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New version of Hermite-Hadamard inequality for co-ordinated convex function via generalized conformable integrals

Mehmet Eyüp Kiriş[a], Gözde Bayrak[a,*]

[a]*Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar-Turkey*

Abstract. In this study, a new version of the Hermite-Hadamard inequality for coordinated convex functions is established with the help of generalized conformable fractional integrals. With this approach, a new lower and upper limit was determined and contributed to the generalization of inequality. Additionally, the research contains remarkable findings based on Riemann Liouville and Riemann integral concepts.

1. Introduction

Convex functions are a fundamental and widely-used mathematical concept in various fields of analysis and optimization. Convex functions have notable properties, including the fact that the slope between any two points is either increasing or constant, making them valuable in optimization problems to find minimum or maximum values. On the other hand, Convexity theory has a very important place in the field of inequalities. One of the prominent results in the field of inequalities is the Hermite-Hadamard inequality, which is valid for convex functions (see, e.g., [9], [21, p.137], [13]). These inequalities state that if $\chi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\lambda_1, \lambda_2 \in I$ with $\lambda_1 < \lambda_2$ then,

$$\chi\left(\frac{\lambda_1+\lambda_2}{2}\right) \leq \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta)d\delta \leq \frac{\chi(\lambda_1)+\chi(\lambda_2)}{2}. \tag{1}$$

If χ is concave, the inequality that is stated above is provided reversely. The Hermite-Hadamard inequality establishes a relationship between the mean value of a convex function on an interval and the extremes of that interval. It serves as a powerful tool in various comprehensive analysis and has applications in various fields. This subject has attracted the attention of many mathematicians in recent years. Especially in recent years, many generalizations and extensions of this inequality have been created. The definition of convexity is widely used when creating new versions of the Hermite-Hadamard inequality. To define convexity on coordinates let us first consider a bidimensional interval $\Delta := [\lambda_1, \lambda_2] \times [\mu_1, \mu_2]$ in $\mathbb{R}^2$ with $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. A formal definition for co-ordinated convex function may be stated as follows:

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* Corresponding author: Gözde Bayrak
Email addresses: mkiris@gmail.com (Mehmet Eyüp Kiriş), gozdebayrak92@gmail.com (Gözde Bayrak)

Definition 1.1. *[10] A function $\chi : \Delta \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \Delta$ and $t, s \in [0,1]$, if it satifies the following inequality:*

$$\begin{aligned} & \chi(t\lambda_1 + (1-t)\,\lambda_2, s\mu_1 + (1-s)\,\mu_2) \\ \leq\ & ts\,\chi(\lambda_1, \mu_1) + t(1-s)\chi(\lambda_1, \mu_2) + s(1-t)\chi(\lambda_2, \mu_1) + (1-t)(1-s)\chi(\lambda_2, \mu_2). \end{aligned} \tag{2}$$

In [10], Dragomir proved the Hermite-Hadamard inequality, which formed the basis of this article and is valid for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $\chi : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} \chi\left(\frac{\lambda_1+\lambda_2}{2}, \frac{\mu_1+\mu_2}{2}\right) \leq\ & \frac{1}{2}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi\left(\delta, \frac{\mu_1+\mu_2}{2}\right) d\delta \right. \\ & \left. + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi\left(\frac{\lambda_1+\lambda_2}{2}, \rho\right) d\rho \right] \\ \leq\ & \frac{1}{(\lambda_2-\lambda_1)(\mu_2-\mu_1)}\int_{\lambda_1}^{\lambda_2}\int_{\mu_1}^{\mu_2} \chi(\delta, \rho)\, d\rho\, d\delta \\ \leq\ & \frac{1}{4}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta, \mu_1)\, d\delta + \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta, \mu_2)\, d\delta \right. \\ + & \left. \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_1, \rho)\, d\rho + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_2, \rho)\, d\rho \right] \\ \leq\ & \frac{\chi(\lambda_1, \mu_1) + \chi(\lambda_1, \mu_2) + \chi(\lambda_2, \mu_1) + \chi(\lambda_2, \mu_2)}{4}. \end{aligned} \tag{3}$$

The above inequalities are sharp. The inequalities in (3) hold in the reverse direction if the mapping χ is a co-ordinated concave mapping.

For several results concerning Hermite-Hadamard type inequality for co-ordinated convex functions, we refer the reader to [3], [5], [6], [26].

