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Solutions of Riesz-Caputo fractional derivative problems involving anti-periodic boundary conditions

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Abstract. This article deals with the investigation concerning the existence and uniqueness of anti-periodic boundary value solutions for a kind of Riesz-Caputo fractional differential equations. The equation is as follows

$$ {}_0^{RC}D_l^{\zeta}\varpi(\tau) + \mathfrak{T}\left(\tau, \varpi(\tau), {}_0^{RC}D_l^{\eta}\varpi(\tau)\right) = 0, \tau \in \mathcal{J} := [0, l], $$
$$ a_1\varpi(0) + b_1\varpi(l) = 0, a_2\varpi'(0) + b_2\varpi'(l) = 0, a_3\varpi''(0) + b_3\varpi''(l) = 0, $$

where $2 < \zeta \leq 3$ and, $1 < \eta \leq 2$, ${}_0^{RC}D_l^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\zeta, \eta\}$, $\mathfrak{T} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and a_i, b_i are non-negative constants with $a_i > b_i$, $i = 1, 2, 3$. Uniqueness is demonstrated using Banach contraction principle, and existence is demonstrated employing the fixed point theorems of Schaefer and Krasnoselskii. Our results are supported by suitable numerical illustrations.

1. Introduction

Fractional calculus is an interesting branch of mathematical sciences which covers plenty of different prospects of defining real number or complex number powers of the differentiation operator and of the integration operator and flourishing a broader calculus for aforementioned operators generalizing the classical one. In fact, the fractional order models provide a more compelling explanation of memory and genetic processes than integral order models. A lot of contributions have been made to different fractional differential equations and inclusions. Of late, fractional differential equations or inclusions with anti-periodic boundary value problems (APBVP, in short) have been the subject of extensive research due to their widespread use in many fields. There are plenty of interesting articles and books related to fractional differential equations, see [1, 3–5, 7, 9–12, 15, 20, 22, 24, 32, 33] as well as the references within.

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In recent years, Riemann-Liouville and Caputo derivatives have been widely used in research on fractional boundary and initial value problems. These fractional operators, are one-sided and can only change the past or the future. The Riesz space fractional operator, in contrast to the other fractional operators, is a two-sided operator that captures both past and present non-local memory effects. This is significant because present states in mathematical models of physical processes on finite domains are influenced by both forgotten and upcoming memory consequences. In the anomalous diffusion problem, as an example, the Riesz fractional derivative is applied to explain the memory outcomes in past and as well as the future agglomerations [25, 26, 28]. In 2017, Chen et al.[17] investigated the following fractional APBVP

$$\begin{aligned} &{}_0^{RC}\mathcal{D}_{\mathcal{L}}^{\zeta}\varpi(\tau) = \Psi(\tau,\varpi(\tau)), \quad \zeta \in (0,1], \quad 0 \le \tau \le \mathcal{L}, \\ &\varpi(0) = a, \varpi(\mathcal{L}) = b, \end{aligned} \tag{1.1}$$

where ${}_0^{RC}\mathcal{D}_{\mathcal{L}}^{\zeta}$ is the Riesz-Caputo derivative (RCD, in short) and $\Psi : [0,\mathcal{L}]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, Gu et al. [19] used the Leray-Schauder and Krasnoselskii fixed point theorems to demonstrate the existence of positive solutions. In addition, Chen et al. [16] also discussed the following fractional APBVP

$$\begin{aligned} {}_0^{RC}\mathcal{D}_{\mathcal{L}}^{\zeta}\varpi(\tau) &= \Psi(\tau,\varpi(\tau)), \quad \tau \in [0,\mathcal{L}], \quad 1 < \zeta \le 2, \\ \varpi(0) + \varpi(\mathcal{L}) = 0, &\qquad \varpi'(0) + \varpi'(\mathcal{L}) = 0, \end{aligned}$$

