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Spectral properties for unbounded block operator matrices via polynomially Riesz perturbations

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Abstract. As well-known, the perturbation theory of polynomially Riesz operators is an attractive way to characterize certain spectral analysis in Fredholm theory, it is also a tool of great significance in the matrix framework. The first aim of this paper is to find some new arguments of perturbations allowing us to provide some original left-right Fredholm properties of 3×3 unbounded block operator matrix form defined with maximal domain and to provide an amelioration and a continuation of the recent work invested by Abdmouleh, Khlif and Walha in [Spectral description of Fredholm operators via polynomially Riesz operators perturbation, Georgian Math. J. 29(3) (2022), 317-333.] in the context of the spectral analysis in Fredholm theory of the last 3×3 block operator matrices. Our second goal is to express the incidence of some essential spectra of the before-cited model of operator matrices involving the theory of polynomially Riesz operators perturbation. Our approach allows us to present a new description in the theory of unbounded operator matrices via a new technique and new arguments of perturbations coined as polynomially Riesz perturbations.

1. Introduction

Spectral theory is an essential part of functional analysis. In recent years, it has witnessed an explosive development and it has various applications in many sections of mathematics and physics including function theory, matrix theory, control theory, differential and integral equations and complex analysis (see [5, 7, 9] and references therein). The operator concept is one of the most general in the branch of functional analysis that studies the properties of operators and the application of operators to the solution of various problems in the mathematical physics field. For example, we can refer the readers to references [2, 5, 10].

Strictly speaking, the study of spectral problems in view of the theory of Fredholm operators and their derivative sets has been extensively increased and studied in the mathematical framework. The before-mentioned notion of operators has led to compelling advances which supply a boost for the study of many different subjects likewise: in the theory of perturbations, in theoretical physics, in the exploration of various classes of singular integral equations and getting lovely properties of certain differential operators. Among the works in this direction we quote, for example, [4, 15].

Further there are many types of spectra, both for bounded or unbounded linear operators, with impressive applications, for example the approximate point spectrum, essential spectrum, local spectrum, etc.

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Specifically, the invariance of essential spectra for bounded linear operators acting on Banach spaces was advocated firstly in [12].

Afterwards, the class of polynomially Riesz operators, dates back to 2006, and was anticipated by K. Latrach, M. Paoli and M. A. Taoudi [11] where they characterized the concept of polynomially Riesz strongly continuous semigroups and later by S. C. Ž. Žlatanović et al. in [17–19]. Such notion of operators appears viewed as a generalization of the class of Fredholm perturbations, polynomially Fredholm perturbations, Riesz operators and polynomially compact operators extremely developed in the literature [1, 3, 15, 16].

Hereafter, the theory of operator matrices has attracted the attention of several mathematicians and researchers because of their rich applications in different areas of pure and applied mathematics (see [2, 8]). For this reason and over the last two decades, this kind of theory occurs as a new line of attack in spectral theory and remains as an impressive tool in the proof of the interaction problems between Fredholm operators and their derivative classes and their corresponding essential spectra. In particular, many authors in [2], have paid attention to the research of the issue related to the spectral characteristics of unbounded 2×2 operators matrices with mixed and maximal domain which developed due to powerful classes of two-sided ideal of the set of bounded operators.

The main objective of the manuscript is to investigate under new sufficient assumptions involving the concept of polynomially Riesz operators in order to resolve the invariance problem of some essential spectra of an unbounded 3×3 block operator matrix defined with maximal domain which is not taken considerably in the literature. By way of explanation, the common tool in this investigation is based on the use of invertible modulo compact operators as well as the one sided invertible operators called left and right Fredholm operators. This exploration involves a graceful use of the properties of Riesz operators introduced by S. R. Caradus et al. in [4] simultaneously with the concept of polynomially Riesz operators in order to characterize the interaction between upper, lower semi Fredholm, left-right Fredholm and left-right Weyl essential spectra of unbounded 3×3 block operator matrix defined with maximal domain and their diagonal entries.

In other words, an impressive perspective and a powerful approach of the concept of polynomially Riesz operators perturbations with their properties are investigated to present some left-right Fredholm spectral properties under less conditions related the components entries of the operator matrix $\mathcal{M}_0$ (see Theorem 3.6), and an exact description of some essential spectra of $\mathcal{M}$ is shown in Corollary 3.7 (see Section 3, for more details). Thus, these results appear as natural taking note of scientific progress in this field.

2. Basic concepts and mathematical tools

This section contains basic definitions and results that we will need in the sequel.

For the reader's convenience and in order to clarify our subsequent development in the next section, we start with the following list by introducing the usual notations and symbols needed later.

Notations and symbols:

E :	Banach space,
$\mathcal{D}(T)$:	the domain of T,
$\mathrm{Ker}(T)$:	the null space of T,
$\mathrm{R}(T)$:	the range of T,
$\alpha(T) := \dim\big(\mathrm{Ker}(T)\big)$	the nullity of T, defined as the dimension of $\mathrm{Ker}(T)$,
$\beta(T) := \mathrm{codim}\big(\mathrm{R}(T)\big)$	the deficiency of T, defined as the codimension of $\mathrm{R}(T)$,
$\sigma(T)$:	the spectrum set of T,
$r_s(T)$:	the resolvent set of T.

Definition 2.1.

(i) We define the set of upper, respectively, lower semi-Fredholm operators on E as:

$$\Phi_+(\mathrm{E}) := \{T \in \mathcal{C}(\mathrm{E}) : \quad \alpha(T) < \infty \text{ and } \mathrm{R}(T) \text{ is closed in E}\}$$

resp.

$$\Phi_{-}(\mathrm{E}) := \{T \in \mathcal{C}(\mathrm{E}) : \ \beta(T) < \infty \text{ and } \mathrm{R}(T) \text{ is closed in E}\}.$$

(ii) Set of Fredholm operators on E is defined by:

$$\Phi(\mathrm{E}) := \Phi_{+}(\mathrm{E}) \cap \Phi_{-}(\mathrm{E}).$$

(iii) For an operator $T \in \Phi(\mathrm{E})$, the index of T is defined by the number:

$$i(T) = \alpha(T) - \beta(T).$$

After this, and using the previous definitions, the following sets will be essential clarified as:

Definition 2.2.

