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Some Caputo-Fabrizio fractional integral inequalities with applications

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Abstract. Fractional calculus provides a significant generalization of classical concepts and overcomes the limitation of classical calculus in dealing with non-differentiability function. Implementing fractional operator to obtain new versions of classical outcomes is very intriguing topic of research in the mathematical analysis. The objective of the present study is to establish novel Hermite-Hadamard integral inequalities for twice differentiable function using Caputo-Fabrizio integral operator. In order to complete task, we start by demonstrating a new identity for Hermite-Hadamard inequality that serve as supporting result for our main finding. It has been observed that the obtained Hermite-Hadamard type inequalities have a relationship with previous results. In addition, we provide application to special means and graphical analysis to evaluate the accuracy of our results.

1. Introduction

Integral inequalities are crucial to our understanding of the universe, and there are numerous simple ways for determining the uniqueness and existence of linear and nonlinear differential equations with symmetry. Researchers in many applied domains, such as convex programming, are very interested in convex functions how crucial they are to the theory of inequality in a variety of applications. Additionally, convex functions are great interest to mathematicians in a wide range of theoretical fields, including probability theory. Its good to start by identifying this kind of function.

Definition 1.1. *See [1]. Let I be a convex subset of a real vector space $\mathbb{R}$ and let $g : I \to \mathbb{R}$ be a function. Then, a function g is said to be convex, if*

$$g(\theta\varkappa_1 + (1-\theta)\varkappa_2) \leq \theta g(\varkappa_1) + (1-\theta) g(\varkappa_2),$$

holds for all $\varkappa_1, \varkappa_2 \in I$ and $\theta \in [0,1]$.

Convex functions are significant in many mathematics fields as an application. They play a crucial role in the research of optimization problems since they distinguish out due to a wide range of useful characteristics.

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Convex functions are also used to establish a historical inequality, where the lower and upper bounds can be expressed as arithmetic means. Understanding the inequality described here is essential for numerical integration because it is utilized in formulas for error estimation such the trapezoidal and midpoint formulas [2–4]. In convex functions theory, Hermite-Hadamard (H-H) inequality is very important which was discovered by C. Hermite and J. Hadamard independently. Let function $g : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on I, where $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, then the following inequality.

$$g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx \leq \frac{g(\varkappa_1)+g(\varkappa_2)}{2}.$$

Several authors discuss the concepts of convexity and celebrated integral inequality. This form of research is focused on investigating the properties of inequalities of the Hadamard, Bullen, Ostrowski, and Simpson types as well as the properties of various inequalities that can be found in the data of static biological systems. This study provided a novel approach and enhanced the literature's new applications. New details on convexity and integral inequality are provided in the following articles [5–7] in several directions.

The characteristics and definition of convexity are advantageous for fractional integral inequalities, which have recently grown to be a very important area of study. Finding solutions to open problems involving fractional-order derivatives is the focus of the recently developed area of applied mathematics known as fractional analysis. Due to the persistent research interest in the subject after the discovery of this solution, mathematicians have found themselves establishing totally new lines of investigation.

In recent years fractional calculus offers a powerful tool for modelling complex phenomena, it is important to note that its application requires a solid understanding of the underlying mathematical principles. Fractional calculus is a specialized branch, contributing to the development of mathematical tools that can be effectively used to address real-world challenges in a variety of domains. The use of fractional integral and derivative operators helps bridge the gap between mathematical models and the real-world complexities encountered in various disciplines. Fractional integral operators and fractional derivative operators have contributed new concepts to fractional analysis in terms of their application areas and spaces [8–10]. In all of these do not have same singularity, locality, or kernel qualities with each other. This extension is not only limited to derivatives but also applies to fractional integrals, providing a comprehensive set of tools for analyzing and modelling diverse systems in the real world. The versatility of fractional integrals makes them applicable across a diverse range of disciplines, contributing to advancements in modeling, analysis, and problem-solving in science and engineering. The theory is established through the introduction of the most useful fractional integral operators, Riemann-Liouville fractional integrals, there are other fractional integral operators, such as the Caputo fractional integral, which is another widely used formulation in fractional calculus to see these articles [11]. For anyone who is interested in learning more about fractional integral and derivative operators, the articles [13–16, 20–33] are an excellent place to begin.

2. Preliminaries.

Definition 2.1. *See [34] Let $0 < s \leq 1$. A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is called s-φ-convex with respect to bifunction $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$(briefly φ-convex), if*

$$g(\theta\varkappa_1 + (1-\theta)\varkappa_2) \leq g(\varkappa_2) + \theta^s \varphi(g(\varkappa_1), g(\varkappa_2)), \tag{1}$$

for all $\varkappa_1, \varkappa_2 \in I$ and $\theta \in [0,1]$. Furthermore, g is called φ-quasi-convex if

$$g(\theta\varkappa_1 + (1-\theta)\varkappa_2) \leq \max\left[g(\varkappa_2), g(\varkappa_2) + \varphi(g(\varkappa_1), g(\varkappa_2)\right],$$

for all $\varkappa_1, \varkappa_2 \in I$ and $\theta \in [0,1]$.

Remark 2.2. *If we take $s = 1$, in (2.1), then we have the definition of φ-convex function and moreover if we take $\varphi(\varkappa_1, \varkappa_2) = \varkappa_1 - \varkappa_2$ in (2.1), then the definition of s-φ-convex function is reduced to the definition of $s-$convex function in the first sense.*

Next, we will give examples for above definition.

Example 2.3. *Let $g(x) = x^2$, then g is convex and $\frac{1}{2}$-φ-convex with $\varphi(\varkappa_1, \varkappa_2) = 2\varkappa_1 + \varkappa_2$; indeed,*

$$\begin{aligned} g(\theta\varkappa_1 + (1-\theta)\varkappa_2) &= (\theta\varkappa_1 + (1-\theta)\varkappa_2)^2 \\ &= \theta^2\varkappa_1^2 + 2\theta(1-\theta)\varkappa_1\varkappa_2 + (1-\theta)^2\varkappa_2^2 \\ &\leq \varkappa_2^2 + \theta\varkappa_1^2 + 2\theta\varkappa_1\varkappa_2 \\ &= \varkappa_2^2 + \theta^{\frac{1}{2}}\left[\theta^{\frac{1}{2}}\varkappa_1^2 + 2\theta^{\frac{1}{2}}\varkappa_1\varkappa_2\right]. \end{aligned}$$

on the other hand;

$$\begin{aligned} 0 \leq \theta \leq 1 \Longrightarrow 0 < \theta^{\frac{1}{2}} < 1 \\ \Longrightarrow \quad \theta^{\frac{1}{2}}\varkappa_1^2 + 2\theta^{\frac{1}{2}}\varkappa_1\varkappa_2 \leq \varkappa_1^2 + 2\varkappa_1\varkappa_2 \leq \varkappa_1^2 + \varkappa_1^2 + \varkappa_2^2. \end{aligned}$$

Hence,

$$\begin{aligned} & g(\theta\varkappa_1 + (1-\theta)\varkappa_2) \leq \varkappa_2^2 + \theta^{\frac{1}{2}}\left[2\varkappa_1^2 + \varkappa_2^2\right] \\ = \quad & g(\varkappa_2) + \theta^{\frac{1}{2}}\varphi(g(\varkappa_1), g(\varkappa_2)). \end{aligned}$$