where ${}_0^{RC}\mathcal{D}_{\mathcal{L}}^{\zeta}$ is the Riesz-Caputo derivative $1 < \zeta \le 2$ and $\Psi : [0,\mathcal{L}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to ϖ and τ. By means of novel fractional Gronwall inequalities and a few fixed point results, the authors proposed some existence results of the solutions for the aforementioned kind of fractional differential equations under different conditions. Of late, Naas et al. [27] discussed the existence and uniqueness of solutions of the succeeding kind of fractional differential equations concerning Riesz-Caputo derivative

$$\begin{aligned} {}_0^{RC}D_l^{\zeta}\varpi(\tau) + \mathfrak{T}\left(\tau,\varpi(\tau),{}_0^{RC}D_l^{\eta}\varpi(\tau)\right) = 0, \quad \tau \in \mathcal{J} := [0,l], \\ \varpi(0) + \varpi(l) = 0, \quad \mu\varpi'(0) + \sigma\varpi'(l) = 0, \end{aligned}$$

where $1 < \zeta \le 2, 0 < \eta \le 1, {}_0^{RC}D_l^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\zeta,\eta\}$, $\mathfrak{T} : \mathcal{J}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and μ,σ are non-negative constants with $\mu > \sigma$. They used the Banach fixed point theorem for uniqueness, and Schaefer and Krasnoselskii's theorem to obtain existence results.

Motivated by their interesting findings, in our manuscript, we enquire for the existence and uniqueness of solutions of the subsequent type of fractional differential equations concerning Riesz-Caputo derivative

$$\begin{aligned} {}_0^{RC}D_l^{\zeta}\varpi(\tau) + \mathfrak{T}\left(\tau,\varpi(\tau),{}_0^{RC}D_l^{\eta}\varpi(\tau)\right) = 0, \tau \in \mathcal{J} := [0,l], \\ a_1\varpi(0) + b_1\varpi(l) = 0, a_2\varpi'(0) + b_2\varpi'(l) = 0, a_3\varpi''(0) + b_3\varpi''(l) = 0, \end{aligned} \tag{1.2}$$

where $2 < \zeta \le 3$, $1 < \eta \le 2$, ${}_0^{RC}D_l^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\zeta,\eta\}$, and $\mathfrak{T} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Intent readers are referred to a few recent publications that deal with a Riesz-Caputo derivative [2, 13, 14, 18, 21, 23, 27, 29–31]. The class of fractional differential equations with boundary conditions involving Riesz-Caputo fractional derivatives studied in the research article has various applications across different fields.

By studying the existence and uniqueness of solutions for this class of equations, researchers can better understand the behaviour of various complex systems in different fields and develop more accurate predictive models.

This paper is arranged in a way such that Section 2 includes definitions and some of the most fundamental results in the literature of fractional calculus. In Section 3, Schaefer and Krasnoselskii fixed point theorems are employed to obtain the existence of solutions of problem (1.2) using Riesz-Caputo derivatives. Further, Banach fixed point theorem employed to guarantee the uniqueness of the obtained fixed points. Besides, the results are verified by two illustrative numerical examples. Finally, in Section 4, we come up with the conclusion and a problem for future research ventures.

2. Preliminaries

In this section, we recall a few fundamental concepts, and findings from the literature which are crucial in view of our article. Suppose $\beta > 0$, and $n-1 < \beta \leq n$, $n \in \mathbb{N}$ and $n = \lceil v \rceil$, where $\lceil . \rceil$ the ceiling of a number.

Definition 2.1. *([6, 28]) Let $\mathfrak{T} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For $0 \leq \tau \leq l$, the classical Riesz-Caputo fractional derivative is defined by*

$$
\begin{aligned}
{}^{RC}_{0}D^{\zeta}_{l}\mathfrak{T}(\tau) &= \frac{1}{\Gamma(n-\zeta)}\int_{0}^{l} |\tau-\varphi|^{n-\zeta-1}\mathfrak{T}^{(n)}(\varphi)d\varphi \\
&= \frac{1}{2}\left({}^{C}_{0}D^{\zeta}_{\tau} + (-1)^{n}\,{}^{C}_{\tau}D^{\zeta}_{l}\right)\mathfrak{T}(\tau),
\end{aligned}
$$