(i) the sets of left and right Fredholm operators on E, respectively, are defined as:

$$\Phi^{\ell}(\mathrm{E}) := \{T \in \Phi_{+}(\mathrm{E}) : \ \mathrm{R}(T) \ \text{ is a complemented subset of } \ \mathrm{E}\}$$

and

$$\Phi^{r}(\mathrm{E}) := \{T \in \Phi_{-}(\mathrm{E}) : \ \mathrm{Ker}(T) \ \text{ is a complemented subset of } \ \mathrm{E}\}.$$

(ii) The sets of left and right Weyl operators on E are defined respectively by:

$$\mathcal{W}^{\ell}(\mathrm{E}) := \{T \in \mathcal{C}(\mathrm{E}) : T \in \Phi^{\ell}(\mathrm{E}) \text{ and } i(T) \leq 0\}$$

and

$$\mathcal{W}^{r}(\mathrm{E}) := \{T \in \mathcal{C}(\mathrm{E}) : T \in \Phi^{r}(\mathrm{E}) \text{ and } i(T) \geq 0\}.$$

Consequently, we deduce the set of Weyl operators on E, denoted by $\mathcal{W}(\mathrm{E})$, defined as:

$$\mathcal{W}(\mathrm{E}) := \mathcal{W}^{\ell}(\mathrm{E}) \cap \mathcal{W}^{r}(\mathrm{E}) := \{T \in \Phi(\mathrm{E}) : \ i(T) = 0\}.$$

In order to translate the above results in terms of essential spectra, the following definition may be essential.

Definition 2.3. Let $T \in \mathcal{C}(\mathrm{E})$. We define:

(i) The upper (resp. lower) Fredholm essential spectrum of T, denoted by $\sigma^{+}_{ess}(T)$ (resp. $\sigma^{-}_{ess}(T)$), as the following set:

$$\sigma^{+}_{ess}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \Phi_{+}(\mathrm{E})\} \text{ (resp. } \sigma^{-}_{ess}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \Phi_{-}(\mathrm{E})\}).$$

(ii) The left (resp. right) Fredholm essential spectrum of T, denoted by $\sigma^{\ell}_{ess}(T)$ (resp. $\sigma^{r}_{ess}(T)$), as the following set:

$$\sigma^{\ell}_{e}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \Phi^{\ell}(\mathrm{E})\} \ \text{ (resp. } \sigma^{r}_{ess}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \Phi^{r}(\mathrm{E})\}).$$

(iii) The left (resp. right) Weyl spectrum of T, denoted by $\sigma^{\ell}_{w}(T)$ (resp. $\sigma^{r}_{w}(T)$) as:

$$\sigma^{\ell}_{w}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \mathcal{W}^{\ell}(\mathrm{E})\} \ \text{ (resp. } \sigma^{r}_{w}(T) := \{\eta \in \mathbb{C} : \eta - T \notin \mathcal{W}^{r}(\mathrm{E})\}).$$

Moreover, the main tool of this paper is based on the following definition concerning Fredholm operators. For the sake of convenience, we refer to the books of V. Müller [13] and M. Schechter [14].

Definition 2.4. Let E be a Banach space.

(i) An operator $T \in \mathcal{L}(\mathrm{E})$ is said to have a left Fredholm inverse if there exists $T^{\ell} \in \mathcal{L}(\mathrm{E})$ such that $I - T^{\ell}T \in \mathcal{K}(\mathrm{E})$.

(ii) An operator $T \in \mathcal{L}(\mathrm{E})$ is said to have a right Fredholm inverse if there exists $T^{r} \in \mathcal{L}(\mathrm{E})$ such that $I - TT^{r} \in \mathcal{K}(\mathrm{E})$.

Remark 2.5.
According to Definition 2.4 with Equation (2.1) in [2], we can extend the notion of left and right Fredholm inverses to the case of closed densely defined operators. For more details, see [2, Section 2].

Hereafter, let $T \in \mathcal{C}(\mathrm{E})$ and $\mathrm{E}_T := (\mathcal{D}(T), \|.\|_T)$ designates a Banach space endowed with the graph norm $\|.\|_T$ (that is, $\|x\|_T := \|x\| + \|Tx\|$). We define these sets $\Theta_{T,\ell}(\mathrm{E})$ and $\Theta_{T,r}(\mathrm{E})$ by:

$$\Theta_{T,\ell}(\mathrm{E}) := \{T^{\ell} \in \mathcal{L}(\mathrm{E}, \mathrm{E}_T) \ : \ T^{\ell} \text{ is a left Fredholm inverse of } T\},$$

$$\Theta_{T,r}(\mathrm{E}) := \{T^{r} \in \mathcal{L}(\mathrm{E}, \mathrm{E}_T) \ : \ T^{r} \text{ is a right Fredholm inverse of } T\}.$$

By refereing to the stability problems of diverse essential spectra of closed densely defined linear operators acting on Banach spaces, several classes of Fredholm perturbations allow us to treat this kind of problem. To achieve this goal, we need to introduce some of these classes.

Definition 2.6. Let E be a Banach space and assume that $T \in \mathcal{L}(\mathrm{E})$.

(i) An operator T is said to be weakly compact if $T(S)$ is relatively weakly compact in E, for every bounded $S \subset \mathrm{E}$. Note that the class of weakly compact operators is a closed two-sided ideal of $\mathcal{L}(\mathrm{E})$ containing $\mathcal{K}(\mathrm{E})$ (see [6]).

(ii) An operator T is called a Riesz operator if $\lambda - T \in \Phi(X)$ for all scalars $\lambda \neq 0$.

(iii) We call that T is a polynomially Riesz operator on E if there exists a nonzero complex polynomial p(.) such that $\mathrm{p}(T)$ is a Riesz operator.

The set of polynomially Riesz operators will be defined as follows:

$$\mathcal{PR}(\mathrm{E}) := \{T \in \mathcal{L}(\mathrm{E}) : \text{ there exist p(.) such that } \mathrm{p}(T) \text{ is a Riesz operator}\}.$$

(iv) We define the minimal polynomial of Riesz operators on E as the nonzero polynomial p(.) of least degree and leading coefficient 1 such that $\mathrm{p}(T)$ is a Riesz operator defined as:

$$\mathrm{p}(z) := \prod_{i=1}^{n} (z - \lambda_i),$$

for which λ_i be a root of p(.).

As a continuation in this direction, let recall the following proposition on polynomially Riesz operators which is crucial for our aim originating from the work of K. Latrach et al. in [11].

Proposition 2.7. Define the following subset $\mathcal{E}_{\mathcal{PR}}(\mathrm{E})$ of $\mathcal{PR}(\mathrm{E})$ as:

$$\mathcal{E}_{\mathcal{PR}}(\mathrm{E}) := \{T \in \mathcal{PR}(\mathrm{E}) : \text{ the minimal polynomial p(.) of } T \text{ satisfies } \mathrm{p}(-1) \neq 0\}.$$

If $T \in \mathcal{E}_{\mathcal{PR}}(\mathrm{E})$, then $I + T \in \Phi(\mathrm{E})$ and $i(I + T) = 0$.

At the end of this section, we summarize in the following list some classes that are needed repeatedly in the sequel.