Fractional calculus ([18], [22], [20], [14] [4]) a branch that extends traditional derivatives and integrals to non-integer orders; it is a powerful tool for solving complex problems in many applied fields such as physics, engineering and signal processing. Thanks to the fractional integral operators used, the generalization of many integral inequalities has been introduced to the literature. More than one fractional operator has been defined so far: Caputo, Riemann-Liouville, Hadamard, Katugampola are a few of them. Conformable fractional calculus is a particular aspect of fractional calculus that we examine and allows us to integrate congruent fractional derivatives. These integrals have vital applications in modeling real-world phenomena such as diffusion processes, wave propagation, and viscoelastic materials. The concept of harmonic fractional calculus enriches the framework of fractional calculus, offers new methods for dealing with complex systems, and provides valuable information in various scientific research. The conformable fractional integral operator was used throughout this study. For all this, please see [16] [2], [1], [12], [28], [17], [25] The known definition of conformable fractional integral account is as follows:

Definition 1.3. *[14] For $\xi \in L_1[\eta_1, \eta_2]$, the conformable fractional integral operator ${}^{\beta}I^{\alpha}_{\eta_1+}\xi(\delta)$ and ${}^{\beta}I^{\alpha}_{\eta_2-}\xi(\delta)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are presented by*

$${}^{\beta}\mathcal{I}^{\alpha}_{\eta_1+}\xi(\delta) = \frac{1}{\Gamma(\beta)}\int_{\eta_1}^{\delta} \left(\frac{(\delta-\eta_1)^{\alpha} - (t-\eta_1)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\xi(t)}{(t-\eta_1)^{1-\alpha}}\,dt, \quad t > \eta_1 \tag{4}$$

and

$$ {}^{\beta}\mathcal{I}^{\alpha}_{\eta_2-}\xi(\delta) = \frac{1}{\Gamma(\beta)} \int_{\delta}^{\eta_2} \left(\frac{(\eta_2-\delta)^{\alpha} - (\eta_2-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\xi(t)}{(\eta_2-t)^{1-\alpha}} dt, \quad t < \eta_2, \tag{5} $$

respectively.

If we consider $\alpha = 1$, then the fractional integral in (4) reduces to the Riemann-Liouville fractional integral.

Definition 1.4. *[7] Let $\xi \in L_1([\eta_1, \eta_2] \times [\vartheta_1, \vartheta_2])$ and let $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, α, $\beta \in \mathbf{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. The generalized conformable fractional integral of order α, β of $\xi(\delta, \rho)$ is defined by;*

$$ \left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi \right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\eta_1}^{\delta} \int_{\vartheta_1}^{\rho} \left(\frac{(\delta-\eta_1)^{\gamma_1} - (t-\eta_1)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \times \left(\frac{(\rho-\vartheta_1)^{\gamma_2} - (s-\vartheta_1)^{\gamma_2}}{\gamma_2} \right)^{\beta-1} \frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}} dsdt \right], \tag{6} $$

$$ \left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi \right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\delta}^{\eta_2} \int_{\vartheta_1}^{\rho} \left(\frac{(\eta_2-\delta)^{\gamma_1} - (\eta_2-t)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \times \left(\frac{(\rho-\vartheta_1)^{\gamma_2} - (s-\vartheta_1)^{\gamma_2}}{\gamma_2} \right)^{\beta-1} \frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}} dsdt \right], \tag{7} $$

$$ \left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi \right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\eta_1}^{\delta} \int_{\rho}^{\vartheta_2} \left(\frac{(\delta-\eta_1)^{\gamma_1} - (t-\eta_1)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \times \left(\frac{(\vartheta_2-\rho)^{\gamma_2} - (\vartheta_2-s)^{\gamma_2}}{\gamma_2} \right)^{\beta-1} \frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}} dsdt \right], \tag{8} $$

and

$$ \left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi \right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\delta}^{\eta_2} \int_{\rho}^{\vartheta_2} \left(\frac{(\eta_2-\delta)^{\gamma_1} - (\eta_2-t)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \times \left(\frac{(\vartheta_2-\rho)^{\gamma_2} - (\vartheta_2-s)^{\gamma_2}}{\gamma_2} \right)^{\beta-1} \frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}} dsdt \right], \tag{9} $$

the generalized conformable fractional integrals.