where ${}^{C}_{0}D^{\zeta}_{\tau}$ and ${}^{C}_{\tau}D^{\zeta}_{l}$ are the Caputo derivative of left and right, respectively defined by

$$
\begin{aligned}
{}^{C}_{0}D^{\zeta}_{\tau}\mathfrak{T}(\tau) &= \frac{1}{\Gamma(n-\zeta)}\int_{0}^{\tau} (\tau-\varphi)^{n-\zeta-1}\mathfrak{T}^{(n)}(\varphi)d\varphi, \\
{}^{C}_{\tau}D^{\zeta}_{l}\mathfrak{T}(\tau) &= \frac{(-1)^{n}}{\Gamma(n-\zeta)}\int_{\tau}^{l} (\varphi-\tau)^{n-\zeta-1}\mathfrak{T}^{(n)}(\varphi)d\varphi.
\end{aligned}
$$

Remark 2.2. *([16, 17]) When $\mathfrak{T}(\tau) \in \mathfrak{C}(\mathcal{J})$ and $0 < \zeta \leq 1$, we have*

$$
{}^{RC}_{0}D^{\zeta}_{l}\mathfrak{T}(\tau) = \frac{1}{2}\left({}^{C}_{0}D^{\zeta}_{\tau} - {}^{C}_{\tau}D^{\zeta}_{l}\right)\mathfrak{T}(\tau),
$$

and when $\mathfrak{T}(\tau) \in \mathfrak{C}^{2}(\mathcal{J})$ and $1 < \zeta \leq 2$, then we have

$$
{}^{RC}_{0}D^{\zeta}_{l}\mathfrak{T}(\tau) = \frac{1}{2}\left({}^{C}_{0}D^{\zeta}_{\tau} + {}^{C}_{\tau}D^{\zeta}_{l}\right)\mathfrak{T}(\tau).
$$

Definition 2.3. *([17]) Let $\mathfrak{T} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The notions of Riemann-Liouville fractional integrals of order ζ are defined as*

$$
\begin{aligned}
{}_{0}I^{\zeta}_{\tau}\mathfrak{T}(\tau) &= \frac{1}{\Gamma(\zeta)}\int_{0}^{\tau} (\tau-\varphi)^{(\zeta-1)}\mathfrak{T}(\varphi)d\varphi, \\
{}_{\tau}I^{\zeta}_{l}\mathfrak{T}(\tau) &= \frac{1}{\Gamma(\zeta)}\int_{\tau}^{l} (\varphi-\tau)^{(\zeta-1)}\mathfrak{T}(\varphi)d\varphi, \\
{}_{0}I^{\zeta}_{l}\mathfrak{T}(\tau) &= \frac{1}{\Gamma(\zeta)}\int_{0}^{l} |\varphi-\tau|^{(\zeta-1)}\mathfrak{T}(\varphi)d\varphi.
\end{aligned}
$$

Lemma 2.4. ([27],[8]) *Let $\mathfrak{T}(\tau) \in \mathfrak{C}^{n}(\mathcal{J})$. Then*

$$
{}_{0}I^{\zeta}_{\tau}\,{}^{C}_{0}D^{\zeta}_{\tau}\mathfrak{T}(\tau) = \mathfrak{T}(\tau) - \sum_{j=0}^{n-1}\frac{\mathfrak{T}^{(j)}(0)}{j!}(\tau-0)^{j}
$$

and

$$
{}_{\tau}I^{\zeta}_{l}\,{}^{C}_{\tau}D^{\zeta}_{l}\mathfrak{T}(\tau) = (-1)^{n}\left[\mathfrak{T}(\tau) - \sum_{j=0}^{n-1}\frac{(-1)^{j}\mathfrak{T}^{(j)}(l)}{j!}(l-\tau)^{j}\right].
$$