Classes:

$\mathcal{L}(\mathrm{E})$:	the set of all bounded linear operators in E,
$\mathcal{C}(\mathrm{E})$:	the class of densely defined closed linear operators on E,
$\mathcal{K}(\mathrm{E})$:	the closed ideal of compact operators in $\mathcal{L}(\mathrm{E})$,
$\mathcal{K}^{\mathrm{P}}(\mathrm{E}) := \{T \in \mathcal{L}(\mathrm{E}) : T^n \in \mathcal{K}(\mathrm{E}) \text{ for some } n \in \mathbb{N}\}$	the class of power compact operators,
$\mathcal{WC}(\mathrm{E})$:	the class of weakly compact operators on E,
$\mathcal{QC}(\mathrm{E})$:	the class of quasi-compact operators on E,
$\mathcal{PC}(\mathrm{E})$:	the class of polynomially compact operators on E,
$\mathcal{R}(\mathrm{E})$:	the class of Riesz operators R acting on E (that is, $R - \eta I$ is Fredholm for every non-zero complex η),
$\mathcal{PR}(\mathrm{E})$:	the class of polynomially Riesz operators on E.

The relationship between classes given in the precedent definitions was studied in [4, 6, 11, 13, 14, 16], and is recapitulated in the following diagram. (Arrows signify inclusions).

$$\begin{array}{ccccccc} \mathcal{WC}(\mathrm{E}) & \longleftarrow & \mathcal{K}(\mathrm{E}) & \longrightarrow & \mathcal{R}(\mathrm{E}) & \longrightarrow & \mathcal{E}_{\mathcal{PR}}(\mathrm{E}) \\ & & \downarrow & & & & \downarrow \\ \mathcal{QC}(\mathrm{E}) & \longleftarrow & \mathcal{K}^{\mathrm{P}}(\mathrm{E}) & \longrightarrow & \mathcal{PC}(\mathrm{E}) & \longrightarrow & \mathcal{PR}(\mathrm{E}) \end{array}$$

3. Spectral analysis of unbounded 3 × 3 block operator matrix via polynomially Riesz operators

The aim of this section is to formulate new criterions of perturbations on the entries of an model of unbounded 3×3 block operator matrix defined with maximal domain acting in the product of Banach spaces $\mathfrak{A} := \mathrm{E} \times \mathrm{E} \times \mathrm{E}$. Our interest allows us to formulate new techniques to analyze the description of some essential spectra of the closure of such matrix forms under the general concept of perturbations. To explain in details this interest, we consider in $\mathfrak{A}$ the following 3×3 block operator matrix :

$$\mathcal{M}_0 := \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix},$$

defined with maximal domain

$$\mathcal{D}(\mathcal{M}_0) := \prod_{i=1}^{3} \mathcal{D}(A_i) \cap \mathcal{D}(B_i) \cap \mathcal{D}(C_i).$$

Each operator entries of such kind of operator matrix have an appropriate domain and act on their corresponding spaces as:

$$\begin{array}{lll} A_1 : \mathcal{D}(A_1) \subset \mathrm{E} \to \mathrm{E} & A_2 : \mathcal{D}(A_2) \subset \mathrm{E} \to \mathrm{E} & A_3 : \mathcal{D}(A_3) \subset \mathrm{E} \to \mathrm{E} \\ B_1 : \mathcal{D}(B_1) \subset \mathrm{E} \to \mathrm{E} & B_2 : \mathcal{D}(B_2) \subset \mathrm{E} \to \mathrm{E} & B_3 : \mathcal{D}(B_3) \subset \mathrm{E} \to \mathrm{E} \\ C_1 : \mathcal{D}(C_1) \subset \mathrm{E} \to \mathrm{E} & C_2 : \mathcal{D}(C_2) \subset \mathrm{E} \to \mathrm{E} & C_3 : \mathcal{D}(C_3) \subset \mathrm{E} \to \mathrm{E}. \end{array}$$

Note that, in general, the operators occurring as entries in $\mathcal{M}_0$ are unbounded and that $\mathcal{M}_0$ is neither a closed nor a closable operator, even if its entries are closed.

In what follows, we will assume that all entries obey to the following hypotheses:

($\mathcal{H}_1$) A_1 is a densely defined linear operator on E with non-empty resolvent set $r_s(A_1)$.

($\mathcal{H}_2$) The operator B_1 (resp. C_1) verifies that $\mathcal{D}(A_1) \subset \mathcal{D}(B_1)$ (resp. $\mathcal{D}(A_1) \subset \mathcal{D}(C_1)$) and for some (hence for all) $\mu \in r_s(A_1)$, the operator $B_1(\mu - A_1)^{-1}$ (resp. $C_1(\mu - A_1)^{-1}$) is bounded.

- In particular, if B_1 (resp. C_1) is closable, then it follows from the closed graph theorem that $P_1(\mu)$ (resp. $P_2(\mu)$) is bounded.
- Define the operators $P_i(\mu), i = \{1, 2\}$, respectively as:

$$P_1(\mu) := B_1(\mu - A_1)^{-1} \text{ and } P_2(\mu) := C_1(\mu - A_1)^{-1}, \text{ for } \mu \in r_s(A_1).$$

($\mathcal{H}_3$) The operator A_2 (resp. A_3) is densely defined on E (resp. E) and for some (hence for all) $\mu \in r_s(A_1)$, the operator $(\mu - A_1)^{-1}A_2$ (resp. $(\mu - A_1)^{-1}A_3$) is bounded on its domain.

Remark 3.1. Keeping into account from ($\mathcal{H}_3$), we derive from the use of closed graph theorem that:
$Q_1(\mu) := \overline{(\mu - A_1)^{-1}A_2}$ and $Q_2(\mu) := \overline{(\mu - A_1)^{-1}A_3}$ are two bounded operators.

($\mathcal{H}_4$) The lineal $\mathcal{D}(A_2) \cap \mathcal{D}(B_2)$ is dense in E and for some (hence for all) $\mu \in r_s(A_1)$, the operator $B_2 - B_1(\mu - A_1)^{-1}A_2$ is closed. Set for $\mu \in r_s(A_1)$, the first Schur complement of the matrix operator $\mathcal{M}_0$ as:

$$S_1(\mu) := B_2 - B_1(\mu - A_1)^{-1}A_2.$$

($\mathcal{H}_5$) $\mathcal{D}(A_3) \subset \mathcal{D}(B_3)$ and the operator $B_3 - B_1(\mu - A_1)^{-1}A_3$ is bounded on its domain, for some $\mu \in r_s(A_1)$ and therefore for all $\mu \in r_s(A_1)$.

To formulate our interest, define

$$Q_3(\mu) := (\mu - S_1(\mu))^{-1}\overline{(B_3 - B_1(\mu - A_1)^{-1}A_3)} \in \mathcal{L}(\mathrm{E}),$$

for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.