Similar to Definition 2 and Definition 3, we introduce the following fractional integrals:

Definition 1.5. *Let $\xi \in L_1([\eta_1, \eta_2] \times [\vartheta_1, \vartheta_2])$ and let $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, α, $\beta \in \mathbf{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. In this case, the following equations can be written:*

$$ \left({}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi \right)\left(\delta, \frac{\vartheta_1+\vartheta_2}{2} \right) = \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\delta} \left(\frac{(\delta-\eta_1)^{\gamma_1} - (t-\eta_1)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \frac{\xi\left(t, \frac{\vartheta_1+\vartheta_2}{2}\right)}{(t-\eta_1)^{1-\gamma_1}} dt, \ \delta > \eta_1, \tag{10} $$

$$ \left({}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi \right)\left(\delta, \frac{\vartheta_1+\vartheta_2}{2} \right) = \frac{1}{\Gamma(\alpha)} \int_{\delta}^{\eta_2} \left(\frac{(\eta_2-\delta)^{\gamma_1} - (\eta_2-t)^{\gamma_1}}{\gamma_1} \right)^{\alpha-1} \frac{\xi\left(t, \frac{\vartheta_1+\vartheta_2}{2}\right)}{(\eta_2-t)^{1-\gamma_1}} dt, \ \delta < \eta_2, \tag{11} $$

$$\left({}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\rho\right)=\frac{1}{\Gamma(\beta)}\int_{\vartheta_1}^{\rho}\left(\frac{(\rho-\vartheta_1)^{\gamma_2}-(s-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi\left(\frac{\eta_1+\eta_2}{2},s\right)}{(s-\vartheta_1)^{1-\gamma_2}}ds,\ \rho>\vartheta_1, \tag{12}$$

and

$$\left({}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\rho\right)=\frac{1}{\Gamma(\beta)}\int_{\rho}^{\vartheta_2}\left(\frac{(\vartheta_2-\rho)^{\gamma_2}-(\vartheta_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi\left(\frac{\eta_1+\eta_2}{2},s\right)}{(\vartheta_2-s)^{1-\gamma_2}}ds,\ \rho<\vartheta_2. \tag{13}$$

Meanwhile, Hyder et al., in a study they conducted in 2022, established a new version of the Hermite-Hadamard inequality using generalized Riemann-Liouville fractional integrals for convex functions. This Theorem, which is the basis of this study, is as follows:

Theorem 1.6. *[11] Assume ξ is a convex function that goes from $[\eta_1,\eta_2]$ into $\mathbb{R}$. Then, for $Re(\beta)>0$ and $\alpha\in[0,1]$, the inequalities below are valid for the generalized Riemann-Liouville fractional integrals.*

$$\xi\left(\frac{\eta_1+\eta_2}{2}\right)\le\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(\eta_2-\eta_1)^{\alpha\beta}}\left[{}^{\beta}_{\eta_1}I^{\alpha}\xi\left(\frac{\eta_1+\eta_2}{2}\right)\ +^{\beta}I^{\alpha}_{\eta_2}\xi\left(\frac{\eta_1+\eta_2}{2}\right)\right]\le\frac{\xi(\eta_1)+\xi(\eta_2)}{2}. \tag{14}$$

For some results connected with fractional integral inequalities see [23], [27], [8], [15], [19], [24].

The purpose of this article is to establish the Hermite-Hadamard-type inequality for co-ordinated convex mappings using the generalized conformable fractional integral operator. Then, new lower and upper limits will be determined for the new Hermite-Hadamard inequality, and a more general version will be created.

2. New Version of Hermite-Hadamard Inequality for Co-Ordinated Convex Functions

In this part, we obtain a new version of Hermite-Hadamard inequality that is applicable to the conformable fractional integrals.

Theorem 2.1. *Assume ξ is a co-ordinated convex function that goes from $[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]$ into $\mathbb{R}$ and let $\gamma_1\neq 0$, $\gamma_2\neq0$, α, $\beta\in(0,1]$, $Re(\alpha)>0$ and $Re(\beta)>0$. The following inequality holds for generalized conformable fractional integrals.*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (15)\\
\le\ &\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\le\ &\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}$$

Proof. Since ξ is co-ordinated convex on $[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]$, for $t,\ s\in[0,1]$, we can write

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
=\ &\xi\left(\frac{1}{4}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)+\frac{1}{4}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right.\\
&\left.+\frac{1}{4}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)+\frac{1}{4}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right).
\end{aligned}$$