Example 2.4. *Let $g(x) = x^3$, then g is not convex but is φ-convex with $\varphi(\varkappa_1, \varkappa_2) = 3\varkappa_2^2(\varkappa_1 - \varkappa_2) + 3\varkappa_2(\varkappa_1 - \varkappa_2)^2 + (\varkappa_1 - \varkappa_2)^3$; indeed,*

$$\begin{aligned} g(\theta\varkappa_1 + (1-\theta)\varkappa_2) &= ((1-\theta)\varkappa_2 + \theta\varkappa_1)^3 = (\theta(\varkappa_1 - \varkappa_2) + \varkappa_2)^3 \\ &= \varkappa_2^3 + 3\varkappa_2^2\theta(\varkappa_1 - \varkappa_2) + 3\varkappa_2\theta^2(\varkappa_1 - \varkappa_2)^2 + \theta^3(\varkappa_1 - \varkappa_2)^3 \\ &= g(\varkappa_2) + \theta\left[3\varkappa_2^2(\varkappa_1 - \varkappa_2) + 3\varkappa_2\theta(\varkappa_1 - \varkappa_2)^2 + \theta^2(\varkappa_1 - \varkappa_2)^3\right] \\ &\leq g(\varkappa_2) + \theta\left[3\varkappa_2^2(\varkappa_1 - \varkappa_2) + 3\varkappa_2(\varkappa_1 - \varkappa_2)^2 + (\varkappa_1 - \varkappa_2)^3\right] \\ &= g(\varkappa_2) + \theta\varphi(g(\varkappa_1), g(\varkappa_2)). \end{aligned}$$

Example 2.5. *Let $g : [\varkappa_1, \varkappa_2] \to \mathbb{R}$, $0 < \varkappa_1 < \varkappa_2$, with $g(x) = 2$. we obviously see that g is a φ- quasi-convex with $\varphi(\varkappa_1, \varkappa_2) = \varkappa_1 - \varkappa_2$.*

Theorem 2.6. *[35]Suppose that $g : [0, \infty) \to [0, \infty)$ is s-convex function in the second kind, where $s \in (0, 1)$, and let $\varkappa_1, \varkappa_2 \in [0, \infty)$, $\varkappa_1 < \varkappa_2$. If $g \in L[\varkappa_1, \varkappa_2]$, then the following inequality holds:*

$$\begin{aligned} 2^{s-1}g\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) &\leq \frac{B(\alpha)}{\alpha(\varkappa_2 - \varkappa_1)}\left[\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k) + \left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k) - \frac{2(1-\alpha)}{\alpha(\varkappa_2 - \varkappa_1)}g(k)\right] \\ &\leq \frac{g(\varkappa_1) + g(\varkappa_2)}{s+1}. \end{aligned}$$

Lemma 2.7. *[35]Let I be a real interval such that $\varkappa_1, \varkappa_2 \in I^0$, the interior of I with $\varkappa_1 < \varkappa_2$. Let $g : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. If $g' \in L[\varkappa_1, \varkappa_2]$, then following equality holds:*

$$\frac{g(\varkappa_1) + g(\varkappa_2)}{2} - \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(u)\,du = \frac{\varkappa_2 - \varkappa_1}{2}\int_0^1 (1 - 2\theta)\,g'(\theta\varkappa_1 + (1-\theta)\varkappa_2)\,d\theta.$$

Lemma 2.8. *[35]Let I be a real interval such that $\varkappa_1, \varkappa_2 \in I^0$, (the interior of I) with $\varkappa_1 < \varkappa_2$. Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, $\varkappa_1, \varkappa_2 \in I^0$ with $\varkappa_1 < \varkappa_2$. If $g' \in L[\varkappa_1, \varkappa_2]$, and $0 \leq \alpha \leq 1$, then following equality holds:*

$$
\begin{aligned}
& \frac{\varkappa_2-\varkappa_1}{2}\int_0^1 (1-2\theta)\, g'(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta-\frac{2(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}g(k)\\
= & \frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left(\left({}_{\varkappa_1}^{CF}I^{\alpha}g\right)(k)+\left({}^{CF}I_{\varkappa_2}^{\alpha}g\right)(k)\right),
\end{aligned}
$$

where $k\in[\varkappa_1,\varkappa_2]$ and $B(\alpha)>0$ is a normalization function. To additionally encourage the conversation of this article, we present the definition of the Riemann-Liouville fractional operator.

Definition 2.9. *See [36]. Let* $[\varkappa_1,\varkappa_2]\to\mathbb{R}$. *Then, Riemann-Liouville fractional integrals* $I_{\varkappa_1^+}^{\alpha}g(\theta)$ *and* $I_{\varkappa_2^-}^{\alpha}g(\theta)$ *of order* $\alpha>0$ *are defined by:*

$$
\begin{aligned}
I_{\varkappa_1^+}^{\alpha}g(x) &= \frac{1}{\Gamma(\alpha)}\int_{\varkappa_1}^{x}(\theta-x)^{\alpha-1}g(\theta)\,d\theta, x>\varkappa_1,\\
I_{\varkappa_2^-}^{\alpha}g(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{\varkappa_2}(x-\theta)^{\alpha-1}g(\theta)\,d\theta, x<\varkappa_2.
\end{aligned}
$$

where $\frac{1}{\Gamma.}$ *is the Gamma function.*

Definition 2.10. *See [37, 38] . Let* $g\in H^1(\varkappa_1,\varkappa_2)$, $\varkappa_1<\varkappa_2$, $\alpha\in[0,1]$, *then the notion of left and right Caputo-Fabrizio fractional integrals are defined by,*

$$
\begin{aligned}
\left({}_{\varkappa_1}^{CF}I^{\alpha}g\right)(x) &= \frac{1-\alpha}{B(\alpha)}g(x)+\frac{\alpha}{B(\alpha)}\int_{\varkappa_1}^{x}g(\theta)\,d\theta\\
\left({}^{CF}I_{\varkappa_2}^{\alpha}g\right)(x) &= \frac{1-\alpha}{B(\alpha)}g(x)+\frac{\alpha}{B(\alpha)}\int_{x}^{\varkappa_2}g(\theta)\,d\theta,
\end{aligned}
$$

where $B(\alpha)>0$ *is a normalization function that satisfies* $B(0)=B(1)=1$.

In this study, we have proved general identity for twice differentiable function through well known fractional integral operator Caputo-Fabrizio. It is also show that the newly established inequalities are the generalization of comparable inequalities in the literature. Finally, we provide few application to some special mean, and also show the graphical behavior of new errors bounds.

3. Main Results

This section explains how to use the Caputo-Fabrizio fractional integral operator to derive a new identity for differentiable convex functions. Then, taking this identity into consideration, numerous improvements are shown using some basic integral inequalities.

Lemma 3.1. *Let* $I\subset\mathbb{R}$ *be an open interval,* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$. *If* $g:I\to\mathbb{R}$ *is a twice differentiable mapping such that* g'' *is integrable and* $0\le\eta\le 1$.*Then the following equality holds:*

$$
\begin{aligned}
& (\eta-1)\,g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\\
& +\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}_{\varkappa_1}^{CF}I^{\alpha}g\right)(k)+\left({}^{CF}I_{\frac{\varkappa_1+\varkappa_2}{2}}^{\alpha}g\right)(k)\right\}+\left\{\left({}_{\frac{\varkappa_1+\varkappa_2}{2}}^{CF}I^{\alpha}g\right)(k)+\left({}^{CF}I_{\varkappa_2}^{\alpha}g\right)(k)\right\}\right]\\
= & (\varkappa_2-\varkappa_1)^2\int_0^1 k(\theta)\,g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta,
\end{aligned}
$$

where,

$$k(\theta)=\begin{cases}\frac{1}{2}\theta(\theta-\eta), & 0\le\theta\le\frac{1}{2};\\ \frac{1}{2}(1-\theta)(1-\eta-\theta), & \frac{1}{2}\le\theta\le 1.\end{cases}$$

where $k\in[\varkappa_1,\varkappa_2]$ and $B(\alpha)$ is a normalization function.