($\mathcal{H}_6$) The operator C_2 satisfies that $\mathcal{D}(A_2) \subset \mathcal{D}(C_2)$ and for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$, we suppose that:

$$P_3(\mu) := (C_2 - C_1(\mu - A_1)^{-1}A_2)(\mu - S_1(\mu))^{-1} \in \mathcal{L}(\mathrm{E}).$$

($\mathcal{H}_7$) Assume that:
(**i**) $\mathcal{D}(A_3) \subset \mathcal{D}(C_3)$.
(**ii**) For some (hence for all) $\mu \in r_s(A_1) \cap\ r_s(S_1(\mu))$, the second Schur complement of the matrix $\mathcal{M}_0$,

$$S_2(\mu) := C_3 - C_1(\mu - A_1)^{-1}A_3 - P_3(\mu)[B_3 - B_1(\mu - A_1)^{-1}A_3]$$

is assumed to be closable. Therefore, its closure will be denoted by $\overline{S}_2(\mu)$.

All assumptions cited above are used to describe the closure of the operator $\mathcal{M}_0$ which are really strong and fruitful to develop our purpose.

Theorem 3.2. [8, Theorem 3.1]
Assume that the hypotheses ($\mathcal{H}_1$)-($\mathcal{H}_6$) are satisfied. Then, the operator $\mathcal{M}_0$ is closable in $\mathfrak{A}$ if and only if $S_2(\mu)$ is closable on E, for some $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.
Moreover, for such μ, the closure of $\mathcal{M}_0$ denoted by $\mathcal{M}$ and described as follows:

$$\mathcal{M} := \mu - \Pi_P\, \mathfrak{D}\, \Pi_Q\,, \tag{3.1}$$

where

$$\Pi_P := \begin{pmatrix} I & 0 & 0 \\ P_1(\mu) & I & 0 \\ P_2(\mu) & P_3(\mu) & I \end{pmatrix}, \ \Pi_Q := \begin{pmatrix} I & Q_1(\mu) & Q_2(\mu) \\ 0 & I & Q_3(\mu) \\ 0 & 0 & I \end{pmatrix}$$

and the diagonal operator matrix $\mathfrak{D} := \mathrm{diag}(\mathfrak{D}_i)$ with diagonal operator entries given by $\mathfrak{D}_1 := \mu - A_1$, $\mathfrak{D}_2 := \mu - S_1(\mu)$ and $\mathfrak{D}_3 := \mu - \overline{S}_2(\mu)$, respectively for $i = \{1, 2, 3\}$.

Remark 3.3. Let $(\tau, \mu) \in \big(\mathbb{C}, r_s(A_1) \cap r_s(S_1(\mu))\big)$ and assume that the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_7$) are fulfilled. Thus, the factorization used in Theorem 3.2 plays a substantial role below to write the operator matrix $\tau - \mathcal{M}$ as well:

$$\tau - \mathcal{M} := \Pi_P(\mu)\, \mathfrak{D}(\tau)\, \Pi_Q(\mu) + (\mu - \tau)\mathcal{G}(\mu), \tag{3.2}$$

where the bounded operators matrices $\mathfrak{D}(\tau)$ and $\mathcal{G}(\mu)$ are given by:

$$\mathfrak{D}(\tau) := \begin{pmatrix} \tau - A_1 & 0 & 0 \\ 0 & \tau - S_1(\mu) & 0 \\ 0 & 0 & \tau - \overline{S}_2(\mu) \end{pmatrix}$$

and

$$\mathcal{G}(\mu) := \begin{pmatrix} 0 & Q_1(\mu) & Q_2(\mu) \\ P_1(\mu) & P_1(\mu)Q_1(\mu) & P_1(\mu)Q_2(\mu) + Q_3(\mu) \\ P_2(\mu) & P_2(\mu)Q_1(\mu) + P_3(\mu) & P_2(\mu)Q_2(\mu) + P_3(\mu)Q_3(\mu) \end{pmatrix}.$$

It is well known that $\mathcal{R}(\mathrm{E})$ is not an ideal of $\mathcal{L}(\mathrm{E})$ and the class of Riesz operators verify the Riesz-Schauder theory of compact operators. In the spirit of the previously-defined class of operators, we start with the following lemma which can be found in [4, 16].

Lemma 3.4. Let E be a Banach space. Assume that T and S are two commuting operators of $\mathcal{L}(\mathrm{E})$. Then, we have:

(i) If $T \in \mathcal{R}(\mathrm{E})$, then $TS \in \mathcal{R}(\mathrm{E})$.

(ii) If $(T, S) \in \mathcal{R}^2(\mathrm{E})$, then $T + S \in \mathcal{R}(\mathrm{E})$.

Before moving to study the essential spectra of this kind of operator matrix via polynomially Riesz operators perturbation, the following proposition may be essential.

Proposition 3.5. Let consider the following diagonal operator matrix denoted by $\mathfrak{D}'$ expressed as follows:

$$\mathfrak{D}' := \mathrm{diag}(A, B, C).$$

Then, we have:

(**i**) Suppose that for each $A \in \Phi^{\ell}(\mathrm{E})$, $B \in \Phi^{\ell}(\mathrm{E})$ and $C \in \Phi^{\ell}(\mathrm{E})$, there exists $A^{\ell} \in \Theta_{A,\ell}(\mathrm{E})$, $B^{\ell} \in \Theta_{B,\ell}(\mathrm{E})$ and $C^{\ell} \in \Theta_{C,\ell}(\mathrm{E})$.

Then, we conclude that

$$\mathfrak{D}'_{\ell} := \begin{pmatrix} A^{\ell} & 0 & 0 \\ 0 & B^{\ell} & 0 \\ 0 & 0 & C^{\ell} \end{pmatrix} \in \Theta_{\mathfrak{D}',\ell}(\mathfrak{A}).$$

(**ii**) Suppose that for each $A \in \Phi^{r}(\mathrm{E})$, $B \in \Phi^{r}(\mathrm{E})$ and $C \in \Phi^{r}(\mathrm{E})$, there exists $A^{r} \in \Theta_{A,r}(\mathrm{E})$, $B^{r} \in \Theta_{B,r}(\mathrm{E})$ and $C^{r} \in \Theta_{C,r}(\mathrm{E})$.

Then, we conclude that

$$\mathfrak{D}'_{r} := \begin{pmatrix} A^{r} & 0 & 0 \\ 0 & B^{r} & 0 \\ 0 & 0 & C^{r} \end{pmatrix} \in \Theta_{\mathfrak{D}',r}(\mathfrak{A}).$$

(**iii**) Moreover, assume that $(\mathcal{U}, \mathcal{V})$ are bounded and boundedly invertible. Then,

$$\mathcal{V}^{-1}\mathfrak{D}'_{\ell}\,\mathcal{U}^{-1} \in \Theta_{\mathcal{U}\mathfrak{D}'\mathcal{V},\ell}(\mathfrak{A})$$
$$\Big(\text{resp. } \mathcal{V}^{-1}\mathfrak{D}'_{r}\,\mathcal{U}^{-1} \in \Theta_{\mathcal{U}\mathfrak{D}'\mathcal{V},r}(\mathfrak{A}\Big).$$

Proof. The results may be obvious from the use of Definition 2.4 with the fact that $\mathfrak{D}'_{\ell}$ and $\mathfrak{D}'_{r}$ are both diagonal operator matrices. □

In order to obtain our main result, we start by introducing some general assumptions on the entries of the operator matrix $\mathcal{M}_0$ that are required to provide a new characterization of some essential spectra of $\mathcal{M}$ involving the theory of polynomially Riesz operators perturbations and the concept of Fredholm inverse.

