = g_i$ and if $U(h_i : Y \to X_i) = kg_i : UY \to b \to U(X_i)$, then there exists a lift $\bar{k} : Y \to X$ of $k : UY \to UX$, i.e., $U(\bar{k}) = k$. This latter condition means that every U-source has an initial lift [1], p.333 or [30], p. 17. It is well known that the existence of initial lifts of arbitrary U-source is equivalent to the existence of final lifts (the dual of the initial lifts) for arbitrary U-sink [1], p.335.

A topological functor $U : \mathcal{E} \to$ **Set** is said to be normalized if there is only one structure on the empty set and on a point [3], p.334. A non set-based topological functor is said to be normalized if the constant objects, i.e., subterminals have a unique structure [9], p.592.

Recall from [28], p.279 that a topological functor $U : \mathcal{E} \to$ **Set** has a left adjoint $D :$ **Set** $\to \mathcal{E}$, where $D(e)$ is obtained as the final lift of the empty sink on e. An object of the form $e = DUe$ is called a discrete object in $\mathcal{E}$. An object e in $\mathcal{E}$ is discrete if and only if every map $U(e) \to U(c)$ lifts to a map $e \to c$ for each object c in $\mathcal{E}$ [28] or [30], p.28.

Also, a topological functor $U : \mathcal{E} \to$ **Set** has a right adjoint $ID :$ **Set** $\to \mathcal{E}$, where $ID(b)$ is obtained as the initial lift of the empty source on b [1], p.336. An object b in $\mathcal{E}$ is indiscrete of and only if every map $U(c) \to U(b)$ lifts to a map $c \to b$ for each object c in $\mathcal{E}$ [30], p.28.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. M is called a subspace of X if there exists monomorphism $i : M \to X$ that is an initial lift (i.e., an embedding) and we denote it by $M \subset X$.

Let B be a non-empty set and let $B^2 \bigvee_\Delta B^2$ be taking two distinct copies of B^2 identified along Δ. The map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(a,b)_1 = (a,b,b)$ and $S(a,b)_2 = (a,a,b)$ and the map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(a,b)_1 = (a,b,a)$ and $A(a,b)_2 = (a,a,b)$.

The fold map $\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\nabla((a,b)_i) = (a,b)$ for $i = 1,2$ [3], p.337.

Let $X \in Ob(\mathcal{E})$ with $U(X) = B$, where $\mathcal{E}$ is a set based topological category.

Let S_W (resp. A_W) be the initial lift of the U-source S (resp. A) : $B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$ and $(B^2 \bigvee_\Delta B^2)'$ be the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$, where $i_k : B^2 \to B^2 \coprod B^2, k = 1,2$ are the canonical injection maps and $q : B^2 \coprod B^2 \to B^2 \bigvee_\Delta B^2$ is the quotient map.

Definition 2.1. (1) If X does not contain an indiscrete subspace with (at least) two points, then X is said to be a T_0 object [26].

(2) If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(X^3)$ (resp. $S : B^2 \vee_\Delta B^2 \to U(X^3)$) is discrete, then X is said to be a $\overline{T_0}$ (resp. T_1) object [3], p.338.

(3) If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)'$ is discrete, then X is said to be a T_0' object [3], p.338.

(4) If $S_W = (B^2 \bigvee_\Delta B^2)'$ (resp. $S_W = A_W$), then X is said to be a $PreT_2'$ (resp. $Pre\overline{T}_2$) object.

(5) If X is $Pre\overline{T}_2$ (resp. $PreT_2'$) and $\overline{T}_0$, then X is said to be a $\overline{T}_2$ (resp. LT_2) object [3], p.338.

(6) If X is $Pre\overline{T}_2$ and T_0' (resp. T_0), then X is said to be a KT_2 (resp. NT_2) object [5], p.42.

Remark 2.2. (1) In **Top** (the category of topological spaces and continuous functions), by Remark 1.6 of [5], $\overline{T}_0$, T_0', and T_0 (resp. NT_2, $\overline{T}_2$, and KT_2) reduce to T_0 (resp. T_2) axiom. A topological space is $PreT_2' = Pre\overline{T}_2 = PreT_2$ if and only if for any two distinct points, if there is a neighborhood of one missing the other, then the points have disjoint neighborhoods [3], p.338. There is no implication between $Pre\overline{T}_2$ and each of T_0 and T_1. Take the integers set Z with indiscrete and cofinite topologies. In the realm of $PreT_2$ topological spaces, by Theorem 3.5 of [9], all T_0, T_1, and T_2 spaces are equivalent.

(2) In any topological category, by Theorem 3.1 of [7], $PreT_2'$ (resp. LT_2) implies $Pre\overline{T}_2$ (resp. KT_2) and there is no implication between T_0 and each of $Pre\overline{T}_2, \overline{T}_2$, and KT_2.

3. Separation and compactness

Rel denotes the category of directed graphs (relation spaces) where objects are sets with a binary relation and where morphisms $f : (A_1, R) \to (B_1, S)$ are functions with $f(a)Sf(b)$ if aRb for all $a, b \in A_1$ [15], p.56.

The category **RRel** (resp. **Prord**) of spatial graphs (reflexive relation spaces) (resp. reflexive and transitive relation spaces (preordered set)) is the full subcategory of **Rel** [15], page 57.

Note that every filter convergence space and pretopological space induce a graph and spatial graph, respectively [15], page 57. Also, there is a one to one relation between topological space and a transitive spatial graph [15], page 59. There is a fully faithful functor from preordered sets to topological spaces that equips a preordered set with its Alexandroff topology.

Proposition 3.1. *An epimorphism $f : (A_1, R) \longrightarrow (B_1, S)$ is final in* **Rel** *and* **RRel** *(resp.* **Prord***) if and only if for all $s, t \in B_1$, sSt holds in B_1 precisely when there exist $a, b \in A_1$ such that aRb, $f(a) = s$ and $f(b) = t$ (resp. there exist $a_i \in B_1, i = 1, 2, ...n$ with $a = a_1Sa_2Sa_3Sa_4...Sa_n = b$ such that for each $k = 1, 2, ...n - 1$ there are $c_k, c_{k+1} \in A_1$ such that $f(c_k) = a_k$, $f(c_{k+1}) = a_{k+1}$ and c_kRc_{k+1}) [30].*

A source $f_i : (A_1, R) \longrightarrow (B_i, R_i), i \in I$ is initial in **Rel** *(resp.* **RRel** *or* **Prord***) iff, for all $a, b \in A_1$, aRb holds exactly when $f_i(a)R_if_i(b)$ for all $i \in I$ [15], p.57 or [30], p.21.*

The discrete structure (A, R) on A in **Rel** *is given by R = the empty relation $=\emptyset$ and the discrete structure (A, R) on A in* **RRel** *is given by aRb iff $a = b$ for all a, b in A.*

Rel, **RRel** *and* **Prord** *are topological categories [15], p.57.*

The category **PBorn** of prebornological spaces has as objects $(A_1, \mathcal{F})$, where $\mathcal{F}$ is a family of subsets of A_1 that contains all finite subsets of A_1 and is closed under finite unions and as morphisms $f : (A_1, \mathcal{F}) \to (B_1, \mathcal{G})$ are the functions such that $f(C) \in \mathcal{G}$ if $C \in \mathcal{F}$ [27], p. 530. Furthermore, if $\mathcal{F} \neq \emptyset$ and $\mathcal{F}$ is hereditary closed, then $(A_1, \mathcal{F})$ is called a bornological space [27], p.530 or [30], p.21. The category **Born** bornological spaces is the full subcategory of **PBorn** and they are topological categories [27, 30].