The aforementioned definitions lead us to the following,

$$
{}_{0}I^{\zeta}_{l}\,{}^{RC}_{0}D^{\zeta}_{l}\mathfrak{T}(\tau) = \frac{1}{2}\,{}_{0}I^{\zeta}_{l}\left({}^{C}_{0}D^{\zeta}_{\tau} + (-1)^{n}\,{}^{C}_{\tau}D^{\zeta}_{l}\right)\mathfrak{T}(\tau)
$$

$$
\begin{aligned}
&= \frac{1}{2}{}_0I_l^{\zeta}{}_0^C D_\tau^{\zeta}\mathfrak{T}(\tau) + \frac{(-1)^n}{2}{}_0I_l^{\zeta}{}_\tau^C D_l^{\zeta}\mathfrak{T}(\tau) \\
&= \frac{1}{2}\left({}_0I_\tau^{\zeta}{}_0^C D_\tau^{\zeta} + {}_\tau I_l^{\zeta}{}_0^C D_\tau^{\zeta}\right)\mathfrak{T}(\tau) + \frac{(-1)^n}{2}\left({}_0I_\tau^{\zeta}{}_\tau^C D_l^{\zeta} + {}_\tau I_l^{\zeta}{}_\tau^C D_l^{\zeta}\right)\mathfrak{T}(\tau) \\
&= \frac{1}{2}\left({}_0I_\tau^{\zeta}{}_0^C D_\tau^{\zeta} + (-1)^n_\tau I_l^{\zeta}{}_\tau^C D_l^{\zeta}\right)\mathfrak{T}(\tau).
\end{aligned}
$$

In particular, if $2 < \zeta \leq 3$ *and* $\mathfrak{T}(\tau) \in \mathfrak{C}^3(\mathcal{J})$, *then*

$$
\begin{aligned}
{}_0I_l^{\zeta}{}_0^{RC}D_l^{\zeta}\mathfrak{T}(\tau) = {}&\mathfrak{T}(\tau) - \frac{1}{2}(\mathfrak{T}(0) + \mathfrak{T}(l)) - \frac{1}{2}(\mathfrak{T}'(0) + \mathfrak{T}'(l))\tau + \frac{l}{2}\mathfrak{T}'(l) \\
&- \frac{1}{4}(\mathfrak{T}''(0) + \mathfrak{T}''(l))\tau^2 - \frac{1}{4}\mathfrak{T}''(l)[l^2 - 2l\tau].
\end{aligned} \tag{2.1}
$$

3. Main Outcomes

Here we derive a few interesting findings concerning a certain kind of Riesz-Caputo fractional differential equations. We make way into this section by proving the succeeding result.

Lemma 3.1. *Consider* $\hbar \in \mathfrak{C}(\mathcal{J}, \mathbb{R})$ *and* $\varpi \in \mathfrak{C}^3(\mathcal{J})$. *A function* ϖ *is a solution of the Riesz-Caputo fractional differential equation*

$$
\begin{aligned}
&{}_0^{RC}D_l^{\zeta}\varpi(\tau) + \hbar(\tau) = 0, \tau \in [0, l], 2 < \zeta \leq 3, \\
&a_1\varpi(0) + b_1\varpi(l) = 0, a_2\varpi'(0) + b_2\varpi'(l) = 0, a_3\varpi''(0) + b_3\varpi''(l) = 0,
\end{aligned} \tag{3.1}
$$

if and only if ϖ *is a solution of the following fractional integral equation:*

$$
\begin{aligned}
\varpi(\tau) = {}&\frac{\chi_1 l^2 + \chi_2 l\tau + \chi_3\tau^2}{\Gamma(\zeta - 2)}\int_0^l (l - s)^{(\zeta-3)}\hbar(s)ds + \frac{\chi_4\tau + \chi_5 l}{\Gamma(\zeta - 1)}\int_0^l (l - s)^{(\zeta-2)}\hbar(s)ds \\
&- \frac{1}{\Gamma(\zeta)}\int_0^\tau (\tau - s)^{(\zeta-1)}\hbar(s)ds - \frac{1}{\Gamma(\zeta)}\int_\tau^l (s - \tau)^{(\zeta-1)}\hbar(s)ds,
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
\chi_1 &= \frac{2b_3b_2\alpha_1 - 2b_3a_2\alpha_2 + 2a_3a_2\alpha_1 + a_3\alpha_1\beta_1}{2\alpha_1\beta_1\gamma_1}, \\
\chi_2 &= \frac{2b_3a_2\alpha_1 - 2b_3b_2\alpha_1 + 2a_3\alpha_1\beta_1}{2\alpha_1\beta_1\gamma_1}, \\
\chi_3 &= \frac{\gamma_2\alpha_1\beta_1}{2\alpha_1\beta_1\gamma_1}, \\
\chi_4 &= \frac{\alpha_1\beta_2}{2\alpha_1\beta_1}, \\
\chi_5 &= \frac{-b_2\alpha_2 - a_2\alpha_1}{2\alpha_1\beta_1}, \\
(a_1 + b_1) &= \alpha_1, \\
(a_1 - b_1) &= \alpha_2, \\
(a_2 + b_2) &= \beta_1, \\
(a_2 - b_2) &= \beta_2, \\
(a_3 + b_3) &= \gamma_1, \\
(a_3 - b_3) &= \gamma_2.
\end{aligned}
$$