For $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$, we suppose:

$(\mathbf{Z}_1)$ $S_1^\ell P_1(\mu) = P_1(\mu)A_1^\ell$. $(\mathbf{Z}_1')$ $S_1^r P_1(\mu) = P_1(\mu)A_1^r$.

$(\mathbf{Z}_2)$ $S_2^\ell P_2(\mu) = P_2(\mu)A_1^\ell$. $(\mathbf{Z}_2')$ $S_2^r P_2(\mu) = P_2(\mu)A_1^r$.

$(\mathbf{Z}_3)$ $S_2^\ell P_3(\mu) = P_3(\mu)S_1^\ell$. $(\mathbf{Z}_3')$ $S_2^r P_3(\mu) = P_3(\mu)S_1^r$.

$(\mathbf{Z}_4)$ $S_1^\ell Q_3(\mu) = Q_3(\mu)S_2^\ell$. $(\mathbf{Z}_4')$ $S_1^r Q_3(\mu) = Q_3(\mu)S_2^r$.

$(\mathbf{Z}_5)$ $A_1^\ell Q_2(\mu) = Q_2(\mu)S_2^\ell$. $(\mathbf{Z}_5')$ $A_1^r Q_2(\mu) = Q_2(\mu)S_2^r$.

$(\mathbf{Z}_6)$ $A_1^\ell Q_1(\mu) = Q_1(\mu)S_1^\ell$. $(\mathbf{Z}_6')$ $A_1^r Q_1(\mu) = Q_1(\mu)S_1^r$.

$(\mathbf{Z}_7)$ $P_1(\mu) = P_2(\mu)$. $(\mathbf{Z}_7')$ $P_2(\mu) = P_1(\mu)$.

On the basis of the above hypotheses, left-right Fredholm properties of the operator matrix $\tau - \mathcal{M}$ in terms of polynomially Riesz operators are given in the following theorem.

Theorem 3.6. Let $\tau \in \mathbb{C}$ and assume that the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ are fulfilled, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$. Thus, we have:

(i) If $A_1^\ell \in \Theta_{\tau-A_1,\ell}(\mathrm{E})$, $S_1^\ell \in \Theta_{\tau-S_1(\mu),\ell}(\mathrm{E})$ and $S_2^\ell \in \Theta_{\tau-\overline{S}_2(\mu),\ell}(\mathrm{E})$ such that the assumptions $(\mathbf{Z}_1)$-$(\mathbf{Z}_7)$ are satisfied, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.
Then, we obtain:

$$\mathcal{G}(\mu) \in \mathcal{R}(\mathfrak{A}) \ \Rightarrow\ \tau - \mathcal{M} \in \Phi^\ell(\mathfrak{A}) \text{ with } \ i\Big(\tau - \mathcal{M}\Big) = i\Big(\mathfrak{D}(\tau)\Big).$$

(ii) If $A_1^r \in \Theta_{\tau-A_1,r}(\mathrm{E})$, $S_1^r \in \Theta_{\tau-S_1(\mu),r}(\mathrm{E})$ and $S_2^r \in \Theta_{\tau-\overline{S}_2(\mu),r}(\mathrm{E})$ such that the assumptions $(\mathbf{Z}_1')$-$(\mathbf{Z}_7')$ are satisfied, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.
Then, we obtain:

$$\mathcal{G}(\mu) \in \mathcal{R}(\mathfrak{A}) \ \Rightarrow\ \tau - \mathcal{M} \in \Phi^r(\mathfrak{A}) \text{ with } i\Big(\tau - \mathcal{M}\Big) = i\Big(\mathfrak{D}(\tau)\Big).$$

Proof. Assume that the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ are fulfilled, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$. Let consider $\tau \in \mathbb{C}$ for which $\tau \neq \mu$. Thus, from the Frobenius-Schur factorization used in Remark 3.3, we infer that the operator matrix $\tau - \mathcal{M}$ may be written as:

$$\begin{aligned}\tau - \mathcal{M} &:= \Pi_P(\mu)\ \mathfrak{D}(\tau)\ \Pi_Q(\mu) + (\mu - \tau)\mathcal{G}(\mu)\\ &= \mathcal{T} + \mathcal{J},\end{aligned}$$

where $\mathcal{T} := \Pi_P \mathfrak{D}(\tau)\Pi_Q$ and $\mathcal{J} := (\mu - \tau)\mathcal{G}(\mu)$.
Hence, to obtain the desired result, it is remains to prove it by the use of Theorem 3.1 in [2] for the bounded operators $\mathcal{T}$ and $\mathcal{J}$.

(i) Suppose, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$, that $A_1^\ell \in \Theta_{\tau-A_1,\ell}(\mathrm{E})$, $S_1^\ell \in \Theta_{\tau-S_1(\mu),\ell}(\mathrm{E})$ and $S_2^\ell \in \Theta_{\tau-\overline{S}_2(\mu),\ell}(\mathrm{E})$. In order to achieve our goal, we will proceed by the following steps:

• Step I : Proofing that $\mathcal{T}$ has a left Fredholm inverse.

Since Π_P and Π_Q are two bounded and boundedly invertible operators with the fact that A_1^ℓ, S_1^ℓ and S_2^ℓ are left Fredholm inverses of the operators $\tau - A_1$, $\tau - S_1(\mu)$ and $\tau - \overline{S}_2(\mu)$, respectively, we deduce from Proposition 3.5 that:

$$\mathcal{T}^\ell := \Pi_Q^{-1}\ \mathfrak{D}_\ell'\ \Pi_P^{-1} \in \Theta_{\mathcal{T},\ell}(\mathfrak{A}),$$

where $\mathfrak{D}_\ell' \in \Theta_{\mathfrak{D}(\tau),\ell}(\mathfrak{A})$.

• Step II : Proofing that $-\mathcal{J}\mathcal{T}^\ell \in \mathcal{E}_{\mathcal{PR}}(\mathfrak{A})$.