Let $\bigvee_x^\infty B$ be taking countably many disjoint copies of B and identifying them at the point $x \in B$. The map $A_x^\infty : \bigvee_x^\infty B \to B^\infty$ (resp. $\nabla_x^\infty : \bigvee_x^\infty B \longrightarrow B$) is given by $A_x^\infty(a_i) = (x, ..., x, a, x, x, ...)$ (resp. $\nabla_x^\infty(a_i) = a$ for all $i \in I$), where a_i is in the i-th component of $\bigvee_x^\infty B$ and B^∞ is the countable product of B, where I is the index set $\{i : a_i$ is in the i-th component of $\bigvee_x^\infty B\}$ [4], p. 386.

Let $X \in Ob(\mathcal{E})$ with $U(X) = B$ and $N \subset X$, where $\mathcal{E}$ is a set based topological category.

Definition 3.2. (1) If the initial lift of the U-source $\nabla_x^\infty : \vee_x^\infty B \to UD(B)$ and $A_x^\infty : \vee_x^\infty B \to U(X^\infty)$ is discrete, then $\{x\}$ is said to be closed [3], p.336.

(2) If $\{*\}$, the image of N, is closed in X/N or $N = \emptyset$, then N is said to be closed, where X/N is the final lift of the epi U-sink $Q : U(X) \to B/N = (B \backslash N) \cup \{*\}$, identifying N with a point * [3], p.336

(3) If N^C, the complement of N, is closed, then N is said to be open [8], p.492.

(4) If the image of each closed (open) subobject of X is a closed (open) subobject of Y, then $f : X \longrightarrow Y$ is said to be closed (open) [6], p.225.

(5) If the projection map $\pi_2 : X \times Z \longrightarrow Z$ is closed for each object Z in $\mathcal{E}$, then X is said to be a compact object [6], p.225.

(6)If the only subsets of X both open and closed are X and $\emptyset$, then X is said to be strongly connected [8], p.493.

(7) If the only strongly connected subspaces of X are singletons and $\emptyset$, then X is said to be strongly hereditarily disconnected [12], p.290.

In **Top**, compactness (resp. strong connectedness, strong hereditary disconnectedness, openness, and closedness) coincides with the usual compactness [6], p.225 (resp. connectedness [8], p.493, hereditary disconnectedness [12], p.290, openness [8], p.493, and closedness [3], p.336).

Recall from [11] that the closure $cl_X(N)$ of N is the intersection of all closed sets in X containing N and the notion of closedness induces a closure operator in some categories [11–13, 17, 22–24, 29].

Theorem 3.3. *(1) $f : (A, R_1) \to (B, R)$ is a morphism of* **Rel***, then f reflects discreteness i.e., if (B, R) is discrete, then so is (A, R_1).*

(2) Every subset of a graph is both open and closed.

(3) Every graph is compact and strongly hereditarily disconnected.

Proof. (1) If (B, R) is discrete i.e., $R = \emptyset$ but (A, R_1) is not discrete i.e., $R_1 \neq \emptyset$, then there exists a, b in A such that aR_1b. Since $f : (A, R_1) \to (B, R)$ is a morphism in **Rel**, it follows that $f(a)Rf(b)$, a contradiction. Hence $R_1 = \emptyset$.

(2) Let $(B, R) \in Ob(\textbf{Rel})$ and $N \subset B$. If $N = \emptyset$, then in view of Definition 3.2, N is closed. If $N = \{x\}$ for some $x \in B$, then let R_1 be the initial structure on $\vee_x^\infty B$ induced by $\nabla_x^\infty : \vee_x^\infty B \to (B, \emptyset)$ and $A_x^\infty : \vee_x^\infty B \to (B^\infty, R^\infty)$, where $\emptyset$ is the discrete relation on B and R^∞ is the product relation on B^∞.

By part (1), we must have $R_1 = \emptyset$ and by Definition 3.2, $\{x\}$ is closed in (B, R).

If N has cardinality at least 2, then $\{*\}$ is closed in B/N and by Definition 3.2, N is closed. Since N^C is closed, by Definition 3.2, N is open.

(3) It follows from part (2) and Definition 3.2. □

Theorem 3.4. *Let $(B, S), (A, R) \in Ob(\textbf{RRel})$.*

(1) (A, R) is compact if and only if for every $x \in A$ there exist $a, b \in A$ with xRa and bRx.

(2) If (A, R) is KT_2, then $N \subset A$ is open if and only if it is closed.

(3) If M is closed subset of a compact KT_2 spatial graph (A, R), then M is compact.

(4) A compact subset of KT_2 spatial graph need not be closed.

(5) Product of compact spatial graphs is compact.

(6) Let $f : (A,R) \longrightarrow (B,S)$ be a morphism in **RRel**.
(i) If (A,R) is compact, then $f(A)$ is compact.
(ii) If (A,R) is compact and (B,S) is KT_2, then f need not be closed.

Proof. (1) Suppose (A,R) is compact with $A \neq \emptyset$. Let $B = \{\infty\} \cup A$ where $\infty \notin A$.

Define a relation $T \subset B \times B$ as for $a, b \in B$ aTb if and only if $a = \infty$ or $b = \infty$ or aRb for $a, b \in A$. Note that (B,T) is a spatial graph. Let $K = \{(x,x) : x \in A\} \subset A \times B$. By Theorem 2.4 of [12], the closure $cl_{A\times B}(K)$ of K is closed and since (A,R) is compact, $\pi_2(cl_{A\times B}(K))$ is a closed set in B. Note that $A \subset \pi_2(cl_{A\times B}(K))$ and $sT\infty$ and ∞Tt for $s,t \in A \subset \pi_2(cl_{A\times B}(K))$. Since $\pi_2(cl_{A\times B}(K))$ is closed, by Theorem 3.8 of [11], $\infty \in \pi_2(cl_{A\times B}(K))$ and there exist $(b,b),(a,a) \in K$ and $x \in A$ with $(x,\infty) \in cl_{A\times B}(K)$, $(x,\infty),((a,a)) \in Z$, and $((b,b),(x,\infty)) \in Z$, where Z is the product structure on $A \times B$. By Proposition 3.1, we must have xRa and bRx.