Proof.

$$\begin{aligned}
I &= (\varkappa_2-\varkappa_1)^2\int_0^1 k(\theta)\,g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
I &= (\varkappa_2-\varkappa_1)^2\int_0^{\frac{1}{2}}\frac{1}{2}\theta(\theta-\eta)\,g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta+\\
&\quad(\varkappa_2-\varkappa_1)^2\int_{\frac{1}{2}}^1\frac{1}{2}(1-\theta)(1-\eta-\theta)\,g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
I &= I_1+I_2.
\end{aligned}$$

Now we integration I_1 as:

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}\frac{1}{2}\theta(\theta-\eta)\,g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
&= \frac{1}{2}\left[\theta(\theta-\eta)\frac{g'(\theta\varkappa_1+(1-\theta)\varkappa_2)}{\varkappa_1-\varkappa_2}\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}(2\theta-\eta)\frac{g'(\theta\varkappa_1+(1-\theta)\varkappa_2)}{\varkappa_1-\varkappa_2}d\theta\right]\\
&= \frac{1}{4(\varkappa_1-\varkappa_2)}\left(\frac{1}{2}-\eta\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{2(\varkappa_1-\varkappa_2)}\left[\frac{(1-\eta)\,g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)}{(\varkappa_1-\varkappa_2)}+\frac{\eta g(\varkappa_2)}{\varkappa_1-\varkappa_2}\right]\\
&\quad-\frac{1}{(\varkappa_1-\varkappa_2)^2}\int_0^{\frac{1}{2}}g(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
&= \frac{1}{4(\varkappa_2-\varkappa_1)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{(\eta-1)}{2(\varkappa_2-\varkappa_1)^2}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad-\frac{\eta}{2(\varkappa_2-\varkappa_1)^2}g(\varkappa_2)+\frac{1}{(\varkappa_2-\varkappa_1)^3}\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{\varkappa_2}g(u)\,du\\
&= \frac{1}{4(\varkappa_2-\varkappa_1)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{(\eta-1)}{2(\varkappa_2-\varkappa_1)^2}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad-\frac{\eta}{2(\varkappa_2-\varkappa_1)^2}g(\varkappa_2)+\frac{1}{(\varkappa_2-\varkappa_1)^3}\left[\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{k}g(u)\,du+\int_k^{\varkappa_2}g(u)\,du\right]. \qquad (2)
\end{aligned}$$

Multiply equality (2) with $\frac{\alpha(\varkappa_2-\varkappa_1)^3}{B(\alpha)}$ and add $\frac{2(1-\alpha)}{B(\alpha)}g(k)$, we get

$$\begin{aligned}
&\frac{\alpha(\varkappa_2-\varkappa_1)^3}{B(\alpha)}I_1+\frac{2(1-\alpha)}{B(\alpha)}g(k)\\
&= \frac{\alpha(\varkappa_2-\varkappa_1)^2}{4B(\alpha)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{\alpha(\eta-1)(\varkappa_2-\varkappa_1)}{2B(\alpha)}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\
&\quad-\frac{\alpha(\varkappa_2-\varkappa_1)}{2B(\alpha)}\eta g(\varkappa_2)+\frac{\alpha}{B(\alpha)}\left[\int_{\frac{\varkappa_1+\varkappa_2}{2}}^{k}g(u)\,du+\frac{(1-\alpha)}{B(\alpha)}g(k)+\int_k^{\varkappa_2}g(u)\,du+\frac{(1-\alpha)}{B(\alpha)}g(k)\right]
\end{aligned}$$

$$= \frac{\alpha(\varkappa_2-\varkappa_1)^2}{4B(\alpha)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{\alpha(\eta-1)(\varkappa_2-\varkappa_1)}{2B(\alpha)}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)$$
$$-\frac{\alpha(\varkappa_2-\varkappa_1)}{2B(\alpha)}\eta g(\varkappa_2)+\left[\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right]$$

$$(\varkappa_2-\varkappa_1)^2\int_0^{\frac{1}{2}}\frac{1}{2}\theta(\theta-\eta)g''(\theta\varkappa_1+(1-\theta)\varkappa_2)d\theta+\frac{2(1-\alpha)}{B(\alpha)}g(k)$$
$$= \frac{(\varkappa_2-\varkappa_1)}{4}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{1}{2}(\eta-1)g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{2}\eta g(\varkappa_2)$$
$$+\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right]. \tag{3}$$

Analogously I_2 as:

$$I_2 = \int_{\frac{1}{2}}^1\frac{1}{2}(1-\theta)(1-\eta-\theta)g''(\theta\varkappa_1+(1-\theta)\varkappa_2)d\theta$$
$$= \frac{1}{2}\left[(1-\theta)(1-\eta-\theta)\frac{g'(\theta\varkappa_1+(1-\theta)\varkappa_2)}{\varkappa_1-\varkappa_2}\Bigg|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^1(2\theta+\eta-2)\frac{g'(\theta\varkappa_1+(1-\theta)\varkappa_2)}{\varkappa_1-\varkappa_2}d\theta\right]$$
$$= -\frac{1}{4(\varkappa_1-\varkappa_2)}\left(\frac{1}{2}-\eta\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{2(\varkappa_1-\varkappa_2)}\left[\frac{\eta g(\varkappa_1)}{\varkappa_1-\varkappa_2}-\frac{(1-\eta)g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)}{(\varkappa_1-\varkappa_2)}\right]$$
$$+\frac{1}{(\varkappa_1-\varkappa_2)^2}\int_{\frac{1}{2}}^1 g(\theta\varkappa_1+(1-\theta)\varkappa_2)d\theta$$
$$= -\frac{1}{4(\varkappa_2-\varkappa_1)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{(\eta-1)}{2(\varkappa_2-\varkappa_1)^2}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)$$
$$-\frac{\eta}{2(\varkappa_2-\varkappa_1)^2}g(\varkappa_1)+\frac{1}{(\varkappa_2-\varkappa_1)^3}\int_{\varkappa_1}^{\frac{\varkappa_1+\varkappa_2}{2}}g(u)du$$
$$= -\frac{1}{4(\varkappa_2-\varkappa_1)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{(\eta-1)}{2(\varkappa_2-\varkappa_1)^2}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)$$
$$-\frac{\eta}{2(\varkappa_2-\varkappa_1)^2}g(\varkappa_1)+\frac{1}{(\varkappa_2-\varkappa_1)^3}\left[\int_{\varkappa_1}^{k}g(u)du+\int_{k}^{\frac{\varkappa_1+\varkappa_2}{2}}g(u)du\right]. \tag{4}$$

Multiply equality (4) with $\frac{\alpha(\varkappa_2-\varkappa_1)^3}{B(\alpha)}$ and add $\frac{2(1-\alpha)}{B(\alpha)}g(k)$, we have

$$\frac{\alpha(\varkappa_2-\varkappa_1)^3}{B(\alpha)}I_2+\frac{2(1-\alpha)}{B(\alpha)}g(k)$$
$$= -\frac{\alpha(\varkappa_2-\varkappa_1)^2}{4B(\alpha)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{\alpha(\eta-1)(\varkappa_2-\varkappa_1)}{2B(\alpha)}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)$$
$$-\frac{\alpha(\varkappa_2-\varkappa_1)}{2B(\alpha)}\eta g(\varkappa_1)+\frac{\alpha}{B(\alpha)}\left[\int_{\varkappa_1}^{k}g(u)du+\frac{(1-\alpha)}{B(\alpha)}g(k)+\int_{k}^{\frac{\varkappa_1+\varkappa_2}{2}}g(u)du+\frac{(1-\alpha)}{B(\alpha)}g(k)\right]$$
$$= -\frac{\alpha(\varkappa_2-\varkappa_1)^2}{4B(\alpha)}\left(\eta-\frac{1}{2}\right)g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+\frac{\alpha(\eta-1)(\varkappa_2-\varkappa_1)}{2B(\alpha)}g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)$$
$$-\frac{\alpha(\varkappa_2-\varkappa_1)}{2B(\alpha)}\eta g(\varkappa_1)+\left[\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right]$$

$$
\begin{aligned}
& (\varkappa_2-\varkappa_1)^2 \int_{\frac{1}{2}}^{1} \frac{1}{2}(1-\theta)(1-\eta-\theta)\, g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\, d\theta + \frac{2(1-\alpha)}{B(\alpha)} g(k) \\
= & -\frac{(\varkappa_2-\varkappa_1)}{4}\left(\eta-\frac{1}{2}\right) g'\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + \frac{1}{2}(\eta-1)\, g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{1}{2}\eta g(\varkappa_1) \\
& + \frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left({}^{CF}_{\varkappa_1}I^{\alpha} g\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}} g\right)(k)\right]. \qquad (5)
\end{aligned}
$$