A short computation reveals that $\mathcal{T}^\ell\mathcal{J}$ may be expressed as:

$$\mathcal{T}^\ell\mathcal{J} := \begin{pmatrix} \imath_{11}(\mu) & \imath_{12}(\mu) & \imath_{13}(\mu)\\ \imath_{21}(\mu) & \imath_{22}(\mu) & \imath_{23}(\mu)\\ \imath_{31}(\mu) & \imath_{32}(\mu) & \imath_{33}(\mu)\end{pmatrix},$$

where:

$$\begin{aligned}
\imath_{11}(\mu) &:= -Q_1(\mu)S_1^{\ell}P_1(\mu) - Q_1(\mu)Q_3(\mu)S_2^{\ell}P_3(\mu)P_1(\mu) + Q_2(\mu)S_2^{\ell}P_3(\mu)P_1(\mu)\\
&\quad + Q_1(\mu)Q_3(\mu)S_2^{\ell}P_2(\mu) - Q_2(\mu)S_2^{\ell}P_2(\mu),\\
\imath_{12}(\mu) &:= A_1^{\ell}Q_1(\mu) + Q_1(\mu)Q_3(\mu)S_2^{\ell}P_3(\mu) - Q_2(\mu)S_2^{\ell}P_3(\mu),\\
\imath_{13}(\mu) &:= A_1^{\ell}Q_2(\mu) - Q_1(\mu)S_1^{\ell}Q_3(\mu),\\
\imath_{21}(\mu) &:= S_1^{\ell}P_1(\mu) + Q_3(\mu)S_2^{\ell}P_3(\mu)P_1(\mu) - Q_3(\mu)S_2^{\ell}P_2(\mu),\\
\imath_{22}(\mu) &:= -Q_3(\mu)S_2^{\ell}P_3(\mu),\\
\imath_{23}(\mu) &:= S_1^{\ell}Q_3(\mu),\\
\imath_{31}(\mu) &:= -S_2^{\ell}P_3(\mu)P_1(\mu) + S_2^{\ell}P_2(\mu),\\
\imath_{32}(\mu) &:= S_2^{\ell}P_3(\mu),\\
\imath_{33}(\mu) &:= 0.
\end{aligned}$$

On the other side, we will also calculate $-\mathcal{J}\mathcal{T}^{\ell}$ as:

$$\mathcal{J}\mathcal{T}^{\ell} := -\begin{pmatrix} \jmath_{11}(\mu) & \jmath_{12}(\mu) & \jmath_{13}(\mu)\\ \jmath_{21}(\mu) & \jmath_{22}(\mu) & \jmath_{23}(\mu)\\ \jmath_{31}(\mu) & \jmath_{32}(\mu) & \jmath_{33}(\mu)\end{pmatrix},$$

where:

$$\begin{aligned}
\jmath_{11}(\mu) &:= -Q_1(\mu)S_1^{\ell}P_1(\mu) - Q_1(\mu)Q_3(\mu)S_2^{\ell}P_3(\mu)P_1(\mu) + Q_2(\mu)S_2^{\ell}P_3(\mu)P_1(\mu)\\
&\quad + Q_1(\mu)Q_3(\mu)S_2^{\ell}P_2(\mu) - Q_2(\mu)S_2^{\ell}P_2(\mu),\\
\jmath_{12}(\mu) &:= Q_1(\mu)S_1^{\ell} + Q_1(\mu)Q_3(\mu)S_2^{\ell}P_3(\mu) - Q_2(\mu)S_2^{\ell}P_3(\mu),\\
\jmath_{13}(\mu) &:= Q_2(\mu)S_2^{\ell} - Q_1(\mu)Q_3(\mu)S_2^{\ell},\\
\jmath_{21}(\mu) &:= P_1(\mu)A_1^{\ell} + Q_3(\mu)S_2^{\ell}P_1(\mu)P_3(\mu) - Q_3(\mu)S_2^{\ell}P_2(\mu),\\
\jmath_{22}(\mu) &:= P_1(\mu)Q_2(\mu)S_2^{\ell}P_3(\mu) - P_2(\mu)Q_2(\mu)S_2^{\ell}P_3(\mu) - Q_3(\mu)S_2^{\ell}P_3(\mu),\\
\jmath_{23}(\mu) &:= Q_3(\mu)S_2^{\ell},\\
\jmath_{31}(\mu) &:= P_2(\mu)A_1^{\ell} - P_3(\mu)S_1^{\ell}P_1(\mu),\\
\jmath_{32}(\mu) &:= P_3(\mu)S_1^{\ell},\\
\jmath_{33}(\mu) &:= 0.
\end{aligned}$$

We keep into account that the assumptions $(\mathbf{Z}_1)$-$(\mathbf{Z}_7)$ are satisfied, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$. Then, we obtain that $\mathcal{T}^{\ell}\mathcal{J} = \mathcal{J}\mathcal{T}^{\ell}$.

Now, according to the fact that $\mathcal{T}^{\ell} \in \mathcal{L}(\mathfrak{A})$ and $\mathcal{G}(\mu) \in \mathcal{R}(\mathfrak{A})$, we conclude in view of Lemma 3.4 that:

$$-\mathcal{J}\mathcal{T}^{\ell} \in \mathcal{R}(\mathfrak{A}) \subset \mathcal{E}_{\mathcal{PR}}(\mathfrak{A}).$$

- <u>Step III</u> : Proofing that $\tau - \mathcal{M} \in \Phi^{\ell}(\mathfrak{A})$ with $i\Big(\tau - \mathcal{M}\Big) = i\Big(\mathfrak{D}(\tau)\Big)$.

The use of Theorem 3.1 in [2] with respect to the fact that $-\mathcal{J}\mathcal{T}^{\ell} \in \mathcal{E}_{\mathcal{PR}}(\mathfrak{A})$, affirms that:

$$\mathcal{T} + \mathcal{J} \in \Phi^{\star}(\mathfrak{A}) \quad \text{with} \quad i\Big(\mathcal{T} + \mathcal{J}\Big) = i\Big(\mathcal{T}\Big).$$

for which $\Phi^{\star}(\mathfrak{A}) := \{\Phi_+(\mathfrak{A}),\ \Phi^{\ell}(\mathfrak{A})\}$.

Since Π_P and Π_Q are two bounded and boundedly invertible operators. Hence, we infer that:

$$i\Big(\mathcal{T}\Big) = i\Big(\Pi_P\mathfrak{D}(\tau)\Pi_Q\Big) = i\Big(\mathfrak{D}(\tau)\Big).$$

Thus, due to the reason that $\mathfrak{D}(\tau)$ is a diagonal operator matrix, we deduce that:

$$i\Big(\mathcal{T}\Big) = i\Big(\mathfrak{D}(\tau)\Big) = i\Big(\tau - A_1\Big) + i\Big(\tau - S_1(\mu)\Big) + i\Big(\tau - \overline{S}_2(\mu)\Big).$$

Finally, we conclude that:

$$i\Big(\tau - \mathcal{M}\Big) = i\Big(\tau - A_1\Big) + i\Big(\tau - S_1(\mu)\Big) + i\Big(\tau - \overline{S}_2(\mu)\Big). \tag{3.3}$$

(ii) Let $\tau \in \mathbb{C}$. Assume, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$, that $A_1^r \in \Theta_{\tau - A_1, r}(\mathrm{E})$, $S_1^r \in \Theta_{\tau - S_1(\mu), r}(\mathrm{E})$ and $S_2^r \in \Theta_{\tau - \overline{S}_2(\mu), r}(\mathrm{E})$ such that the conditions $(\mathbf{Z}_1')$-$(\mathbf{Z}_7')$ are fulfilled.