Suppose for every $x \in A$ there exist $a, b \in A$ with xRa and bRx. We need to show that for each spatial graph (B,T), the projection map $\pi_2 : (A\times B, Z) \longrightarrow (B,T)$ is closed. Suppose $K \subset A\times B$ is closed. If $K = \emptyset$, then $\pi_2(K) = \emptyset$ is closed. Suppose $K \neq \emptyset$ and for $s \in B$ there exist $t,u \in \pi_2(K)$ such that sTt and uTs. $t,u \in \pi_2(K)$ implies there exist $a,b \in A$ with $(a,t),(b,u) \in K$. By assumption, we have xRa and bRx for every $x \in A$. By Proposition 3.1, $((x,s),(a,t)) \in Z$ and $((b,u),(x,s)) \in Z$. By Theorem 3.8 of [11], $(x,s) \in K$ since K is closed and hence, $s \in \pi_2(K)$. By Theorem 3.8 of [11], $\pi_2(K)$ is closed and hence, (A,R) is compact.

(2) Suppose (A,R) is KT_2, N is closed, and for each $x \in A$ there exist $c,d \in N^C$ such that xRc and dRx. If xRc, then cRx because (A,R) is KT_2. Since xRc, cRx and N is closed, by Theorem 3.8 of [11], $x \notin N$ i.e., $x \in N^C$. Similarly, if dRx, then $x \in N^C$ and by Theorem 2.3 of [12], N is open.

Conversely, if N is open and for each $x \in A$ there exist $c,d \in N$ with xRc and dRx, then cRx and xRd since (A,R) is KT_2. By Theorem 2.3 of [12], $x \notin N^C$ since N is open. Hence, by Theorem 3.8 of [11], N is closed.

(3) Suppose that M is closed subset of a compact KT_2 spatial graph (A,R) and $x \in M$. Since (A,R) is compact and $x \in A$, by Part (1), there exist $a,b \in A$ with xRa and bRx. Since (A,R) is KT_2, by Theorem 2.4 of [12], aRx, xRb and since M is closed, $a,b \in M$. Let (M,R_1) be the subspace of (A,R). By Proposition 3.1, $xR_1a = xRa$ and $bR_1x = bRx$ and by Part (1), (M,R_1) is compact.

(4) The indiscrete spatial graph (Z,Z^2) is KT_2 and by Part (1), the subset $\{1,2,3,4,5\}$ of Z is compact but by Theorem 3.8 of [11], $\{1,2,3,4,5\}$ is not closed.

(5) Let (A_i,R_i) be compact spatial graphs, $x = (x_1,x_2,...) \in A$, and $(A = \prod_{i\in I} A_i, R)$ be the product space. Since (A_i,R_i) is compact and $x_i \in A_i$ for each $i \in I$, by Part (1), there exist $a_i, b_i \in A_i$ such that $x_iR_ia_i$ and $b_iR_ix_i$. Note that $a = (a_1,a_2,...), b = (b_1,b_2,...) \in A$ and by Proposition 3.1, xRa and bRx. Hence, by Part (1), (A,R) is compact.

(6) Let $f : (A,R) \longrightarrow (B,S)$ be a morphism in **RRel**.

(i) If (A,R) is compact and $y \in f(A)$, then by Part (1), there exist $a,b \in A$ with xRa, bRx and $y = f(x)$. f is a relation preserving mapping implies $f(x)Sf(a)$ and $f(b)Sf(x)$. Hence, by Part (1), $f(A)$ is compact.

(ii) Let $(A = \{r,p\}, R)$ with $R = \{(r,r),(p,p),(r,p)\}$ and $(A = \{r,p\}, A^2)$ be the indiscrete spatial graph. The identity function $id : A \longrightarrow A$ is a relation preserving mapping and by Part (1), (A,R) is compact, (A,A^2) is KT_2, and $\{r\}$ is closed in (A,R) but $id(\{r\}) = \{r\}$ is not closed in (A,A^2). □

Theorem 3.5. *Let $(B,R) \in Ob($***Prord***$)$.*
(1) (B,R) is compact if and only if for every $x \in B$ there exist $a,b \in B$ with xRa and bRx.
(2) The following are equivalent:
(a) (B,R) is T_0'.
(b) R is anti-symmetric.
(c) (B,R) is $\underline{T_0}$.
(d) (B,R) is $\overline{T_0}$.
(3) The following are equivalent:
(a) (B,R) is NT_2.
(b) (B,R) is $\underline{LT_2}$.
(c) (B,R) is $\overline{T}_2$.
(d) (B,R) is KT_2.

Proof. The proof of part (1) is similar to the proof of part (1) of Theorem 3.4.

(2) $(a) \Rightarrow (b)$: Suppose (B,R) is T_0' and for any $x, y \in B$, xRy and yRx. We show that $x = y$. Assume $x \neq y$. Let $s = (x,y)_1$ and $t = (x,y)_2$. Note that $\nabla(s) = (x,y) = \nabla(t)$, $i_1((x,y)) = s, i_2((x,y)) = t, (x,y)R^2(y,y)$ and $(y,y)R^2(x,y)$, where R^2 is the product relation on B^2. Since (B,R) is T_0', by Definition 2.1 and Proposition 3.1, $s = t$. Thus, $x = y$, i.e., R is anti-symmetric.

The equivalence of parts (b) and (c) are given in Theorem 3.8 of [11].

$(b) \Rightarrow (d)$: Suppose that R is anti-symmetric. Let R^3 be the product relation on B^3 and R_1 be the initial structure on the wedge $B^2 \bigvee_\Delta B^2$ induced by

$$A : B^2 \vee_\Delta B^2 \to (B^3, R^3) \quad \text{and} \quad \nabla : B^2 \vee_\Delta B^2 \to (B^2, N)$$

where N is discrete relation on B^2. If sR_1t for any points s and t of the wedge $B^2 \bigvee_\Delta B^2$, then $\pi_1 A(s)R\pi_1 A(t)$, $\pi_2 A(s)R\pi_2 A(t)$, $\pi_3 A(s)R\pi_3 A(t)$, and $\nabla(s) = \nabla(t)$. Since $\nabla(s) = \nabla(t)$, s and t have the form: $(x,y)_i$ and $(x,y)_j$ for some $x, y \in B$ and $i, j \in \{1,2\}$.

If $s = (x,y)_1$ (resp. $(x,y)_2$) and $t = (x,y)_2$ (resp. $(x,y)_1$), then

$$\pi_1 A(s)R\pi_1 A(t) = xRx,$$

$$\pi_2 A(s)R\pi_2 A(t) = yRx \quad (\text{resp.} xRy),$$

$$\pi_3 A(s)R\pi_3 A(t) = xRy \quad (\text{resp.} \quad yRx),$$

and

$$\nabla(s) = (x,y) = \nabla(t).$$

Since R is anti-symmetric, $x = y$ and thus $s = t$.