Now we combining equalities (3) and (5) we get result. □

Theorem 3.2. *Let $I \subset \mathbb{R}$ be an open interval, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. If $g : I \to \mathbb{R}$ is a twice differentiable mapping such that g'' is integrable and $0 \le \eta \le 1$. If $|g''|$ is a s-φ-convex on $[x_1, x_2]$, then the following inequality holds:*

$$
\begin{aligned}
& \Bigg|(\eta-1)\, g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2} - \frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)} + \\
& \frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha} g\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}} g\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha} g\right)(k) + \left({}^{CF}I^{\alpha}_{\varkappa_2} g\right)(k)\right\}\right]\Bigg| \qquad (6) \\
\le\ & (\varkappa_2-\varkappa_1)^2 \begin{cases} \left[(A_1+A_3)\left|g''(\varkappa_1)\right| + \frac{(A_2+A_4)}{2}\varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right] & for 0 \le \eta \le \frac{1}{2}, \\ \left[(A_5+A_7)\left|g''(\varkappa_1)\right| + \frac{(A_6+A_8)}{2}\varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right] & for \frac{1}{2} \le \eta \le 1. \end{cases}
\end{aligned}
$$

Where,

$$
\begin{aligned}
A_1 &= \int_0^{\eta} (\theta(\eta-\theta))\, d\theta + \int_{\eta}^{\frac{1}{2}} (\theta(\theta-\eta))\, d\theta = \frac{1}{24}\left(1-3\eta+8\eta^3\right). \\
A_2 &= \int_0^{\eta} (\theta(\eta-\theta))\, \theta^s + \int_{\eta}^{\frac{1}{2}} (\theta(\theta-\eta))\, \theta^s d\theta = \frac{2^{-3-s}\left(2+s-6\eta-2s\eta+2^{4+s}\eta^{3+s}\right)}{(s+2)(s+3)}. \\
A_3 &= \int_{\frac{1}{2}}^{1-\eta} (1-\theta)(1-\eta-\theta)\, d\theta + \int_{1-\eta}^{1} (1-\theta)(\theta+\eta-1)\, d\theta \\
&= \frac{1}{24}\left(1-3\eta+8\eta^3\right). \\
A_4 &= \int_{\frac{1}{2}}^{1-\eta} (1-\theta)(1-\eta-\theta)\, \theta^s d\theta + \int_{1-\eta}^{1} (1-\theta)(\theta+\eta-1)\, \theta^s d\theta \\
&= \frac{8(1-\eta)^{2+s}(2+\eta+s\eta) + 2^{-s}\left(14-7s-s^2+2(3+s)^2\eta\right)}{2^3(s+1)(s+2)(s+3)} + \\
&\quad \frac{8\left(-2+(3+s)\eta+(1-\eta)^{2+s}(2+\eta+s\eta)\right)}{2^3(s+1)(s+2)(s+3)}. \\
A_5 &= \int_0^{\frac{1}{2}} \theta(\eta-\theta)\, d\theta = \frac{\eta}{8} - \frac{1}{24}. \\
A_6 &= \int_0^{\frac{1}{2}} \theta(\eta-\theta)\, \theta^s d\theta = 2^{-3-s}\left(\frac{2\eta}{s+2} - \frac{1}{s+3}\right). \\
A_7 &= \int_{\frac{1}{2}}^{1} (1-\theta)(1-\eta-\theta)\, d\theta = \frac{\eta}{8} - \frac{1}{24}. \\
A_8 &= \int_{\frac{1}{2}}^{1} (1-\theta)(1-\eta-\theta)\, \theta^s
\end{aligned}
$$

$$= \frac{2^{-3-s}\left(14+7s+s^2-18\eta-12s\eta-2s\eta-2s^2\eta+2^{3+s}\left(-2+(3+s)\eta\right)\right)}{(s+1)(s+2)(s+3)}.$$

$$\begin{aligned}
A_9 &= \int_0^{\eta}(\theta(\eta-\theta))\,d\theta+\int_{\eta}^{\frac{1}{2}}(\theta(\theta-\eta))\,d\theta \\
&\quad+\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)\,d\theta+\int_{1-\eta}^{1}(1-\theta)(\theta+\eta-1)\,d\theta \\
&= \frac{\left(1-3\eta+8\eta^3\right)}{12}. \\
A_{10} &= \int_0^{\frac{1}{2}}\theta(\eta-\theta)\,d\theta+\int_{\frac{1}{2}}^{1}(1-\theta)(1-\eta-\theta)\,d\theta \\
&= \frac{3\eta-1}{12}.
\end{aligned}$$

Proof. Using the Lemma 3.1 and the s-φ-convexity of $\left|g''\right|$, we get

$$\begin{aligned}
&\left|(\eta-1)g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}+\right. \\
&\left.\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\right| \\
\le\;& (\varkappa_2-\varkappa_1)^2\int_0^1|k(\theta)|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta \\
=\;& \frac{(\varkappa_2-\varkappa_1)^2}{2}\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta \\
&+\frac{(\varkappa_2-\varkappa_1)^2}{2}\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta \\
=\;& \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[I_1+I_2\right],
\end{aligned} \tag{7}$$

where we assume that $0\le\eta\le\frac{1}{2}$ using the s-φ-convex of $\left|g''\right|$, we get

$$\begin{aligned}
I_1 &\le \int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\
&= \int_0^{\eta}(\theta(\eta-\theta))\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\
&\quad+\int_{\eta}^{\frac{1}{2}}(\theta(\theta-\eta))\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\
&= A_1\left|g''(\varkappa_1)\right|+A_2\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|\right),
\end{aligned} \tag{8}$$

similarly, we have

$$\begin{aligned}
I_2 &\le \int_{\frac{1}{2}}^{\eta}\left|(1-\theta)(1-\eta-\theta)\right|\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\
&\quad\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta
\end{aligned}$$

$$+\int_{1-\eta}^{1}(1-\theta)(\theta+\eta-1)\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta$$
$$= A_3\left|g''(\varkappa_1)\right|+A_4\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right), \quad (9)$$

using equality (8) and (9) in (7) we get the first inequality (6) holds.

where we assume that $\frac{1}{2}\le\eta\le 1$, using the s-φ-convex of $\left|g''\right|$, we get

$$\begin{aligned} I_1 &\le \int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\ &= \int_0^{\frac{1}{2}}\theta(\eta-\theta)\left|g''(\varkappa_1)\right|+\int_0^{\frac{1}{2}}\theta(\eta-\theta)\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right) \\ &= A_5\left|g''(\varkappa_1)\right|+A_6\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right), \end{aligned} \quad (10)$$

similarly

$$\begin{aligned} I_2 &\le \int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right|\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)d\theta \\ &= \int_{\frac{1}{2}}^{1}(1-\theta)(1-\eta-\theta)\left|g''(\varkappa_1)\right|d\theta \\ &\quad +\int_{\frac{1}{2}}^{1}(1-\theta)(1-\eta-\theta)\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)d\theta \\ &= A_7\left|g''(\varkappa_1)\right|+A_8\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)d\theta, \end{aligned} \quad (11)$$

using equality (10) and (11) in (7) we get the second inequality of (6). This completes the proof. □

Corollary 3.3. *Theorem 3.2 with $B(0)=B(1)=1$ and $s=\alpha=1$, $\eta=0$, we have*

$$\begin{aligned} &\left|g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}g(x)\,dx\right| \\ &\le \frac{(\varkappa_2-\varkappa_1)^2}{24}\left[\left|g''(\varkappa_1)\right|+\frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right]. \end{aligned} \quad (12)$$

Remark 3.4. *Inequality (12) with $\varphi\left(\left|g''(\varkappa_1),g''(\varkappa_2)\right|\right)=\left|g''(\varkappa_2)\right|-\left|g''(\varkappa_1)\right|$, we have*

$$\left|g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}g(x)\,dx\right|\le\frac{(\varkappa_2-\varkappa_1)^2}{48}\left[\left|g''(\varkappa_1)\right|+\left|g''(\varkappa_2)\right|\right].$$

This inequality is obtained by Sarikaya et al. [29] Proposition 1.