Obviously, the use of Proposition 3.5 with Remark 3.3 in view of the fact that Π_P and Π_Q are two bounded and boundedly invertible operators shows that $\mathcal{T}^r := \Pi_Q^{-1}\, \mathfrak{D}'_r\, \Pi_P^{-1}$ is a right Fredholm inverse of $\mathcal{T}$.
Furthermore, a short computation signifies that:

$$\mathcal{J}\mathcal{T}^r := \begin{pmatrix} \mathfrak{f}_{11}(\mu) & \mathfrak{f}_{12}(\mu) & \mathfrak{f}_{13}(\mu) \\ \mathfrak{f}_{21}(\mu) & \mathfrak{f}_{22}(\mu) & \mathfrak{f}_{23}(\mu) \\ \mathfrak{f}_{31}(\mu) & \mathfrak{f}_{32}(\mu) & \mathfrak{f}_{33}(\mu) \end{pmatrix} \quad \text{and} \quad \mathcal{T}^r\mathcal{J} := \begin{pmatrix} \mathfrak{g}_{11}(\mu) & \mathfrak{g}_{12}(\mu) & \mathfrak{g}_{13}(\mu) \\ \mathfrak{g}_{21}(\mu) & \mathfrak{g}_{22}(\mu) & \mathfrak{g}_{23}(\mu) \\ \mathfrak{g}_{31}(\mu) & \mathfrak{g}_{32}(\mu) & \mathfrak{g}_{33}(\mu) \end{pmatrix},$$

where:

$$\begin{aligned}
\mathfrak{f}_{11}(\mu) := \mathfrak{g}_{11}(\mu) := &-Q_1(\mu)S_1^\ell P_1(\mu) - Q_1(\mu)Q_3(\mu)S_2^\ell P_3(\mu)P_1(\mu) + Q_2(\mu)S_2^\ell P_3(\mu)P_1(\mu)\\
&+ Q_1(\mu)Q_3(\mu)S_2^\ell P_2(\mu) - Q_2(\mu)S_2^\ell P_2(\mu),
\end{aligned}$$

$\mathfrak{f}_{12}(\mu) := Q_1(\mu)S_1^\ell + Q_1(\mu)Q_3(\mu)S_2^\ell P_3(\mu) - Q_2(\mu)S_2^\ell P_3(\mu),$

$\mathfrak{g}_{12}(\mu) := A_1^\ell Q_1(\mu) + Q_1(\mu)Q_3(\mu)S_2^\ell P_3(\mu) - Q_2(\mu)S_2^\ell P_3(\mu),$

$\mathfrak{f}_{13}(\mu) := Q_2(\mu)S_2^\ell - Q_1(\mu)Q_3(\mu)S_2^\ell,$

$\mathfrak{g}_{13}(\mu) := A_1^\ell Q_2(\mu) - Q_1(\mu)S_1^\ell Q_3(\mu),$

$\mathfrak{f}_{21}(\mu) := P_1(\mu)A_1^\ell + Q_3(\mu)S_2^\ell P_1(\mu)P_3(\mu) - Q_3(\mu)S_2^\ell P_2(\mu),$

$\mathfrak{g}_{21}(\mu) := S_1^\ell P_1(\mu) + Q_3(\mu)S_2^\ell P_3(\mu)P_1(\mu) - Q_3(\mu)S_2^\ell P_2(\mu),$

$\mathfrak{f}_{22}(\mu) := P_1(\mu)Q_2(\mu)S_2^\ell P_3(\mu) - P_2(\mu)Q_2(\mu)S_2^\ell P_3(\mu) - Q_3(\mu)S_2^\ell P_3(\mu),$

$\mathfrak{g}_{22}(\mu) := -Q_3(\mu)S_2^\ell P_3(\mu),$

$\mathfrak{f}_{23}(\mu) := Q_3(\mu)S_2^\ell,$

$\mathfrak{g}_{23}(\mu) := S_1^\ell Q_3(\mu),$

$\mathfrak{f}_{31}(\mu) := P_2(\mu)A_1^\ell - P_3(\mu)S_1^\ell P_1(\mu),$

$\mathfrak{g}_{31}(\mu) := -S_2^\ell P_3(\mu)P_1(\mu) + S_2^\ell P_2(\mu),$

$\mathfrak{f}_{32}(\mu) := P_3(\mu)S_1^\ell,$

$\mathfrak{g}_{32}(\mu) := S_2^\ell P_3(\mu),$

$\mathfrak{f}_{33}(\mu) := \mathfrak{g}_{33}(\mu) := 0.$

Hence, based on the assumptions $(\mathbf{Z}_1')$-$(\mathbf{Z}_7')$, we deduce that:

$$\mathcal{J}\mathcal{T}^r = \mathcal{T}^r\mathcal{J}.$$

As a consequence, we have $\mathcal{J} \in \mathcal{R}(\mathfrak{A})$ which commutes with the bounded operator $\mathcal{T}^r$. Then, we obtain according to Lemma 3.4 that:

$$-\mathcal{T}^r\mathcal{J} \in \mathcal{R}(\mathfrak{A}) \subset \mathcal{E}_{\mathcal{PR}}(\mathfrak{A}).$$

Directly, Theorem 3.1 in [2] proves that:

$$\tau - \mathcal{M} \in \Phi^\star(\mathfrak{A}) \quad \text{with} \quad i\Big(\tau - \mathcal{M}\Big) = i\Big(\mathcal{T} + \mathcal{J}\Big) = i\Big(\mathcal{T}\Big).$$

for which $\Phi^\star(\mathfrak{A}) := \{\Phi_-(\mathfrak{A}),\ \Phi^r(\mathfrak{A})\}$.
Thus, asserts that:

$$i\Big(\tau - \mathcal{M}\Big) = i\Big(\mathcal{T}\Big) = i\Big(\mathfrak{D}(\tau)\Big) = i\Big(\tau - A_1\Big) + i\Big(\tau - S_1(\mu)\Big) + i\Big(\tau - \overline{S}_2(\mu)\Big), \tag{3.4}$$

where Π_P and Π_Q are bounded and boundedly invertible operators and $\mathfrak{D}(\tau)$ is a diagonal operator matrix. □

However, in Corollary 3.7 below we express some essential spectra of the operator matrix $\mathcal{M}$ involving the concept of polynomially Riesz perturbations.