If $s = (x,y)_1$ (resp. $(x,y)_2$) and $t = (x,y)_1$ ((resp.$(x,y)_2$)), then $s = t$. Hence, by Definition 2.1 and Proposition 3.1, (B,R) is $\overline{T_0}$.

$(d) \Rightarrow (a)$: Suppose (B,R) is $\overline{T_0}$. Let $(B^2 \bigvee_\Delta B^2)'$ be the final lift of

$$q \circ i_1, q \circ i_2 : (B^2, R^2) \to B^2 \vee_\Delta B^2,$$

where R_1 is the initial structure on the wedge $B^2 \bigvee_\Delta B^2$ induced by

$$id : B^2 \vee_\Delta B^2 \to (B^2 \vee_\Delta B^2)' \quad \text{and} \quad \nabla : B^2 \vee_\Delta B^2 \to (B^2, N)$$

where N is discrete relation on B^2. If sR_1t for $s, t \in B^2 \bigvee_\Delta B^2$, then, in particular, $\nabla(s) = (a,b) = \nabla(t)$ for some $a, b \in B$. If $a = b$, then $s = t$.

If $s = (a,b)_1$ (resp. $(a,b)_2$) and $t = (a,b)_2$ (resp. $(a,b)_1$) with $a \neq b$, then by Proposition 3.1, there exists $(d,d) \in B^2$ such that $(a,b)R^2(d,d)$ and $(d,d)R^2(a,b)$ with $i_k((a,b)) = (a,b)_k = s, i_n((a,b)) = (a,b)_n = t$ for $k, n = 1,2$ and $k \neq n$. By Proposition 3.1, we have aRd, bRd, dRa, and dRb. Since R is transitive, aRb and aRb. Note that

$$\pi_1 A(s)R\pi_1 A(t) = aRa,$$

$$\pi_2 A(s)R\pi_2 A(t) = bRa \quad (\text{resp.} \quad aRb),$$

$$\pi_3 A(s)R\pi_3 A(t) = aRb \quad (\text{resp.} \quad bRa),$$

and

$$\nabla(s) = (a,b) = \nabla(t)$$

and thus $s = t$ since (B,R) is $\overline{T_0}$.

If $s = (a,b)_k$ and $t = (a,b)_k$ for $k = 1,2$, then $s = t$. Thus, R_1 is discrete and by 2.1, (B,R) is T_0'.

(3) By Theorems 6.3 and 6.4 of [10], (B,R) is $Pre\overline{T}_2$ if and only if (B,R) is $PreT_2'$ and hence, the result follows from Part (2). □

4. Stone spaces

We define extremally disconnected and various forms of Stone objects in a topological category. Moreover, we characterize Stone graphs, Stone (transitive) spatial graphs, Stone prebornological spaces, and investigate some of their properties.

Let $X \in Ob(\mathcal{E})$ and $cl_X(N)$ be the closure of N [11], where $N \subset X$ and $U : \mathcal{E} \to \mathbf{Set}$ is topological.

Next, we introduce Stone $SHNT_2$ ($SHKT_2$, $SHLT_2$ or $SH\overline{T}_2$) objects in a topological category.

Definition 4.1. (1) If the closure of every open subspace of X is open, then X is said to be an extremally disconnected object.

(2) If X is KT_2, compact, and strongly hereditarily disconnected, then X is called a Stone $SHKT_2$ object.

(3) If X is NT_2, compact, and strongly hereditarily disconnected, then X is called a Stone $SHNT_2$ object.

(4) If X is LT_2, compact, and strongly hereditarily disconnected, then X is called a Stone $SHLT_2$ object.

(5) If X is $\overline{T}_2$, compact, and strongly hereditarily disconnected, then X is called a Stone $SH\overline{T}_2$ object.

Theorem 4.2. *(1) Every Stone $SHLT_2$ (resp. $SH\overline{T}_2$) object is a Stone $SHKT_2$ object.*

(2) In the realm of $PreT_2'$ objects, the following are equivalent:

(i) A Stone $SHLT_2$ object.

(ii) A Stone $SH\overline{T}_2$ object.

(iii) A Stone $SHKT_2$ object.

Proof. Let $Y \in Ob(\mathcal{E})$, where $U : \mathcal{E} \to \mathbf{Set}$ is topological.

(1) If Y is a Stone $SHLT_2$ object, then, in particular, Y is LT_2 and by Definition 2.1, Y is $PreT_2'$ and $\overline{T}_0$. By Theorem 2.4 of [5] and Theorem 3.1 of [7], Y is T_0' and $Pre\overline{T_2}$, respectively. Therefore, by Definition 2.1, Y is KT_2 and hence, Y is a Stone $SHKT_2$ object.

If Y is a Stone $SH\overline{T}_2$ object, then Y is $\overline{T}_2$, i.e., Y is $Pre\overline{T}_2$ and $\overline{T}_0$. Since Y is $\overline{T_0}$, by Theorem 2.4 of [5], Y is T_0'. Therefore, by Definition 2.1, Y is KT_2 and hence, Y is Stone $SHKT_2$ object.

(2) $(i) \Rightarrow (ii)$.If Y is a ELT_2 object, then by Definition 4.1, Y is ET_2, i.e., Y is $PreT_2'$ and $\overline{T}_0$. By Theorem 3.1 of [7], Y is $Pre\overline{T_2}$ and by Definition 2.1, Y is $\overline{T}_2$ and hence, Y is $E\overline{T}_2$.

By Part (1), we get $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ If Y is a EKT_2 object, then, in particular, Y is $Pre\overline{T_2}$ and by Definition 2.1, the initial lift of the U-sources $S : B^2 \vee_\Delta B^2 \to U(Y^3)$ and $A : B^2 \vee_\Delta B^2 \to U(Y^3)$, where $U(Y) = B$, are same, i.e., $S_W = A_W$. By the assumption that Y is $PreT_2'$, we get $(B^2 \bigvee_\Delta B^2)' = S_W$, where $(B^2 \bigvee_\Delta B^2)'$ is the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ (A, S, q, and $i_k, k = 1,2$ are defined in Section 2). Consequently, $(B^2 \bigvee_\Delta B^2)' = A_W$. Since Y is T_0' (because Y is KT_2), then the initial lift of $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)'$ is discrete. Thus, the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(Y^3)$ is discrete, i.e., Y is $\overline{T}_0$ and by Definition 2.1, Y is LT_2. Hence, Y is ELT_2. □

Theorem 4.3. *Let $Y \in Ob(\mathbf{PreT_2}(\mathbf{Top}))$. The following are equivalent:*

(i) Y is T_1.

(ii) Y is sober.

(iii) Y is T_0.

(iv) Y is T_2.

Proof. $(i) \Rightarrow (ii)$ Assume a pre-Hausdorff topological space Y is T_1 and N is a nonempty closed subset of Y. If $N = \{x\}$, $x \in Y$, then $cl_Y(\{x\}) = \overline{\{x\}} = \{x\} = N$ since Y is T_1.