With the help of the co-ordinated convexity of ξ, we have

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (16)\\
\le\ & \frac{1}{4}\left(\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)+\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right.\\
&\left.+\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)+\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right)\\
\le\ & \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}$$

If we multiply the inequality (16) by $\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\cdot(1-t)^{\gamma_1-1}\cdot\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\cdot(1-s)^{\gamma_2-1}$ and integrate the resulting inequality on $[0,1]\times[0,1]$, we have

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (17)\\
&\times\left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt\right\}\\
\le\ & \frac{1}{4}\left[\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right.\right.\\
&\left.\times\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\right)\\
&+\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right.\\
&\left.\times\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\right)\\
&+\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right.\\
&\left.\times\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\right)\\
&+\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right.\\
&\left.\left.\times\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\right)\right]\\
\le\ & \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}\\
&\times\left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt\right\}.
\end{aligned}$$

Using the change of the variable, we get

$$\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right. \tag{18}$$

$$\left.\times\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\right)$$

$$= \frac{2}{\eta_2-\eta_1}\frac{2}{\vartheta_2-\vartheta_1}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{1-(\frac{2\eta_1-2u}{\eta_1-\eta_2})^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{2\eta_1-2u}{\eta_1-\eta_2}\right)^{\gamma_1-1}\right.$$

$$\left.\times\left(\frac{1-(\frac{2\vartheta_1-2v}{\vartheta_1-\vartheta_2})^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\left(\frac{2\vartheta_1-2v}{\vartheta_1-\vartheta_2}\right)^{\gamma_2-1}\xi(u,v)dvdu\right]$$

$$= \left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\eta_1+\eta_2}{2}-\eta_1)^{\gamma_1}-(u-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\right.$$

$$\left.\times\left(\frac{(\frac{\vartheta_1+\vartheta_2}{2}-\vartheta_1)^{\gamma_2}-(v-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\xi(u,v)\frac{1}{(v-\vartheta_1)^{1-\gamma_2}}\frac{1}{(u-\eta_1)^{1-\gamma_1}}dvdu\right]$$

$$= \frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right).$$

Similarly we have,

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \tag{19}$$

$$\times\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt$$

$$= \frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right),$$

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \tag{20}$$

$$\times\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt$$

$$= \frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right),$$

and

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \tag{21}$$

$$\times\xi\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt$$

$$= \frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right).$$

On the other side, we have

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt=\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta}. \tag{22}$$

If we substitute the equalities (18)-(21) in (17), then we get

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta} \qquad (23)\\
\leq\ & \frac{1}{4}\left[\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&+\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\left.+\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\ & \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta},
\end{aligned}$$

which concludes the proof. □

Remark 2.2. *In Theorem 3, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, the following inequalities are achieved*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (24)\\
\leq\ & \frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^{\alpha}(\vartheta_2-\vartheta_1)^{\beta}}\left[{}^{1,1}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{1,1}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\ & \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}$$

Remark 2.3. *In Theorem 3, if we choose $\gamma_1 = 1$, $\gamma_2 = 1$, $\alpha = 1$ and $\beta = 1$, then we have,*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (25)\\
\leq\ & \frac{\Gamma(2)\Gamma(2)}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left[{}^{1,1}I^{1,1}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{1,1}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{1,1}I^{1,1}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{1,1}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
=\ & \frac{1}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\xi(t,s)dsdt+\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\xi(t,s)dsdt\right.
\end{aligned}$$

$$+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\xi(t,s)dsdt+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\xi(t,s)dsdt\Bigg]$$

$$\leq \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.$$

Then the inequalities (25) become the inequalities (3).

Theorem 2.4. *Let $\xi:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be co-ordinated convex on $\Delta:=[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]$ in $\mathbb{R}^2$ with $0\leq\eta_1<\eta_2$, $0\leq\vartheta_1<\vartheta_2$ and $\xi\in L_1(\Delta)$. Then the inequality below holds.*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (26)\\
\leq\ & \frac{2^{\gamma_1\alpha-2}\Gamma(\alpha+1)\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
&+\frac{2^{\gamma_2\beta-2}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\ & \frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\ & \frac{2^{\gamma_1\alpha-3}\Gamma(\alpha+1)\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)+{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\right.\\
&\left.+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)\right]\\
&+\frac{2^{\gamma_1\beta-3}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\ & \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}$$