Corollary 3.5. *Theorem 3.2 with $B(0)=B(1)=1$ and $s=\alpha=1$, $\eta=\frac{1}{3}$, we have*

$$\begin{aligned} &\left|\frac{1}{6}\left[g(\varkappa_1)+4g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+g(\varkappa_2)\right]-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}g(x)\,dx\right| \\ &\le \frac{(\varkappa_2-\varkappa_1)^2}{81}\left[\left|g''(\varkappa_1)\right|+\frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right]. \end{aligned} \quad (13)$$

Remark 3.6. *Inequality (13) with $g(\varkappa_1)=g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)=g(\varkappa_2)$, we have*

$$\left|g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}g(x)\,dx\right|\le\frac{(\varkappa_2-\varkappa_1)^2}{81}\left[\left|g''(\varkappa_1)\right|+\frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right].$$

This inequality is obtained by Miguel Vivas-Cortez et al. [7] Corollary 1.

Remark 3.7. *Inequality (13) with* $\varphi\left(\left|g''(\varkappa_1), g''(\varkappa_2)\right|\right) = \left|g''(\varkappa_2)\right| - \left|g''(\varkappa_1)\right|$, *we have*

$$\left|\frac{1}{6}\left[g(\varkappa_1) + 4g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + g(\varkappa_2)\right] - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{162}\left[\left|g''(\varkappa_1)\right| + \left|g''(\varkappa_2)\right|\right].$$

This inequality is obtained by Sarikaya et al. [29] Proposition 3.

Corollary 3.8. *Theorem 3.2 with* $B(0) = B(1) = 1$ *and* $s = \alpha = 1$, $\eta = 1/2$, *we have*

$$\left|\frac{1}{2}\left[\frac{g(\varkappa_1)+g(\varkappa_2)}{2} + g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right] - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{48}\left[\left|g''(\varkappa_1)\right| + \frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right]. \tag{14}$$

Remark 3.9. *Inequality (14) with* $\varphi\left(\left|g''(\varkappa_1), g''(\varkappa_2)\right|\right) = \left|g''(\varkappa_2)\right| - \left|g''(\varkappa_1)\right|$, *we have*

$$\left|\frac{1}{2}\left[\frac{g(\varkappa_1)+g(\varkappa_2)}{2} + g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right] - \frac{1}{\varkappa_1-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{96}\left[\left|g''(\varkappa_1)\right| + \left|g''(\varkappa_2)\right|\right].$$

This inequality is obtained by Sarikaya et al. [29] Proposition 4.

Corollary 3.10. *Theorem 3.2 with* $B(0) = B(1) = 1$ *and* $s = \alpha = 1$, $\eta = 1$, *we have:*

$$\left|\frac{g(\varkappa_1)+g(\varkappa_2)}{2} - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx\right|$$
$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{12}\left[\left|g''(\varkappa_1)\right| + \frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right]. \tag{15}$$

Remark 3.11. *Inequality (15) with* $\varphi\left(\left|g''(\varkappa_1), g''(\varkappa_2)\right|\right) = \left|g''(\varkappa_2)\right| - \left|g''(\varkappa_1)\right|$, *we have:*

$$\left|\frac{g(\varkappa_1)+g(\varkappa_2)}{2} - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} g(x)\,dx\right| \leq \frac{(\varkappa_2-\varkappa_1)^2}{24}\left[\left|g''(\varkappa_1)\right| + \left|g''(\varkappa_2)\right|\right].$$

This inequality is obtained by Sarikaya et al. [29] Proposition 2.
Another similar result may be extended in the following theorem.

Theorem 3.12. *Let* $I \subset \mathbb{R}$ *be an open interval,* $\varkappa_1, \varkappa_2 \in I$ *with* $\varkappa_1 < \varkappa_2$. *If* $g : I \to \mathbb{R}$ *be a twice differentiable mapping such that* g'' *is integrable and* $0 \leq \eta \leq 1$. *If* $\left|g''\right|^q$ *is a s-φ-convex on* $[\varkappa_1, \varkappa_2]$, $q \geq 1$, *then the following inequality holds:*

$$\left|(\eta-1)\,g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2} - \frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)} + \right.$$

$$\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\Bigg|$$

$$\leq (\varkappa_2-\varkappa_1)^2\left[\frac{1}{24}\left(1-3\eta+8\eta^3\right)\right]^{1-\frac{1}{q}}\times$$

$$\begin{cases}\left[A_9\left|g''(\varkappa_1)\right|^q+(A_2+A_4)\frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right)\right]^{\frac{1}{q}} for 0\leq\eta\leq\frac{1}{2},\\ \left[A_{10}\left|g''(\varkappa_1)\right|^q+(A_6+A_8)\frac{1}{2}\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right)\right]^{\frac{1}{q}} for\frac{1}{2}\leq\eta\leq 1.\end{cases} \tag{16}$$

Where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. *Using the Lemma 3.1 and power mean inequality, we get*

$$\begin{aligned}
&\Bigg|(\eta-1)g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}+\\
&\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\Bigg|\\
\leq\ & (\varkappa_2-\varkappa_1)^2\int_0^1|k(\theta)|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
=\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
&+\frac{(\varkappa_2-\varkappa_1)^2}{2}\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
=\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[\left(\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|^q d\theta\right)^{\frac{1}{q}}+\right.\\
&\left.\left(\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left(\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|\right)^q d\theta\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{17}$$

where we assume that $0\leq\eta\leq\frac{1}{2}$ *using the s-φ-convex of* $\left|g''\right|^q$, *and* $\theta\in[0,1]$

$$\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|^q\leq\left|g''(\varkappa_2)\right|^q+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right),$$

by simple computation

$$\begin{aligned}
I_1\ \leq\ & \int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left[\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)\right]^q d\theta\\
=\ & \left(\int_0^{\eta}\theta(\eta-\theta)+\int_{\eta}^{\frac{1}{2}}\theta(\theta-\eta)\right)\left[\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)\right]^q d\theta\\
=\ & \int_0^{\eta}(\theta(\eta-\theta))\left(\left|g''(\varkappa_1)\right|^q+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right)\right)d\theta\\
&+\int_{\eta}^{\frac{1}{2}}(\theta(\theta-\eta))\left(\left|g''(\varkappa_1)\right|^q+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right)\right)d\theta\\
=\ & A_1\left|g''(\varkappa_1)\right|^q+A_2\varphi\left(\left|g''(\varkappa_1)\right|^q,\left|g''(\varkappa_2)\right|^q\right),
\end{aligned} \tag{18}$$

similarly, we have

$$I_2\ \leq\ \int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left[\left(\left|g''(\varkappa_1)\right|+\theta^s\varphi\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_2)\right|\right)\right)\right]^q d\theta$$

$$\begin{aligned} &= \int_{\frac{1}{2}}^{1-\eta} (1-\theta)(1-\eta-\theta)\left(\left|g''(\varkappa_1)\right|^q + \theta^s \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right)\right) d\theta \\ &\quad + \int_{1-\eta}^{1} (1-\theta)(\theta+\eta-1)\left(\left|g''(\varkappa_1)\right|^q + \theta^s \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right)\right) d\theta \\ &= A_3 \left|g''(\varkappa_1)\right|^q + A_4 \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right), \end{aligned} \tag{19}$$

thus, using (18) -(19) in (17) we obtain the first inequality of (16).
where we assume that $\frac{1}{2} \leq \eta \leq 1$, using the s-$\varphi$-convex of $\left|g''\right|$, we get

$$\begin{aligned} I_1 &\leq \int_0^{\frac{1}{2}} \left|\theta(\theta-\eta)\right| \left[\left(\left|g''(\varkappa_1)\right| + \theta^s \varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right)\right]^q d\theta \\ &= \int_0^{\frac{1}{2}} \theta(\eta-\theta)\left|g''(\varkappa_1)\right|^q + \int_0^{\frac{1}{2}} \theta(\eta-\theta)\theta^s \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right) \\ &= A_5 \left|g''(\varkappa_1)\right| + A_6 \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right), \end{aligned} \tag{20}$$