Suppose N has the cardinality of at least 2. There are $x, y \in Y$ with $x \neq y$. Since Y is T_1, there is a neighborhood of x and y each one missing the other. Since Y is $PreT_2$, the points x and y have disjoint neighborhoods W and V, respectively. Note that $W^C \cap N$ and $V^C \cap N$ are proper closed subsets of N and $N = (W^C \cap N) \cup (V^C \cap N)$, i.e., N is reducible. Thus, nonempty irreducible closed subsets of Y has to be one point sets and Y is sober.

(*ii*) ⇒ (*iii*) and (*iv*) ⇒ (*i*) are trivial.

(*iii*) ⇒ (*iv*). Assume a pre-Hausdorff topological space Y is T_0 and $x, y \in Y$ with $x \neq y$. Since Y is T_0, there is a neighborhood of one missing the other and Y is $PreT_2$ implies the points x and y have disjoint neighborhoods. Hence, Y is T_2. □

Remark 4.4. In **Top**, by Remark 2.2 and Definition 4.1, the notion of extremal disconnectedness coincide with the usual extremal disconnectedness [2], p. 325 or [20], p. 102 and all Stone $SHKT_2$, $SHLT_2$, $SH\overline{T}_2$, and $SHNT_2$ objects reduce to the usual Stone spaces [20], p.70. There is no implication between $Pre\overline{T}_2$ and each of T_0, T_1 and sobriety. Take the Sierpinski space and integers set Z with indiscrete and cofinite topologies. In the realm of $PreT_2$ topological spaces, by Theorem 4.3, all T_0, T_1, T_2, and sober spaces are equivalent.

Theorem 4.5. *Let* $(B, R) \in Ob(\mathbf{Rel})$.

(1) Every graph is strongly hereditarily disconnected and extremally disconnected.

(2) (B, R) *is a Stone* $SHLT_2$ *graph if and only if for every* $z, x, y, w \in B$ *if* xRz, yRz, *and* yRw, *then* $x = y$ *or* $z = w$.

(3) (B, R) *is a Stone* $SHNT_2$ *graph if and only if the following two conditions hold:*

(i) For each $x, y \in B$, *if there exists* $z \in B$ *with* xRz *and* yRz, *then for any* $w \in B$, xRw *if and only if* yRw.

(ii) R *is anti-symmetric or either* $(y, y) \notin R$ *or* $(x, x) \notin R$ *for each* $x, y \in B$.

(4) The following are equivalent:

(i) (B, R) *is a Stone* $SH\overline{T}_2$ *graph.*

(ii) (B, R) *is* $SHKT_2$ *graph.*

(iii) For each $x, y \in B$, *if there exists* $z \in B$ *with* xRz *and* yRz, *then for any* $w \in B$, xRw *if and only if* yRw.

Proof. The proof of part (1) follows from Theorem 3.3 and Definitions 3.2 and 4.1.

(2) Assume (B, R) is a Stone $SHLT_2$ graph and xRz, yRz, and yRw for each $z, x, y, w \in B$.

Note that

$$\pi_1 S(x, y)_1 R \pi_1 S(z, w)_2 = xRz,$$

$$\pi_2 S(x, y)_1 R \pi_2 S(z, w)_2 = yRz,$$

$$\pi_3 S(x, y)_1 R \pi_3 S(z, w)_2 = yRw.$$

Since (B, R) is LT_2, it is $PreT_2'$ and by Proposition 3.1, there exists a pair $(a_1, a_2), (b_1, b_2)$ in B^2 such that $(a_1, a_2)R^2(b_1, b_2)$ and $q \circ i_k(a_1, a_2) = (x, y)_1, q \circ i_k(b_1, b_2) = (z, w)_2$ for $k = 1, 2$. Hence, $(x, y)_1$ and $(z, w)_2$ have to lie in the same component of the wedge and we have to have $x = y$ or $z = w$.

Let $s, t \in B^2 \bigvee_\Delta B^2$. In view of Theorem 3.3 and Part (1), (B, R) is compact and extremally disconnected. The proof that (B, R) is $\overline{T}_0$ is similar to the proof of Part (1) of Theorem 3.3.

We show (B, R) is $PreT_2'$, i.e., (I) and (II) are equivalent, where

(I): there exists a pair $(a_1, a_2), (b_1, b_2)$ in B^2 such that $(a_1, a_2)R^2(b_1, b_2)$, $q \circ i_k(b_1, b_2) = t$, and $q \circ i_k(a_1, a_2) = s$ for $k = 1, 2$

(II): $\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t)$, and $\pi_3 S(s) R \pi_3 S(t)$.

We show (I) implies (II). If $s = (x, y)_k$ and $t = (z, w)_k$ for $k = 1, 2$, then $\pi_1 S(s) R \pi_1 S(t) = xRz$ and $\pi_2 S(s) R \pi_2 S(t) = yRw = \pi_3 S(s) R \pi_3 S(t) = \pi_2 S(s) R \pi_2 S(t)$ and by Proposition 3.1, $(x, y)R^2(z, w)$, $q \circ i_k(z, w) = t$, and $q \circ i_k(x, y) = s$ for $k = 1, 2$.

If $t = (z, w)_2$ and $s = (x, y)_1$, then $\pi_1 S(s) R \pi_1 S(t) = xRz, \pi_2 S(s) R \pi_2 S(t) = yRz$ and $\pi_3 S(s) R \pi_3 S(t) = yRw$. By assumption, we get $y = x$ or $z = w$.

If $y = x$, then $(x, x)R^2(z, w)$ and $q \circ i_2(x, y) = s, q \circ i_2(z, w) = t$. If $z = w$, then $(x, y)R^2(z, z)$, $q \circ i_1(z, w) = t$, and $q \circ i_1(x, y) = s$.

If $t = (z, w)_1$ and $s = (x, y)_2$, then $\pi_1 S(s) R \pi_1 S(t) = xRz, \pi_2 S(s) R \pi_2 S(t) = xRw$ and $\pi_3 S(s) R \pi_3 S(t) = yRw$. By assumption, we get the result.

Next, we show (I) implies (II). If $(x, y)R^2(z, w)$, $q \circ i_k(z, w) = (z, w)_k = t$, and $q \circ i_k(x, y) = (x, y)_k = s$ for $k = 1, 2$, then by Proposition 3.1, xRz and yRw. If $k = 1$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz, \pi_2 S(s) R \pi_2 S(t) = yRw = \pi_3 S(s) R \pi_3 S(t).$$

If $k = 2$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz \pi_2 S(s) R \pi_2 S(t), \pi_3 S(s) R \pi_3 S(t) = yRw.$$

Thus, (B, R) is $PreT_2'$ and it is LT_2. Hence, (B, R) is a Stone $SHLT_2$ graph.

(3) Assume (B, R) is a Stone $SHNT_2$ graph and for each $x, y, w \in B$ there exists $z \in B$ such that xRz and yRz.