Proof. Since $\xi:\Delta\to\mathbb{R}$ is co-ordinated convex function, $\chi_\delta:[\vartheta_1,\vartheta_2]\to\mathbb{R}$, $\chi_\delta(\rho)=\xi(\delta,\rho)$ is convex on $[\vartheta_1,\vartheta_2]$ for all $\delta\in[\eta_1,\eta_2]$. Then by using inequalities (14), we can write

$$\chi_\delta\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\leq\frac{2^{\gamma_2\beta-1}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\chi_\delta\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\chi_\delta\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\leq\frac{\chi_\delta(\vartheta_1)+\chi_\delta(\vartheta_2)}{2},\delta\in[\eta_1,\eta_2]$$

That is,

$$\begin{aligned}
\xi\left(\delta,\frac{\vartheta_1+\vartheta_2}{2}\right)\ \leq\ & \frac{2^{\gamma_2\beta-1}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\right. \qquad (27)\\
&\left.+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right]\leq\frac{\xi(\delta,\vartheta_1)+\xi(\delta,\vartheta_2)}{2}.
\end{aligned}$$

for all $\delta \in [\eta_1, \eta_2]$. Then multiplying both sides of (27) by $\frac{2^{(\gamma_1\alpha-1)}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{1}{(\delta-\eta_1)^{1-\gamma_1}}$ and integrating with respect to δ over $\left[\eta_1, \frac{\eta_1+\eta_2}{2}\right]$ respectively, we have,

$$\frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi\left(\delta,\frac{\vartheta_1+\vartheta_2}{2}\right)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta \tag{28}$$

$$\begin{aligned}
\leq\ & \frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\alpha\beta\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \\
& \times\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\delta-\eta_1)^{1-\gamma_1}(\rho-\vartheta_1)^{1-\gamma_2}}d\rho d\delta\right. \\
& \left.+\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\delta-\eta_1)^{1-\gamma_1}(\vartheta_2-\rho)^{1-\gamma_2}}d\rho d\delta\right] \\
\leq\ & \frac{2^{\gamma_1\alpha-2}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_1)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right. \\
& \left.+\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_2)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right].
\end{aligned}$$

Similarly, let's multiply the integral of (27) by $\frac{2^{(\gamma_1\alpha-1)}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{1}{(\eta_2-\delta)^{1-\gamma_1}}$ and integrate with respect to δ in the interval $\left[\frac{\eta_1+\eta_2}{2},\eta_2\right]$, we have

$$\frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi\left(\delta,\frac{\vartheta_1+\vartheta_2}{2}\right)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta \tag{29}$$

$$\begin{aligned}
\leq\ & \frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\alpha\beta\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \\
& \times\left[\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\eta_2-\delta)^{1-\gamma_1}(\rho-\vartheta_1)^{1-\gamma_2}}d\rho d\delta\right. \\
& \left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\eta_2-\delta)^{1-\gamma_1}(\vartheta_2-\rho)^{1-\gamma_2}}d\rho d\delta\right] \\
\leq\ & \frac{2^{\gamma_1\alpha-2}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_1)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right.
\end{aligned}$$

$$+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_2)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\Bigg].$$

In the same way, from the inequality of (14) as $\chi_\rho : [\eta_1,\eta_2]\to\mathbb{R}$, $\chi_\rho(\delta)=\xi(\delta,\rho)$ is convex function:

$$\chi_\rho\left(\frac{\eta_1+\eta_2}{2}\right)\le\frac{2^{\gamma_1\alpha-1}\Gamma(\alpha+1)\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^\alpha_{\eta_1^+}\chi_\rho\left(\frac{\eta_1+\eta_2}{2}\right)+{}^{\gamma_1}I^\alpha_{\eta_2^-}\chi_\rho\left(\frac{\eta_1+\eta_2}{2}\right)\right]\le\frac{\chi_\rho(\eta_1)+\chi_\rho(\eta_2)}{2},\ \rho\in[\vartheta_1,\vartheta_2]$$