similarly

$$\begin{aligned} I_2 &\leq \int_{\frac{1}{2}}^{1} \left|(1-\theta)(1-\eta-\theta)\right| \left[\left(\left|g''(\varkappa_1)\right| + \theta^s \varphi\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_2)\right|\right)\right)\right]^q d\theta \\ &= \int_{\frac{1}{2}}^{1} (1-\theta)(1-\eta-\theta)\left|g''(\varkappa_1)\right|^q d\theta \\ &\quad + \int_{\frac{1}{2}}^{1} (1-\theta)(1-\eta-\theta)\theta^s \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right) d\theta \\ &= A_7 \left|g''(\varkappa_1)\right|^q + A_8 \varphi\left(\left|g''(\varkappa_1)\right|^q, \left|g''(\varkappa_2)\right|^q\right) d\theta, \end{aligned} \tag{21}$$

thus, using (20) -(21) in (17) we obtain the second inequality of (16).
Note that

$$\begin{aligned} \int_0^{\frac{1}{2}} \left|\theta(\theta-\eta)\right| d\theta &= \int_0^{\eta} \theta(\eta-\theta)\, d\theta + \int_{\eta}^{\frac{1}{2}} \theta(\theta-\eta)\, d\theta \\ &= \frac{1}{24}\left(1-3\eta+8\eta^3\right). \end{aligned} \tag{22}$$

$$\begin{aligned} \int_{\frac{1}{2}}^{1} \left|(1-\theta)(1-\eta-\theta)\right| &= \int_{\frac{1}{2}}^{\eta} (1-\theta)(1-\eta-\theta)\, d\theta + \int_{1-\eta}^{1} (1-\theta)(\theta+\eta-1)\, d\theta \\ &= \frac{1}{24}\left(1-3\eta+8\eta^3\right). \end{aligned} \tag{23}$$

This completes the proof. □

Theorem 3.13. *Let $I \subset \mathbb{R}$ be an open interval, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$. If $g : I \to \mathbb{R}$ be a twice differentiable mapping such that g'' is integrable and $0 \leq \eta \leq 1$. If $\left|g''\right|^q$ is concave on $[\varkappa_1, \varkappa_2]$, for some fixed $q \geq 1$, then the following inequalities holds:*

$$\begin{aligned} &\Bigg|(\eta-1)\, g\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \eta \frac{g(\varkappa_1)+g(\varkappa_2)}{2} - \frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)} + \\ &\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha} g\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}} g\right)(k)\right\} + \left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha} g\right)(k) + \left({}^{CF}I^{\alpha}_{\varkappa_2} g\right)(k)\right\}\right]\Bigg| \end{aligned} \tag{24}$$

$$\leq \frac{(\varkappa_2-\varkappa_1)^2}{2}\begin{cases} \frac{8\eta^3-3\eta+1}{24}\left\{\left|g''\left(\left(\frac{32\eta^4-8\eta+3}{8(8\eta^3-3\eta+1)}\right)\varkappa_1+\left(\frac{64\eta^3-32\eta^4-16\eta+5}{8(8\eta^3-3\eta+1)}\right)\varkappa_2\right)\right|+\right. \\ \left.\left|g''\left(\left(\frac{64\eta^3-32\eta^4-16\eta+5}{8(8\eta^3-3\eta+1)}\right)\varkappa_1+\left(\frac{32\eta^4-8\eta+3}{8(8\eta^3-3\eta+1)}\right)\varkappa_2\right)\right|\right\},\ for 0\leq\eta\leq\frac{1}{2} \\ \frac{3\eta-1}{24}\left\{\left|g''\left(\left(\frac{8\eta-3}{8(3\eta-1)}\right)\varkappa_1+\left(\frac{16\eta+5}{8(3\eta-1)}\right)\varkappa_2\right)\right|+\left|g''\left(\left(\frac{16\eta+5}{8(3\eta-1)}\right)\varkappa_1+\left(\frac{8\eta-3}{8(3\eta-1)}\right)\varkappa_2\right)\right|\right\}\ for\frac{1}{2}\leq\eta\leq 1. \end{cases}$$

Proof. *Using the Lemma 3.1, we have*

$$\begin{aligned}
&\Bigg|(\eta-1)\,g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}+\\
&\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\Bigg|\\
\leq\ & (\varkappa_2-\varkappa_1)^2\int_0^1|k(\theta)|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
\leq\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta+\right.\\
&\left.\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\right]\\
=\ & \frac{(\varkappa_2-\varkappa_1)^2}{2}\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left(\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\right)+\\
&\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left(\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\right).
\end{aligned}$$

Let $0\leq\eta\leq\frac{1}{2}$*, since* $\left|g''\right|^q$ *is concavity on* $[\varkappa_1,\varkappa_2]$*, and the power-mean inequality, and* $\theta\in[0,1]$*, we have*

$$\begin{aligned}
\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|^q &> \theta\left|g''(\varkappa_1)\right|^q+(1-\theta)\left|g''(\varkappa_2)\right|^q\\
&\geq \left(\theta\left|g''(\varkappa_1)\right|+(1-\theta)\left|g''(\varkappa_2)\right|\right)^q,
\end{aligned}$$

hence,

$$\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|\geq\theta\left|g''(\varkappa_1)\right|+(1-\theta)\left|g''(\varkappa_2)\right|.$$

So, $\left|g''\right|$ *is also concave.*
By the Jensen integral inequality, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
\leq\ & \left(\int_0^{\eta}\theta(\eta-\theta)+\int_{\eta}^{\frac{1}{2}}\theta(\theta-\eta)\right)g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta\\
\leq\ & \left(\int_0^{\eta}\theta(\eta-\theta)+\int_{\eta}^{\frac{1}{2}}\theta(\theta-\eta)\right)d\theta\\
&\times\left|g''\left(\frac{\left(\left(\int_0^{\eta}\theta(\eta-\theta)+\int_{\eta}^{\frac{1}{2}}\theta(\theta-\eta)\right)(\theta\varkappa_1+(1-\theta)\varkappa_2)\right)d\theta}{\left(\int_0^{\eta}\theta(\eta-\theta)+\int_{\eta}^{\frac{1}{2}}\theta(\theta-\eta)\right)d\theta}\right)\right|\\
=\ & \frac{8\eta^3-3\eta+1}{24}\left|g''\left(\left(\frac{32\eta^4-8\eta+3}{8(8\eta^3-3\eta+1)}\right)\varkappa_1+\left(\frac{64\eta^3-32\eta^4-16\eta+5}{8(8\eta^3-3\eta+1)}\right)\varkappa_2\right)\right|, \qquad (25)
\end{aligned}$$

similarly,

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right| g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta \\
\leq\;& \left(\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)+\int_{1-\eta}^{1}(1-\theta)(1-\eta-\theta)\right) g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta \\
\leq\;& \left(\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)+\int_{1-\eta}^{1}(1-\theta)(1-\eta-\theta)\right) \\
&\times\frac{\left|g''\left(\left(\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)+\int_{1-\eta}^{1}(1-\theta)(1-\eta-\theta)\right)(\theta\varkappa_1+(1-\theta)\varkappa_2)\right)d\theta\right|}{\left(\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)+\int_{1-\eta}^{1}(1-\theta)(1-\eta-\theta)\right)d\theta} \\
=\;& \frac{8\eta^3-3\eta+1}{24}\left|g''\left(\left(\frac{64\eta^3-32\eta^4-16\eta+5}{8(8\eta^3-3\eta+1)}\right)\varkappa_1+\left(\frac{32\eta^4-8\eta+3}{8(8\eta^3-3\eta+1)}\right)\varkappa_2\right)\right|, \qquad (26)
\end{aligned}$$

using the (25) -(27) we obtain the first inequality of (24).