Note that

$$\pi_1 S(x, y)_1 R \pi_1 S(z, w)_2 = xRz = \pi_1 A(x, y)_1 R \pi_1 A(z, w)_2,$$

$$\pi_2 S(x, y)_1 R \pi_2 S(z, w)_2 = yRz = \pi_2 A(x, y)_1 R \pi_2 A(z, w)_2,$$

$$\pi_3 A(x, y)_1 R \pi_3 A(z, w)_2 = xRw, \pi_3 S(x, y)_1 R \pi_3 S(z, w)_2 = yRw.$$

Since xRz, yRz, and (B, R) is NT_2, in particular, it is $pre\overline{T}_2$, by Proposition 3.1, we must have xRw if and only if yRw.

Suppose there exist $d, c \in B$ with $d \neq c$, $(d, d), (c, d), (c, c) \in R$ and $(d, c) \in R$. Let $D = \{d, c\}$ and R_D be the subrelation on D. By Proposition 3.2, $R_D = D^2$, the indiscrete relation on D^2, a contradiction since (B, R) is NT_2, in particular, it is (B, R) is T_0. Hence, R must be anti-symmetric or for each $y, x \in B$ either $(y, y) \notin R$ or $(x, x) \notin R$.

Assume the conditions hold and $s, t \in B^2 \bigvee_\Delta B^2$. We show (B, R) is HNT_2. In view of Theorem 3.3, (B, R) is compact and extremally disconnected. If R is anti-symmetric or for each $y, x \in B$ either $(y, y) \notin R$ or $(x, x) \notin R$, then by Definition 2.1, (B, R) is T_0.

Let R_A (resp. R_S) be the initial structure on $B^2 \bigvee_\Delta B^2$ induced by $\pi_i A : B^2 \vee_\Delta B^2 \to (B, R)$ (resp. $\pi_i S : B^2 \vee_\Delta B^2 \to (B, R)$), where π_i are the projections maps for $i = 1, 2, 3$. We show (B, R) is $Pre\overline{T}_2$, i.e., $R_A = R_S$.

Let $(s, t) \in R_S$, where $s, t \in B^2 \bigvee_\Delta B^2$. If $s = (x, y)_k$ and $t = (z, w)_k$ for $k = 1, 2$, then by Proposition 3.1,

$$\pi_1 S(s) R \pi_1 S(t) = xRz = \pi_3 A(s) R \pi_3 A(t) = \pi_1 A(s) R \pi_1 A(t),$$

$$\pi_2 S(s) R \pi_2 S(t) = yRw = \pi_3 S(s) R \pi_3 S(t) = \pi_2 A(s) R \pi_2 A(t).$$

Thus, $(s, t) \in R_A$.

If $t = (z, w)_2$ and $s = (x, y)_1$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz = \pi_1 A(s) R \pi_1 A(t)$$

$$\pi_2 S(s) R \pi_2 S(t) = yRz = \pi_2 A(s) R \pi_2 A(t).$$

By assumption, we get $\pi_3 A(s) R \pi_3 A(t) = xRw$ if and only f $\pi_3 S(s) R \pi_3 S(t) = yRw$ and thus, $(s, t) \in R_A$.

If $t = (z, z)$ and $s = (x, y)_1$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz = \pi_1 A(s) R \pi_1 A(t),$$

$$\pi_2 S(s) R \pi_2 S(t) = yRz = \pi_2 A(s) R \pi_2 A(t),$$

$$\pi_3 A(s) R \pi_3 A(t) = xRz,$$

and $\pi_3 S(s) R \pi_3 S(t) = yRz$. Clearly, $\pi_3 A(s) R \pi_3 A(t) = xRz$ iff $\pi_3 S(s) R \pi_3 S(t) = yRz$ and so, $(s, t) \in R_A$.
If $s = (x, x)$ and $t = (z, z)$, then the result is trivial.

If $t = (z, w)_1$ and $s = (x, y)_2$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz = \pi_1 A(s) R \pi_1 A(t),$$

$$\pi_2 S(s) R \pi_2 S(t) = yRz = \pi_2 A(s) R \pi_2 A(t).$$

By assumption, we obtain $\pi_3 A(s) R \pi_3 A(t) = xRw$ iff $\pi_3 S(s) R \pi_3 S(t) = yRw$ and thus, $(s, t) \in R_A$.
Therefore, $R_S \subset R_A$ and the case $R_A \subset R_S$ can be shown similarly.By Definition 2.1, (B, R) is $Pre\overline{T}_2$ and it is NT_2. Hence, by Definition 4.1, (B, R) is a Stone $SHNT_2$ graph.

(4) By Part (1) of Theorem 4.2, we get $(i) \Rightarrow (ii)$.

Note that (B,R) is $\overline{T}_0$ if and only if it is T_0' (see the proof of Theorem 3.3 (1)) and by Definition 2.1, we have $(ii) \Rightarrow (i)$.

By Part (2) and Theorem 3.3, (ii) and (iii) are equivalent. □

Theorem 4.6. *Let $(A,R) \in Ob(\mathbf{RRel})$.*

(1) If (A,R) is KT_2 (resp. $\overline{T}_2$, LT_2 or NT_2), then it is extremally disconnected.

(2) (A,R) is a Stone $SHKT_2$ space if and only if R is symmetric, transitive, and for every $x \in A$ there exist $a, b \in A$ with xRa and bRx.

(3) The following are equivalent.

(i) (A,R) is Stone $SHNT_2$.

(ii) (A,R) is Stone $SH\overline{T}_2$.

(iii) (A,R) is Stone $SHLT_2$.

Proof. (1) If (A,R) is KT_2 and N open in A, then by Theorem 2.4 of [12], the closure $cl_A(N)$ of N is closed and by Theorem 3.4, $cl_A(N)$ is open. Hence, (A,R) is extremally disconnected.

If (A,R) is LT_2 (resp. $\overline{T}_2$ or NT_2) and N open in A, then by Theorem 3.3 of [12], the closure $cl_A(N)$ of N is both closed and open. Hence, (A,R) is extremally disconnected.

(2) Assume (A,R) is a Stone $SHKT_2$ space. Since (A,R) is KT_2, by Theorem 3.2 of [12], R is symmetric and transitive. By Theorem 3.4, for every $x \in A$ there exist $a, b \in A$ with xRa and bRx.

Suppose the conditions hold. We show (A,R) is Stone $SHKT_2$. Since R is symmetric and transitive, in view of Theorem 3.2 of [12], (A,R) is KT_2. By Theorem 3.4, (A,R) is strongly hereditarily disconnected. Since for every $x \in A$ there exist $a, b \in A$ with xRa and bRx, in view of Theorem 3.4, (A,R) is compact and hence, (A,R) is a Stone $SHKT_2$ space.

Part (3) follows from Definition 4.1, Theorem 3.4, and Theorem 3.2 of [12]. □

Theorem 4.7. *Let $(A,R) \in Ob(\mathbf{Prord})$.*

(1) If (A,R) is KT_2 (resp. $\overline{T}_2$, LT_2 or NT_2), then it is strongly hereditarily disconnected and extremally disconnected.