That is,

$$\begin{aligned}\xi\left(\frac{\eta_1+\eta_2}{2},\rho\right)\le\ &\frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^\alpha}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\Bigg[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\rho)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\\&+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\rho)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\Bigg]\le\frac{\xi(\eta_1,\rho)+\xi(\eta_2,\rho)}{2}.\end{aligned}\tag{30}$$

for all $\rho\in[\vartheta_1,\vartheta_2]$. Then multiplying both sides of (30) by $\frac{2^{\gamma_2\beta-1}\beta\gamma_2^\beta}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{1}{(\rho-\vartheta_1)^{1-\gamma_2}}$ and integrating with respect to ρ over $\left[\vartheta_1,\frac{\vartheta_1+\vartheta_2}{2}\right]$, respectively we have,

$$\begin{aligned}&\frac{2^{\gamma_2\beta-1}\beta\gamma_2^\beta}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi\left(\frac{\eta_1+\eta_2}{2},\rho\right)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\\
\le\ &\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\alpha\beta\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\times\Bigg[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\rho-\vartheta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\delta-\eta_1)^{1-\gamma_1}(\rho-\vartheta_1)^{1-\gamma_2}}d\rho d\delta\\
&+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\vartheta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\vartheta_2-\delta)^{1-\gamma_1}(\rho-\vartheta_1)^{1-\gamma_2}}d\rho d\delta\Bigg]\\
\le\ &\frac{2^{\gamma_2\beta-2}\beta\gamma_2^\beta}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\Bigg[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_1,\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\\
&+\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_2,\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\Bigg].\end{aligned}\tag{31}$$

Similarly, let's multiply the integral of (30) by $\frac{2^{\gamma_2\beta-1}\beta\gamma_2^\beta}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{1}{(\vartheta_2-\rho)^{1-\gamma_2}}$ and integrate with respect to ρ in the interval $\left[\frac{\vartheta_1+\vartheta_2}{2}, \vartheta_2\right]$, we have,

$$\begin{aligned} &\frac{2^{\gamma_2\beta-1}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi\left(\frac{\eta_1+\eta_2}{2},\rho\right)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho \\ \leq\ &\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\alpha\beta\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \\ &\times\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\delta-\eta_1)^{1-\gamma_1}(\vartheta_2-\rho)^{1-\gamma_2}}d\rho d\delta\right. \\ &\left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\delta,\rho)}{(\eta_2-\delta)^{1-\gamma_1}(\vartheta_2-\rho)^{1-\gamma_2}}d\rho d\delta\right] \\ \leq\ &\frac{2^{\gamma_2\beta-2}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_1,\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right. \\ &\left.+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_2,\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right]. \end{aligned} \tag{32}$$

If we add the (28), (29), (31) and (32) inequalities and divide by 2, we get,

$$\begin{aligned} &\frac{2^{\gamma_1\alpha-2}\Gamma(\alpha+1)\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right] \\ &+\frac{2^{\gamma_2\beta-2}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right] \\ \leq\ &\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right. \\ &\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right] \\ \leq\ &\frac{2^{\gamma_1\alpha-3}\Gamma(\alpha+1)\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)+{}^{\gamma_1}I^{\alpha}_{\eta_1^+}f\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\right. \\ &\left.+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}f\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)\right] \\ &+\frac{2^{\gamma_1\beta-3}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}f\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}f\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}f\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)\right. \\ &\left.+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}f\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)\right] \end{aligned}$$

which give the second and the third inequalities in (26).

Now, let's write $\delta=\frac{\eta_1+\eta_2}{2}$ on the left side of the (27) inequality, we also have

$$\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{33}$$

$$\leq \frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi\left(\delta,\frac{\vartheta_1+\vartheta_2}{2}\right)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right.$$
$$\left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi\left(\delta,\frac{\vartheta_1+\vartheta_2}{2}\right)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right]$$

and then write $\rho=\frac{\vartheta_1+\vartheta_2}{2}$ to left side of the (30) inequality, we have,

$$\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{34}$$
$$\leq \frac{2^{\gamma_2\beta-1}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\frac{\eta_1+\eta_2}{2},\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\right.$$
$$\left.+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\frac{\eta_1+\eta_2}{2},\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right].$$

When we add the above (33) and (34) inequalities side by side and divide by 2, we get the following inequality:

$$\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{35}$$
$$\leq \frac{2^{\gamma_1\alpha-2}\Gamma(\alpha+1)\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)\right]$$
$$+\frac{2^{\gamma_2\beta-2}\Gamma(\beta+1)\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\right].$$

The inequality in (35) is the first inequality of the Theorem (26).