Proof. Suppose that ϖ satisfies (3.1). By virtue of Lemma 2.4, the equation (2.1) converts to

$$\begin{aligned}\varpi(\tau) &= \frac{1}{2}[\varpi(0)+\varpi(l)] + \frac{\tau}{2}\left[\varpi'(0)+\varpi'(l)\right] - \frac{l}{2}\varpi'(l) + \frac{\tau^2}{4}\left[\varpi''(0)+\varpi''(l)\right] \\ &\quad + \frac{1}{4}\varpi''(l)\left(l^2-2l\tau\right) + \frac{1}{\Gamma(\zeta)}\int_0^\tau (l-s)^{(\zeta-1)}\hbar(s)ds + \frac{1}{\Gamma(\zeta)}\int_\tau^l (s-l)^{(\zeta-1)}\hbar(s)ds. \end{aligned} \tag{3.3}$$

Then,

$$\begin{aligned}\varpi'(\tau) &= \frac{1}{2}\left[\varpi'(0)+\varpi'(l)\right] + \frac{\tau}{2}\left[\varpi''(0)+\varpi''(l)\right] + \frac{1}{2}\varpi''(l)(-l) \\ &\quad + \frac{1}{\Gamma(\zeta-1)}\int_0^\tau (l-s)^{(\zeta-2)}\hbar(s)ds + \frac{1}{\Gamma(\zeta-1)}\int_\tau^l (s-l)^{(\zeta-2)}\hbar(s)ds.\end{aligned}$$

Using boundary conditions on (3.3), we get

$$\begin{aligned}\varpi(0) &= \left(\frac{-b_1}{a_1+b_1}\right)\left[\frac{2l^2b_3b_2}{(a_2+b_2)(a_3+b_3)\Gamma(\zeta-2)} - \frac{l^2b_3}{(a_3+b_3)\Gamma(\zeta-2)}\right]\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds \\ &\quad + \left(\frac{-b_1}{a_1+b_1}\right)\left[\frac{-lb_2}{(a_2+b_2)\Gamma(\zeta-1)}\right]\int_0^l (l-s)^{(\zeta-2)}\hbar(s)ds, \\ \varpi(l) &= \left(\frac{a_1}{a_1+b_1}\right)\left[\frac{2lb_3b_2}{(a_2+b_2)(a_3+b_3)\Gamma(\zeta-2)} - \frac{l^2b_3}{(a_3+b_3)\Gamma(\zeta-2)}\right]\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds \\ &\quad + \left(\frac{a_1}{a_1+b_1}\right)\left[\frac{-lb_2}{(a_2+b_2)\Gamma(\zeta-1)}\right]\int_0^l (l-s)^{(\zeta-2)}\hbar(s)ds \\ &\quad + \frac{2a_1}{(a_1+b_1)\Gamma(\zeta)}\int_0^l (l-s)^{(\zeta-1)}\hbar(s)ds, \\ \varpi'(0) &= \frac{2lb_2b_3}{(a_2+b_2)(a_3+b_3)\Gamma(\zeta-2)}\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds \\ &\quad - \frac{b_2}{(a_2+b_2)\Gamma(\zeta-1)}\int_0^1 (l-s)^{(\zeta-2)}\hbar(s)ds, \\ \varpi'(l) &= \frac{-2la_2b_3}{\Gamma(\zeta-2)(a_2+b_2)(a_3+b_3)}\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds \\ &\quad + \frac{a_2}{(a_2+b_2)\Gamma(\zeta-1)}\int_0^l (l-s)^{(\zeta-2)}\hbar(s)ds \\ \varpi''(0) &= \frac{-2b_3}{(a_3+b_3)\Gamma(\zeta-2)}\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds \text{ and} \\ \varpi''(l) &= \frac{2a_3}{(a_3+b_3)\Gamma(\zeta-2)}\int_0^l (l-s)^{(\zeta-3)}\hbar(s)ds.\end{aligned}$$