Let $\frac{1}{2}\leq\eta\leq 1$ *using the concavity of* $\left|g''\right|^q$*, we have*

$$\begin{aligned}
&\int_{0}^{\frac{1}{2}}\left|\theta(\theta-\eta)\right| g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta \\
\leq\;& \left(\int_{0}^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|d\theta\right)\left|g''\left(\frac{\left(\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|(\theta\varkappa_1+(1-\theta)\varkappa_2)\right)d\theta}{\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|d\theta}\right)\right| \\
=\;& \frac{3\eta-1}{24}\left|g''\left(\left(\frac{8\eta-3}{8(3\eta-1)}\right)\varkappa_1+\left(\frac{16\eta+5}{8(3\eta-1)}\right)\varkappa_2\right)\right|, \qquad (27)
\end{aligned}$$

similarly,

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right| g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\,d\theta \\
\leq\;& \left(\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right|d\theta\right)\left|g''\left(\frac{\left(\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right|(\theta\varkappa_1+(1-\theta)\varkappa_2)\right)d\theta}{\int_{\frac{1}{2}}^{1}\left|(1-\theta)(1-\eta-\theta)\right|d\theta}\right)\right| \\
=\;& \frac{3\eta-1}{24}\left|g''\left(\left(\frac{16\eta+5}{8(3\eta-1)}\right)\varkappa_1+\left(\frac{8\eta-3}{8(3\eta-1)}\right)\varkappa_2\right)\right|, \qquad (28)
\end{aligned}$$

using the (27) -(28) we obtain the second inequality of (24).

This completes the proof. □

Theorem 3.14. *Let* $I\subset\mathbb{R}$ *be an open interval,* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$*. If* $g:I\to\mathbb{R}$ *be a twice differentiable mapping such that* g'' *is integrable and* $0\leq\eta\leq 1$*. If* $\left|g''\right|$ *is a* φ*-quasi-convex on* $[\varkappa_1,\varkappa_2]$*, then the following inequality holds:*

$$\begin{aligned}
&\left|(\eta-1)g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}+\right. \\
&\left.\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\right| \\
\leq\;& \frac{(\varkappa_2-\varkappa_1)^2}{2}\begin{cases}\left(\frac{8\eta^3-3\eta+1}{24}\right)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right] & for\, 0\leq\eta\leq\frac{1}{2}, \\[2mm] \left(\frac{3\eta-1}{24}\right)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right] & for\, \frac{1}{2}\leq\eta\leq 1.\end{cases} \qquad (29)
\end{aligned}$$

Proof. Using the Lemma 3.1 of φ-quasi-convexity of $\left|g''\right|$ we get

$$\begin{aligned}
&\left|(\eta-1)\,g\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\eta\frac{g(\varkappa_1)+g(\varkappa_2)}{2}-\frac{4(1-\alpha)}{\alpha(\varkappa_2-\varkappa_1)}+\right.\\
&\left.\frac{B(\alpha)}{\alpha(\varkappa_2-\varkappa_1)}\left[\left\{\left({}^{CF}_{\varkappa_1}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\frac{\varkappa_1+\varkappa_2}{2}}g\right)(k)\right\}+\left\{\left({}^{CF}_{\frac{\varkappa_1+\varkappa_2}{2}}I^{\alpha}g\right)(k)+\left({}^{CF}I^{\alpha}_{\varkappa_2}g\right)(k)\right\}\right]\right|\\
\leq\;& (\varkappa_2-\varkappa_1)^2\int_0^1 |k(\theta)|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta \qquad (30)
\end{aligned}$$

$$\begin{aligned}
=\;& \frac{(\varkappa_2-\varkappa_1)^2}{2}\int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
&+\frac{(\varkappa_2-\varkappa_1)^2}{2}\int_{\frac{1}{2}}^1\left|(1-\theta)(1-\eta-\theta)\right|\left|g''(\theta\varkappa_1+(1-\theta)\varkappa_2)\right|d\theta\\
=\;& \frac{(\varkappa_2-\varkappa_1)^2}{2}\left[I_1+I_2\right], \qquad (31)
\end{aligned}$$

where we assume that $0\leq\eta\leq\frac{1}{2}$ using the φ-quasi-convexity of $\left|g''\right|$, we get

$$\begin{aligned}
I_1 \leq\;& \int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\\
=\;& \left[\int_0^{\eta}(\theta(\eta-\theta))+\int_{\eta}^{\frac{1}{2}}(\theta(\theta-\eta))\right]\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\\
=\;& \left(\frac{8\eta^3-3\eta+1}{24}\right)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta. \qquad (32)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 \leq\;& \int_{\frac{1}{2}}^{\eta}\left|(1-\theta)(1-\eta-\theta)\right|\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\\
=\;& \left[\int_{\frac{1}{2}}^{1-\eta}(1-\theta)(1-\eta-\theta)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\right.\\
&\left.+\int_{1-\eta}^{1}(1-\theta)(\theta+\eta-1)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\right]\\
=\;& \left(\frac{8\eta^3-3\eta+1}{24}\right)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta. \qquad (33)
\end{aligned}$$

Using equality (32) and (33) in (31) we get the first inequality (29) holds.
where we assume that $\frac{1}{2}\leq\eta\leq 1$, using the φ-quasi-convex of $\left|g''\right|$, we get

$$\begin{aligned}
I_1 \leq\;& \int_0^{\frac{1}{2}}\left|\theta(\theta-\eta)\right|\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\\
=\;& \left[\int_0^{\frac{1}{2}}\theta(\eta-\theta)\right]\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta\\
=\;& \left(\frac{3\eta-1}{24}\right)\max\left[\left(\left|g''(\varkappa_1)\right|,\left|g''(\varkappa_1)\right|+\varphi\left(\left|g''(\varkappa_2)\right|,\left|g''(\varkappa_1)\right|\right)\right)\right]d\theta. \qquad (34)
\end{aligned}$$

similarly

$$\begin{aligned} I_2 &\le \int_{\frac{1}{2}}^{1} \left|(1-\theta)(1-\eta-\theta)\right| \max\left[\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_1)\right| + \varphi\left(\left|g''(\varkappa_2)\right|, \left|g''(\varkappa_1)\right|\right)\right)\right] d\theta \\ &= \int_{\frac{1}{2}}^{1} \left|(1-\theta)(1-\eta-\theta)\right| \max\left[\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_1)\right| + \varphi\left(\left|g''(\varkappa_2)\right|, \left|g''(\varkappa_1)\right|\right)\right)\right] d\theta \\ &= \left(\frac{3\eta-1}{24}\right) \max\left[\left(\left|g''(\varkappa_1)\right|, \left|g''(\varkappa_1)\right| + \varphi\left(\left|g''(\varkappa_2)\right|, \left|g''(\varkappa_1)\right|\right)\right)\right] d\theta. \end{aligned} \tag{35}$$

Using equality (34) and (35) in (31) we get the second inequality of (29). This completes the proof. □

4. Applications to Special Means

We shall consider the following special means:
(a) Arithmetic Mean:

$$A = A(\varkappa_1, \varkappa_2) := \frac{\varkappa_1 + \varkappa_2}{2}, \varkappa_1, \varkappa_2 \ge 0;$$

(b) Geometric Mean:

$$G = G(\varkappa_1, \varkappa_2) := \sqrt{\varkappa_1 \varkappa_2}, \varkappa_1, \varkappa_2 \ge 0;$$

(c) Harmonic Mean:

$$H = H(\varkappa_1, \varkappa_2) := \frac{2\varkappa_1 \varkappa_2}{\varkappa_1 + \varkappa_2}, \varkappa_1, \varkappa_2 \ge 0;$$

(d) Logarithmic Mean:

$$L(\varkappa_1, \varkappa_2) := \frac{\varkappa_2 - \varkappa_1}{\ln \varkappa_2 - \ln \varkappa_1} \varkappa_1, \varkappa_2 > 0, \varkappa_1 \ne \varkappa_2;$$

(e) Generalized logarithmic Mean:

$$L_r^r = L_r^r(\varkappa_1, \varkappa_2) := \left[\frac{\varkappa_2^{r+1} - \varkappa_1^{r+1}}{(r+1)(\varkappa_2 - \varkappa_1)}\right]^{\frac{1}{r}} r \in \mathbb{R} - \{-1, 0\}, \varkappa_1, \varkappa_2 \in \mathbb{R}, \varkappa_1 \ne \varkappa_2.$$