Corollary 3.7. Let $\tau \in \mathbb{C}$ and suppose that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ are satisfied, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$. Thus, we have:

(i) If $A_1^\ell \in \Theta_{\tau - A_1, \ell}(\mathrm{E})$, $S_1^\ell \in \Theta_{\tau - S_1(\mu), \ell}(\mathrm{E})$ and $S_2^\ell \in \Theta_{\tau - \overline{S}_2(\mu), \ell}(\mathrm{E})$ such that the conditions $(\mathbf{Z}_1)$-$(\mathbf{Z}_7)$ are fulfilled, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.
Then, we get:

$$\mathcal{G}(\mu) \in \mathcal{R}(\mathfrak{A}) \quad \Rightarrow \quad \tilde{\sigma}(\mathcal{M}) \subset \tilde{\sigma}(A_1) \cup \tilde{\sigma}(S_1(\mu)) \cup \tilde{\sigma}(\overline{S}_2(\mu)),$$

for $\tilde{\sigma}(.) \in \{\sigma_{ess}^{+}(.), \sigma_{ess}^{\ell}(.), \sigma_{w}^{\ell}(.)\}$.

(ii) If $A_1^r \in \Theta_{\tau - A_1, r}(\mathrm{E})$, $S_1^r \in \Theta_{\tau - S_1(\mu), r}(\mathrm{E})$ and $S_2^r \in \Theta_{\tau - \overline{S}_2(\mu), r}(\mathrm{E})$ such that the conditions $(\mathbf{Z}_1')$-$(\mathbf{Z}_7')$ are fulfilled, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.
Then, we get:

$$\mathcal{G}(\mu) \in \mathcal{R}(\mathfrak{A}) \quad \Rightarrow \quad \tilde{\sigma}(\mathcal{M}) \subset \tilde{\sigma}(A_1) \cup \tilde{\sigma}(S_1(\mu)) \cup \tilde{\sigma}(\overline{S}_2(\mu)),$$

for $\tilde{\sigma}(.) \in \{\sigma_{ess}^{-}(.), \sigma_{ess}^{r}(.), \sigma_{w}^{r}(.)\}$.

Proof. **(i)** We start with the left Weyl spectrum case. Assume that:

$$\tau \notin \sigma_w^\ell(A_1) \cup \sigma_w^\ell(S_1(\mu)) \cup \sigma_w^\ell(\overline{S}_2(\mu)).$$

Hence, $\tau - A_1 \in \mathcal{W}^\ell(\mathrm{E})$, $\tau - S_1(\mu) \in \mathcal{W}^\ell(\mathrm{E})$ and $\tau - \overline{S}_2(\mu) \in \mathcal{W}^\ell(\mathrm{E})$. That would allow us to conclude in view of Theorem 3.6 with (Eq). 3.3 that:

$$\tau - \mathcal{M} \in \mathcal{W}^\ell(\mathfrak{A}),$$

while $i\big(\tau - A_1\big) \leq 0$, $i\big(\tau - S_1(\mu)\big) \leq 0$ and $i\big(\tau - \overline{S}_2(\mu)\big) \leq 0$.
For the upper and left Fredholm essential spectra, the result may be obvious according to Theorem 3.6. Indeed,

$$\tau \notin \tilde{\sigma}(A_1) \cup \tilde{\sigma}(S_1(\mu)) \cup \tilde{\sigma}(\overline{S}_2(\mu)) \quad \Rightarrow \quad \tau \notin \tilde{\sigma}(\mathcal{M}),$$

for $\tilde{\sigma}(.) \in \{\sigma_{ess}^{+}(.), \sigma_{ess}^{\ell}(.)\}$.

(ii) We adopt the same reasoning as the item **(i)** to obtain our desired result. It suffices to use Theorem 3.6 with (Eq). 3.4. □

We close this section by the following question which arises in a natural way from our main result.

Question 3.8. Do the achieved results shown in Corollary 3.7 remain true if we replace its hypotheses by compact arguments?

At the end of this paper, We are unable to decide whether, in the presence of L_1−spaces, the answer to the previous question is affirmative, even in the case of weakly compact assumptions. To explain this:

Let E_1 denotes the space $\mathrm{L}_1(\Omega, d\vartheta)$, where $(\Omega, \Sigma, \vartheta)$ stands for a positive measure space. It is worthy to point out that the incidence of some essential spectra of such kind of 3×3 unbounded operator matrix $\mathcal{M}$ defined with maximal domain is still always true with regard to compact (resp. weakly compact) arguments.

That is if we change the following conditions:

↷ $\mathcal{G}(\mu) \in \mathcal{E}_{\mathcal{PR}}(\mathfrak{A})$.

↷ The conditions $(\mathbf{Z}_1)$-$(\mathbf{Z}_7)$ and $(\mathbf{Z}_1')$-$(\mathbf{Z}_7')$ are fulfilled, for some (hence for all) $\mu \in r_s(A_1) \cap r_s(S_1(\mu))$.

Only by:

↷ The operators $\Big(P_i(\mu), Q_i(\mu)\Big) \in \mathcal{K}^2(\mathrm{E})$ $\Big($resp. $\Big(P_i(\mu), Q_i(\mu)\Big) \in \mathcal{WC}^2(\mathrm{E}_1)\Big)$, for $i = \{1, 2, 3\}$.

We end this paper with the following conjecture.

Conjecture 3.9. Let consider the following block 3×3 of operator matrix

$$\mathcal{M} := \begin{pmatrix} A_1 & B_1 & B_2 \\ C_1 & A_2 & B_3 \\ C_2 & C_3 & A_3 \end{pmatrix}.$$

We ask the following question:

Without considering the case of perturbed upper or lower triangular operator matrix form, what are the conditions that we will impose on the entries components of the operator matrix $\mathcal{M}$ involving the notion of polynomially Riesz operator perturbations to provide that:

$$\tilde{\sigma}(\mathcal{M}) = \bigcup_{i=1}^{3} \tilde{\sigma}(A_i)$$

for $\tilde{\sigma}(.) \in \{\sigma^{+}_{ess}(.), \sigma^{\ell}_{ess}(.), \sigma^{\ell}_{w}(.), \sigma^{-}_{ess}(.), \sigma^{r}_{ess}(.), \sigma^{r}_{w}(.)\}$?

4. Conclusion

The central subject of the work presented in this paper is the spectral analysis of perturbed unbounded 3×3 block operator matrix by means of the concept of polynomially Riesz operators perturbations. Such analysis allows us to derive the interesting results intervening in the theory of Fredholm operators. Our approach allows us to investigate a new technique and a general assumption than provided in the literature in the investigation of some relative essential spectra in Banach space for unbounded block 3×3 operator matrix defined with maximal domain.

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