Substituting all the previous values into (3.3), we get the required (3.2). □

Now we note down the following notations:

$$\Theta_1 = \frac{\chi_1 l^\zeta}{\Gamma(\zeta-1)} + \frac{\chi_5 l^\zeta}{\Gamma(\zeta)} + \frac{2l^\zeta}{\Gamma(\zeta+1)}, \Theta_2 = \frac{\chi_3 l^{(\zeta-\eta)}}{2\Gamma(3-\eta)\Gamma(\zeta-1)}, N_1 = \frac{2l^\zeta}{\Gamma(\zeta+1)}, N_2 = \frac{2l^{(\zeta-\eta-1)}}{\Gamma(\zeta-\eta)}.$$

3.1. First existence result

Let $\mathfrak{C}([0,l],\mathbb{R})$ be a Banach space of all continuous functions defined on $\mathcal{J}=[0,l]$ that are mapped into $\mathbb{R}$. Then Λ be a Banach space defined as,

$$\Lambda=\left\{\varpi:\varpi\in\mathfrak{C}([0,l]),{}^{RC}D^{\delta}\varpi\in\mathfrak{C}([0,l])\right\},$$

equipped with the norm

$$\|\varpi\|_{\Lambda}=\|\varpi\|+\left\|{}^{RC}D^{\delta}\varpi\right\|,$$

where

$$\|\varpi\|=\sup_{\tau\in\mathcal{J}}|\varpi(\tau)|,\left\|{}^{RC}D^{\delta}\varpi\right\|=\sup_{\tau\in\mathcal{J}}\left|{}^{RC}D^{\delta}\varpi(\tau)\right|.$$

Theorem 3.2. *Let Λ be a Banach space. Assume that $\mathcal{T}:\Lambda\to\Lambda$ is completely continuous operator and the set*

$$V=\{\varpi\in\Lambda\mid\varpi=\mu(\mathcal{T}\varpi),0<\mu<1\}$$

is bounded. Then $\mathcal{T}$ has a fixed point in Λ.

Theorem 3.3. *Consider $\mathfrak{T}:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that*

(H_1) there is a positive $\wp$ such that

$$|\mathfrak{T}(\varphi,\varpi,\eta)|<\wp\quad for\quad \varpi,\eta\in\mathbb{R},\quad\tau\in\mathcal{J}.$$

Then the problem (1.2) possesses at least one solution in $\mathcal{J}$.

Proof. Define an operator $\mathcal{T}:\Lambda\to\Lambda$ by

$$\begin{aligned}(\mathcal{T}\varpi)(\tau)=&\frac{\chi_1l^2+\chi_2l\tau+\chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^l(l-\varphi)^{(\zeta-3)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi\\&+\frac{\chi_4\tau+\chi_5l}{\Gamma(\zeta-1)}\int_0^l(l-\varphi)^{(\zeta-2)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi\\&-\frac{1}{\Gamma(\zeta)}\int_0^{\tau}(\tau-\varphi)^{(\zeta-1)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi\\&-\frac{1}{\Gamma(\zeta)}\int_{\tau}^{l}(\varphi-\tau)^{(\zeta-1)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi.\end{aligned}$$

The fact that the operator $\mathcal{T}$ is uniformly continuous and also, $(\mathcal{T}\varpi)(\tau)$ belongs to Λ for every $\tau\in\mathcal{J}$, one can easily verify that $\mathcal{T}$ is well-defined. Now, we utilize Schafer fixed point theorem to illustrate that $\mathcal{T}$ owns a fixed point in Λ. The proof is split into four parts:

Step 1: $\mathcal{T}$ is a continuous mapping on Λ. We conclude that $\mathcal{T}$ is continuous keeping in mind the continuity of $\mathfrak{T}$.