(2) The following are equivalent:

(i) (A,R) is Stone $SHNT_2$.

(ii) (A,R) is Stone $SH\overline{T}_2$.

(iii (A,R) is Stone $SHLT_2$.

(iv) (A,R) is Stone $SHKT_2$.

Proof. (1) If (A,R) is KT_2 (resp. $\overline{T}_2$, LT_2 or NT_2) and N open in A, then by Theorem 3.6 of [10] and Theorem 3.5, N is both open and closed. Thus, (A,R) is strongly hereditarily disconnected and extremally disconnected.

(2) follows from Part (1) and Theorem 3.5. □

Theorem 4.8. *Let* **PBorn** *be the category of prebornological spaces and bounded functions [27].*

(1) Every prebornological space is compact, connected, and extremally disconnected.

(2) A prebornological space $(A_1, \mathcal{F})$ is strongly hereditarily disconnected iff $(A_1, \mathcal{F})$ is a trivial prebornological space, i.e, A has the cardinality of at most 1.

(3) The following are equivalent:

(i) $(A_1, \mathcal{F})$ is Stone $SHKT_2$.

(ii) $(A_1, \mathcal{F})$ is Stone $SHLT_2$.

(iii) $(A_1, \mathcal{F})$ is Stone $SH\overline{T}_2$.

(iv) $(A_1, \mathcal{F})$ is Stone $SHNT_2$.

(4) The following are equivalent:

(i) $(A_1, \mathcal{F})$ is KT_2.

(ii) $(A_1, \mathcal{F})$ is LT_2.

(iii) $(A_1, \mathcal{F})$ is $\overline{T}_2$.

(iv) $(A_1, \mathcal{F})$ is a bornological space.

Proof. (1) By Theorem 3.10 of [4], closed subsets of a prebornological space are only itself and $\emptyset$. Hence, by Definitions 3.2 and 4.1, we get the result.

(2) and (3) follow from part (1) and Definition 3.2.

(4) By Theorem 2.6 of [5], $(A_1, \mathcal{F})$ is $Pre\overline{T_2}$ iff it is $PreT_2'$ if and only if it is a bornological space. The result follows from Theorem 3.6 of [11]. □

Let $T\mathcal{E}$ be the full subcategory of $\mathcal{E}$ consisting of T objects, where $\mathcal{E}$ is a topological category.

Theorem 4.9. *(1)* $\mathbf{SHKT_2Rel} = \mathbf{SH\overline{T}_2Rel} = \mathbf{Pre\overline{T}_2Rel}$ *and they are topological categories.*

(2) $\mathbf{SHLT_2Rel} \subset \mathbf{SHKT_2Rel} = \mathbf{SH\overline{T}_2Rel}$.

(3) $\mathbf{KT_2RRel} = \mathbf{Pre\overline{T}_2RRel} = \mathbf{PreT_2'Prord} = \mathbf{Pre\overline{T}_2Prord}$ *and they are topological categories.*

(4) $\mathbf{SHLT_2RRel} = \mathbf{SHNT_2RRel} = \mathbf{SH\overline{T}_2RRel} \subset \mathbf{SHKT_2RRel}$.

(5) $\mathbf{SH\overline{T}_2PBorn} = \mathbf{SHLT_2PBorn} = \mathbf{SHKT_2PBorn} = \mathbf{SHNT_2PBorn}$.

(6) $\mathbf{SHLT_2Prord} = \mathbf{SH\overline{T}_2Prord} = \mathbf{SHNT_2Prord} = \mathbf{SHKT_2Prord}$.

Proof. The first parts of (1)-(6) follow from Theorems 4.5- 4.8 and the second parts of (1) and (3) follow from Theorem 3.4 of [9] since $\mathbf{Pre\overline{T}_2Rel}$ and $\mathbf{Pre\overline{T}_2RRel}$ are topological categories. □

Remark 4.10. (1) In **Rel**, by Theorem 4.5,

$T_0 \Rightarrow T_0' = \overline{T}_0 = T_1$ and $PreT_2' = LT_2 \Rightarrow NT_2 \Rightarrow KT_2 = \overline{T}_2 = Pre\overline{T}_2 \Rightarrow T_0' = \overline{T}_0 = T_1$.

Let $A = \{p, r, s\}$, $R_1 = \{(r,r), (r,p), (p,p), (p,r)\}$, $R_2 = \{(p,r), (r,r), (p,p)\}$, and $R_3 = \{(p,p), (r,s), (r,p), (p,s)\}$. By Theorem 4.5, (A, R_1) is T_0' but it is not T_0, the indiscrete relation space (Z, Z^2) is KT_2 but it is neither NT_2 nor LT_2 nor T_0, where Z is integers. By Theorem 4.5, (A, R_3) is NT_2 but it is not LT_2 and (A, R_2) is T_0 but it is not KT_2. There is no implication between T_0 and $KT_2 = Pre\overline{T}_2 = \overline{T}_2$.

By Theorem 4.5, $SHLT_2 \Rightarrow SHNT_2 \Rightarrow SHKT_2 = E\overline{T}_2$ and (Z, Z^2) is Stone $SHKT_2$ but it is neither Stone $SHNT_2$ nor $SHLT_2$ and (A, R_3) is Stone $SHNT_2$ but it is not Stone $SHLT_2$.

In the presence of $PreT_2'$ graphs, by Theorem 4.5, all Stone $SHKT_2, SHLT_2, SH\overline{T}_2$, and ENT_2 graphs are equivalent.

By Theorem 4.9, the subcategories $\mathbf{SHKT_2Rel}$ and $\mathbf{SH\overline{T}_2Rel}$ have all limits and colimits.

By Theorem 4.5, every discrete and indiscrete graph is strongly hereditarily disconnected and extremally disconnected.

(2) In **RRel**, by Theorem 4.6, $SHLT_2 = SHNT_2 = SH\overline{T}_2 \Rightarrow SHKT_2$ and (Z, Z^2) is Stone $SHKT_2$ but it is not Stoe $SHNT_2 = SHLT_2$,
and in the presence of $PreT_2'$ spatial graphs, all Stone $SHKT_2, SHLT_2, SH\overline{T}_2$, and $SHNT_2$ spatial graphs are equivalent.

If a spatial graph (A, R) is KT_2 (resp. $\overline{T}_2$, LT_2 or NT_2), then by Theorem 4.6, it is strongly hereditarily disconnected and extremally disconnected. (Z, Z^2) is KT_2 but it is not $NT_2 = \overline{T}_2 = LT_2$. Let $A = \{r, p\}$ with $R = \{(p,p), (r,r), (p,r)\}$. By Theorem 2.3 of [12], all subsets of A are open and closed. Hence, (A, R) is strongly hereditarily disconnected and extremally disconnected but by Theorem 3.2 of [12], it is not KT_2. By Theorem 2.3 of [12], every discrete and indiscrete spatial graph is extremally disconnected.