It is well known that L_r^r is monotonic nondecreasing over $r \in \mathbb{R}$ with $L_{-1} = L$. In particular, we have the following inequalities

$$H \le G \le L \le A.$$

Proposition 4.1. *Let $\varkappa_1, \varkappa_2 \in \mathbb{R}$ with $0 < \varkappa_1 < \varkappa_2$. Then we have*

$$\left|\frac{1}{3}\varkappa_1\left(\varkappa_1^4, \varkappa_2^4\right) + \frac{2}{3}\varkappa_1^4(\varkappa_1, \varkappa_2) - L_5^5(\varkappa_1, \varkappa_2)\right| \le \frac{(\varkappa_2 - \varkappa_1)^2}{27}\left[4\varkappa_1^2 + \varkappa_2^2\right]. \tag{36}$$

Proof. The assertion follows from Theorem 3.2 with $g(x) = \frac{x^4}{12}$, $x \in [\varkappa_1, \varkappa_2]$, $B(0) = B(1) = 1$ and $s = \alpha = 1$, $\eta = \frac{1}{3}$, and a simple computation, where $\left|g''\right|$ is s-φ-convex function with $\varphi(\varkappa_1, \varkappa_2) = 2\varkappa_1 + \varkappa_2$ (See Example 1) .

Proposition 4.2. *Let $\varkappa_1, \varkappa_2 \in \mathbb{R}$ with $0 < \varkappa_1 < \varkappa_2$. Then we have*

$$\left|\frac{1}{3}\varkappa_1\left(\varkappa_1^5, \varkappa_2^5\right) + \frac{2}{3}\varkappa_1^5(\varkappa_1, \varkappa_2) - L_6^6(\varkappa_1, \varkappa_2)\right| \le \frac{10(\varkappa_2 - \varkappa_1)^2}{81}\left[2\varkappa_1^3 + \varkappa_1^9 + \varkappa_2^9\right]. \tag{37}$$

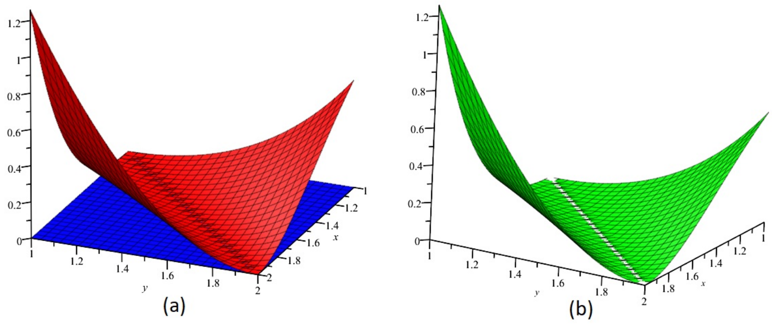

Figure 1: Graphical behaviour of inequality (4.1) (a) $z(\varkappa_1, \varkappa_2)$ and $Z(\varkappa_1, \varkappa_2)$ and $Z(\varkappa_1, \varkappa_2) - z(\varkappa_1, \varkappa_2)$.

Proof. *The assertion follows from Theorem 3.2 with* $g(x) = \frac{x^5}{20}$, $x \in [\varkappa_1, \varkappa_2]$, $B(0) = B(1) = 1$ *and* $s = \alpha = 1$, $\eta = \frac{1}{3}$, *and a simple computation, where* $|g''|$ *is s-φ-convex function with* $\varphi(\varkappa_1, \varkappa_2) = 3\varkappa_2^2(\varkappa_1 - \varkappa_2) + 3\varkappa_2(\varkappa_1 - \varkappa_2)^2 + (\varkappa_1 - \varkappa_2)^3$ *(See Example 2).* □

Proposition 4.3. *Let* $\varkappa_1, \varkappa_2 \in \mathbb{R}$ *with* $0 < \varkappa_1 < \varkappa_2$. *Then we have*

$$\left|\frac{1}{3}\varkappa_1\left(\varkappa_1^2, \varkappa_2^2\right) + \frac{2}{3}\varkappa_1^2(\varkappa_1, \varkappa_2) - L_3^3(\varkappa_1, \varkappa_2)\right| \leq \frac{2(\varkappa_2 - \varkappa_1)^2}{81}. \tag{38}$$

□

Proof. *The assertion follows from Theorem 3.14 with* $g(x) = x^2$, $x \in [\varkappa_1, \varkappa_2]$, *where* $|g''| = 2$, $B(0) = B(1) = 1$ *and* $s = \alpha = 1$, $\eta = \frac{1}{3}$, *and a simple computation, where* $|g''|$ *is φ- quasi-convex function with* $\varphi(\varkappa_1, \varkappa_2) = \varkappa_1 - \varkappa_2$ *(See Example 3)* □

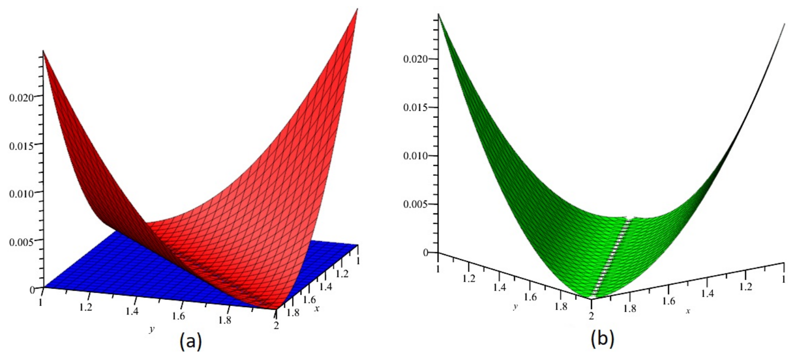

Figure 2: Graphical behaviour of inequality (4.3) (a) $q(\varkappa_1, \varkappa_2)$ and $Q(\varkappa_1, \varkappa_2)$ and $Q(\varkappa_1, \varkappa_2) - q(\varkappa_1, \varkappa_2)$.

5. Graphical Explantation

In order to show the reliability of the obtained inequalities (36) and (38) in the context of φ-convex and φ-quasi convex functions, respectively, we give two three-dimensional graphs. From inequality (36), we define

$$\frac{1}{3}\varkappa_1\left(\varkappa_1^4, \varkappa_2^4\right) + \frac{2}{3}\varkappa_1^4(\varkappa_1, \varkappa_2) - L_5^5(\varkappa_1, \varkappa_2) \quad : \quad = z(\varkappa_1, \varkappa_2)$$

$$\frac{(\varkappa_2-\varkappa_1)^2\left[4\varkappa_1^2+\varkappa_2^2\right]}{27} \quad : \quad = Z(\varkappa_1,\varkappa_2)$$

From inequality (38), we define

$$\frac{1}{3}\varkappa_1\left(\varkappa_1^2,\varkappa_2^2\right)+\frac{2}{3}\varkappa_1^2(\varkappa_1,\varkappa_2)-L_3^3(\varkappa_1,\varkappa_2) \quad : \quad = q(\varkappa_1,\varkappa_2)$$

$$\frac{2(\varkappa_2-\varkappa_1)^2}{81} \quad : \quad = Q(\varkappa_1,\varkappa_2)$$

6. Conclusion

Numerous authors continue to work on various fractional operators to study the integral inequalities that generalized a wide range of different inequalities. One of the similar characteristics in fractional calculus is the restatement of certain integral inequalities for specific function classes using the Caputo-Fabrizio operator. Here, in this article, some inequalities of type Hermite-Hadamard type related to the Caputo-Fabrizio operator on functions whose second derivative is s-φ-convex in the second sense are derived. Furthermore, by analyzing this study, we have established several unique error bounds that were connected to the already published results. For the Caputo-Fabrizio integral operator, numerous fractional versions of various well-known inequalities can be generated, providing the theoretical achievement of extensive work in the field of fractional inequalities. In the future work, similar inequalities would be demonstrated for coordinated convex functions and other types of functions by mathematicians.

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