Step 2: $\mathcal{T}$ maps bounded sets into bounded sets in Λ. For every $\varpi\in\mathcal{B}_r=\{\varpi\in\Lambda:\|\varpi\|_{\Lambda}\leq r\}$ and $\tau\in\mathcal{J}$, we obtain

$$\begin{aligned}|(\mathcal{T}\varpi)(\tau)|\leq&\frac{\chi_1l^2+\chi_2l\tau+\chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^l(l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\&+\frac{\chi_4\tau+\chi_5l}{\Gamma(\zeta-1)}\int_0^l(l-\varphi)^{(\zeta-2)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\&+\frac{1}{\Gamma(\zeta)}\int_0^{\tau}(\tau-\varphi)^{(\zeta-1)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\end{aligned}$$

$$- \frac{1}{\Gamma(\zeta)} \int_{\tau}^{l} (\varphi - \tau)^{(\zeta-1)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi$$
$$\leq \frac{\chi_1 l^{\zeta} \wp}{\Gamma(\zeta - 1)} + \frac{\chi_5 l^{\zeta} \wp}{\Gamma(\zeta)} + \frac{2 l^{\zeta} \wp}{\Gamma(\zeta + 1)}$$

implies that

$$\|(\mathcal{T}\varpi)(\tau)\| \leq \wp\Theta_1 \tag{3.4}$$

and

$$\begin{aligned} \left| {}_0^{RC}D_l^{\eta}(\mathcal{T}\varpi)(\tau) \right| \leq & \frac{1}{\Gamma(\zeta - \eta - 1)} \int_0^{\tau} (\tau - \varphi)^{(\zeta-\eta-2)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{1}{\Gamma(\zeta - \eta - 1)} \int_{\tau}^{l} (\varphi - \tau)^{(\zeta-\eta-2)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{\chi_3 \left[\tau^{(2-\eta)} + (l - \tau)^{(2-\eta)} \right]}{2\Gamma(3 - \eta)\Gamma(\zeta - 2)} \int_0^{l} (l - \varphi)^{(\zeta-3)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & \leq \frac{\chi_3 l^{(\zeta-\eta)} \wp}{2\Gamma(3 - \eta)\Gamma(\zeta - 1)} + \frac{2 l^{(\zeta-\eta-1)} \wp}{\Gamma(\zeta - \eta)} \end{aligned}$$

implies that

$$\left\| {}_0^{RC}D_l^{\eta}(\mathcal{T}\varpi)(\tau) \right\| \leq \wp\Theta_2. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\|(\mathcal{T}\varpi)(\tau)\|_{\Lambda} < \wp\left(\Theta_1 + \Theta_2\right) < \infty,$$

which leads to the fact that $\mathcal{T}$ maps bounded sets to bounded sets on Λ.

Step 3: $\mathcal{T}$ maps bounded sets into equi-continuous sets in Λ. Let $\mathcal{B}_r$ be a bounded set of Λ as defined in Step 2, and let $\varpi \in \mathcal{B}_r$. For each $\tau_1, \tau_2 \in \mathcal{J}, \tau_1 < \tau_2$, we get