By Theorem 3.4, we have:

(i) A compact subset of a KT_2 spatial graph need not be closed.

(ii) Tychonoff Theorem holds in **RRel**.

(iii) Any morphism in **RRel** whose domain is compact and whose codomain is KT_2 need not be closed.

(iv) If (A, R) is KT_2, then (A, R) has a partition consisting of open subsets.

By Theorem 3.2 of [12], the following are equivalent:

(i) (A, R) is $Pre\overline{T}_2$.

(ii) (A, R) is KT_2.

(iii) R is an equivalence relation on B.

(3) In **PBorn**, by Theorem 4.8, all Stone $SHKT_2, SHLT_2, SH\overline{T}_2$, and $SHNT_2$ prebornological spaces are equivalent. By Theorem 4.9, the subcategories $\mathbf{\overline{T}_2PBorn}, \mathbf{LT_2PBorn}$, and $\mathbf{KT_2PBorn}$ are topological categories. Hence, they have all limits and colimits.

By Theorem 4.8, each of discrete and indiscrete prebornological space is extremally disconnected.

(4) In **Prord**, by Theorem 3.5,
$KT_2 = LT_2 = NT_2 = \overline{T}_2 \Rightarrow T_0 = \overline{T}_0 = T_0'$ and by Theorem 4.7, all Stone $SHKT_2, SHLT_2, SH\overline{T}_2$, and $SHNT_2$ transitive spatial graphs are equivalent.

By Theorem 4.7, if a transitive spatial graph (A, R) is KT_2 (resp. $\overline{T}_2$, LT_2 or NT_2), then it is extremally disconnected. (Z, Z^2) is Stone $SHKT_2$. By Theorem 3.6 of [10], each of discrete and indiscrete transitive spatial graph is extremally disconnected. By Theorems 3.4 and 3.5, Tychonoff Theorem holds in **Prord**.

(I) By Theorems 6.3 and 6.4 of [10], the following are equivalent.

(i) (A, R) is $Pre\overline{T}_2$.

(ii) (A, R) is $PreT_2'$.

(iii) R is an equivalence relation on B.

(II) By Theorem 3.5, the following are equivalent.

(i) (A, R) is T_0'.

(ii) (A, R) is $overlineT_0$.

(iii) (A, R) is T_0.

(iv) R is a partial relation on B.

(5) In any topological category, by Theorem 4.2, every Stone $SHLT_2$ (resp. $SH\overline{T}_2$) object is Stone $SHKT_2$ and in the realm of $PreT_2'$ objects, $SHLT_2 = SH\overline{T}_2 = SHKT_2$.

By parts (1) and (3), there is no implication between Stone $SHLT_2$ and Stone $SHNT_2$ objects.

5. Comments

Note that a notion of strong compactness at the level of topological categories was defined in [6]. In **Top**, if a space is T_1, then by Remark 2.2 of [6], compactness and strong compactness (resp. by Remark 2.1 of [6] and Definition 4.1, hereditary disconnectedness and strong hereditary disconnectedness) coincide. Hence, in Definition 4.1, replacing compactness (resp. strong hereditary disconnectedness) with strong compactness (resp. hereditary disconnectedness), many potential extensions of Stone objects may be possible. One form of these extensions may be more useful than another in certain applications but looking for the right extension may be meaningless. In **Top**, each of these extensions of Stone objects reduce to the usual Stone space [20], p.70.

(1) It will be better to characterize each of these extensions of Stone objects in well-known topological categories in order to find out how these forms of Stone objects relate to each other in general topological categories.

The notions of hereditary disconnectedness [16], p. 325 and total disconnectedness [2], p. 154 or [16], p. 369 were also defined in a topological category in [12]. Note that totally disconnected spaces and hereditarily disconnected spaces are referred to as totally separated spaces and totally disconnected spaces, respectively, in [20], p.69. In **Top**, in the presence of compact T_2 topological spaces, the notions of hereditary disconnectedness and total disconnectedness are equivalent [20], p.70.

If a spatial graph is KT_2, then by Theorem 5.2 of [12] and Theorem 4.6, the notions of hereditary disconnectedness and extremal disconnectedness are equivalent and total disconnectedness implies both hereditary disconnectedness and extremal disconnectedness. If a spatial graph is NT_2, then, by Theorem 5.2 and Remark 5.4 of [12] and Theorem 4.6, all the notions of extremal disconnectedness, hereditary disconnectedness and total disconnectedness are equivalent.

By Theorem 4.5, all the notions of extremal disconnectedness, hereditary disconnectedness, and total disconnectedness are equivalent in **Rel**.

In **PBorn**, by Theorem 5.3 of [12] and Theorem 4.8, the notions of hereditary disconnectedness and extremal disconnectedness are equivalent and total disconnectedness implies both hereditary disconnectedness and extremal disconnectedness. In the realm of NT_2 prebornological spaces, by Remark 5.4 and

Theorem 5.3 of [12] and Remark 4.10, all the notions of extremal disconnectedness, hereditary disconnectedness, and total disconnectedness are equivalent.

By Remark 4.10, each of discrete and indiscrete graph (resp. spatial graph, transitive spatial graph, prebornological, bornological, topological) space is extremally disconnected.

(2) Are discrete and indiscrete objects extremally disconnected in a topological category?

(3) How each of extremally, hereditarily, and totally disconnected objects are related to each other in a topological category.

If $U : \mathcal{E} \to$ **Set** is topological and $\mathcal{D}$ is a full subcategory of $\mathcal{E}$ such that the restriction $U_1 = U|_{\mathcal{D}} : \mathcal{D} \to$ **Set** is still topological, then for an object $X \in \mathcal{D}$ we have two notions of Stone objects one with respect to U and one with respect to U_1. One may expect that the two notions may differ. Take $\mathcal{E}$ = **Rel** and $\mathcal{D}$ = **RRel** (Parts (1) and (2) of Remark 4.10).

(4) Under what conditions could these notions be the same?

Recall from [11] that X is called $\overline{T_0}$ (resp. T_0 or T_0') sober if X is $\overline{T_0}$ (resp. T_0 or T_0') and quasi-sober (every nonempty irreducible closed subset of X [13] is the closure of a point).

In **Top**, by Remark 2.2, the abstract notion of sober space reduces with the usual one and in the realm of $PreT_2$ topological spaces, by Theorem 4.3, all T_0, T_1, T_2, and sober spaces are equivalent. Also, Theorem 4.3 can be shown by using Theorem 3.5 of [9] and Lemma of ([20], page 43).

In **Prord**, by Theorem 3.6 of [10] and Theorem 3.5, $\overline{T_0}$ (resp. T_0 or T_0') sober spaces are equivalent.

(5) Does Theorem 4.3 hold in a topological category?

(6) How are sober objects related to each other in a topological category?

(7) Are there any relations between sober objects and T_i objects, $i = 0, 1, 2$, in a topological category?

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