Finally, by using the second inequality in (14) and assuming $\rho=\vartheta_1$ on the right-hand side of the (30) inequality, we have

$$\frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_1)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right. \tag{36}$$
$$\left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_1)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right]\leq\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_2,\vartheta_1)}{2}.$$

And similiarly, assuming $\rho=\vartheta_2$ on the right-hand side of the (30) inequality, we have

$$\frac{2^{\gamma_1\alpha-1}\alpha\gamma_1^{\alpha}}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_2)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right. \tag{37}$$
$$\left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\left(\frac{(\frac{\eta_2-\eta_1}{2})^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\xi(\delta,\vartheta_2)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right]\leq\frac{\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_2)}{2}.$$

Likewise, assuming $\delta = \eta_1$ on the right-hand side of the (27) inequality, we have

$$\frac{2^{\gamma_2\beta-1}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_1,\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\right.$$
$$\left.+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_1,\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right]\leq\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)}{2}. \tag{38}$$

And, assuming $\delta = \eta_2$ on the right-hand side of the (27) inequality, we have

$$\frac{2^{\gamma_2\beta-1}\beta\gamma_2^{\beta}}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\rho-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_2,\rho)}{(\rho-\vartheta_1)^{1-\gamma_2}}d\rho\right.$$
$$\left.+\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2}\left(\frac{(\frac{\vartheta_2-\vartheta_1}{2})^{\gamma_2}-(\vartheta_2-\rho)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(\eta_2,\rho)}{(\vartheta_2-\rho)^{1-\gamma_2}}d\rho\right]\leq\frac{\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{2}. \tag{39}$$

When the inequalities between (36)-(39) are added side by side and divided by 4, the 4*th* inequality of the Theorem (26) is found. □

Remark 2.5. *In Theorem 4, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, the following inequalities are achieved*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (40)\\
\leq\;& \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(\eta_2-\eta_1)^{\alpha}}\left[{}^1I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^1I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
&+\frac{2^{\beta-2}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\left[{}^1I^{\beta}_{\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^1I^{\beta}_{\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\;& \frac{2^{\alpha-1}2^{\gamma\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^{\alpha}(\vartheta_2-\vartheta_1)^{\beta}}\left[{}^{1,1}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{1,1}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{1,1}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\;& \frac{2^{\alpha-3}\Gamma(\alpha+1)}{(\eta_2-\eta_1)^{\alpha}}\left[{}^1I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)+{}^1I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)+{}^1I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\right.\\
&\left.+{}^1I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)\right]\\
&+\frac{2^{\beta-3}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\beta}}\left[{}^1I^{\beta}_{\vartheta_1^+}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^1I^{\beta}_{\vartheta_1^+}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^1I^{\beta}_{\vartheta_2^-}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+\right.\\
&\left.+{}^1I^{\beta}_{\vartheta_2^-}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
\leq\;& \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}$$

Remark 2.6. *In Theorem 4, if we choose $\gamma_1 = 1$, $\gamma_2 = 1$, $\alpha = 1$ and $\beta = 1$,we have,*

$$\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{41}$$

$$\leq \frac{1}{2(\eta_2-\eta_1)}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}} \xi(t,\frac{\vartheta_1+\vartheta_2}{2})dt + \int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2} \xi(t,\frac{\vartheta_1+\vartheta_2}{2})dt\right]$$

$$+\frac{1}{2(\vartheta_2-\vartheta_1)}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}} \xi(\frac{\eta_1+\eta_2}{2},s)ds + \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \xi(\frac{\eta_1+\eta_2}{2},s)ds\right]$$

$$\leq \frac{1}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}} \xi(t,s)dsdt + \int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \xi(t,s)dsdt\right.$$

$$\left.+\int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}} \xi(t,s)dsdt + \int_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \xi(t,s)dsdt\right]$$

$$\leq \frac{1}{4(\eta_2-\eta_1)}\left[\int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}} \xi(t,\vartheta_1)dt + \int_{\eta_1}^{\frac{\eta_1+\eta_2}{2}} \xi(t,\vartheta_2)dt\right]$$

$$+\frac{1}{4(\vartheta_2-\vartheta_1)}\left[\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}} \xi(\eta_1,s)ds + \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \xi(\eta_2,s)ds\right]$$

$$\leq \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.$$

Then the inequalities (41) become the inequalities (3).

3. Conclusion

In this research, we acquired some inequality of Hermite-Hadamard-type for co-ordinated convex functions by means of conformable fractional integrals. In future studies, it is possible to obtain new version of the inequalities we have obtained by using convexity definitions in different types of co-ordinates. Also, new effective and original inequalities can be obtained through different types of fractional integral definitions.

Author contributions

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Conflict interests

The authors declare that they have no conflict interests.

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