$$\begin{aligned} |(\mathcal{T}\varpi)(\tau_2) - (\mathcal{T}\varpi)(\tau_1)| \leq & \frac{\chi_2 l(\tau_2 - \tau_1) + \chi_3(\tau_2^2 - \tau_1^2)}{\Gamma(\zeta - 2)} \int_0^{l} (l - \varphi)^{(\zeta-3)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{\chi_4(\tau_2 - \tau_1)}{\Gamma(\zeta - 1)} \int_0^{l} (l - \varphi)^{(\zeta-2)} \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{1}{\Gamma(\zeta)} \int_0^{\tau_1} \left| (\tau_1 - \varphi)^{(\zeta-1)} - (\tau_2 - \varphi)^{(\zeta-1)} \right| \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{1}{\Gamma(\zeta)} \int_{\tau_1}^{\tau_2} \left| (\zeta - \tau_1)^{(\zeta-1)} - (\tau_2 - \varphi)^{(\zeta-1)} \right| \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ & + \frac{1}{\Gamma(\zeta)} \int_{\tau_2}^{l} \left| (\zeta - \tau_1)^{(\zeta-1)} - (\zeta - \tau_2)^{(\zeta-1)} \right| \left| \mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right) \right| d\varphi \\ \leq & \frac{\chi_2 l^{(\zeta-1)} \left|(\tau_2 - \tau_1)\right|^{(\zeta-1)} + \chi * l^{(\zeta-2)} |(\tau_2^2 - \tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta - 1)} \wp + \frac{\chi_4 l^{(\zeta-1)} \left|(\tau_2 - \tau_1)\right|^{(\zeta-1)}}{\Gamma(\zeta)} \wp \\ & + \frac{\left| (\tau_1^{(\zeta)} - \tau_2^{(\zeta)}) + (\tau_2 - \tau_1)^{(\zeta)} \right|}{\Gamma(\zeta + 1)} \wp \\ & + \frac{\left| ((l - \tau_2)^{(\zeta)} - (l - \tau_1)^{(\zeta)}) + (\tau_2 - \tau_1)^{(\zeta)} \right|}{\Gamma(\zeta + 1)} \wp, \end{aligned}$$

which leads us to

$$\begin{aligned}\|(\mathcal{T}\varpi)(\tau_2)-(\mathcal{T}\varpi)(\tau_1)\| \leq & \frac{\chi_2 l^{(\zeta-1)}\,|(\tau_2-\tau_1)|^{(\zeta-1)}+\chi * l^{(\zeta-2)}|(\tau_2^2-\tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta-1)}\wp \\ &+\frac{\chi_4 l^{(\zeta-1)}\,|(\tau_2-\tau_1)|^{(\zeta-1)}}{\Gamma(\zeta)}\wp+\frac{\left|(\tau_1^{(\zeta)}-\tau_2^{(\zeta)})+(\tau_2-\tau_1)^{(\zeta)}\right|}{\Gamma(\zeta+1)}\wp \\ &+\frac{\left|((l-\tau_2)^{(\zeta)}-(l-\tau_1)^{(\zeta)})+(\tau_2-\tau_1)^{(\zeta)}\right|}{\Gamma(\zeta+1)}\wp, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\left\|{}_0^{RC}D_l^{\eta}(\mathcal{T}\varpi)(\tau_2)-{}_0^{RC}D_l^{\eta}(\mathcal{T}\varpi)(\tau_1)\right\| \\ &\leq \frac{\left|\tau_1^{(\zeta-\eta-1)}-\tau_2^{(\zeta-\eta-1)}\right|}{2\Gamma(\zeta-\eta)}\wp+\frac{\left|((l-\tau_1)^{(\zeta-\eta)}-(l-\tau_2)^{(\zeta-\eta)})+(\tau_2-\tau_1)^{(\zeta-\eta-1)}\right|}{\Gamma(\zeta)}\wp \\ &+\frac{\chi_3\wp}{2\Gamma(\zeta-1)\Gamma(3-\eta)}\left|(\tau_2-\tau_1)^{(\zeta-\eta)}+(l-\tau_1)^{(\zeta-\eta)}-(l-\tau_2)^{(\zeta-\eta)}\right|. \end{aligned} \tag{3.7}$$

Hence adding (3.6) and (3.7),

$$\begin{aligned}&\|(\mathcal{T}\varpi)(\tau_2)-(\mathcal{T}\varpi)(\tau_1)\|_\Lambda \\ &\leq \frac{\chi_2 l^{(\zeta-1)}\,|(\tau_2-\tau_1)|^{(\zeta-1)}+\chi * l^{(\zeta-2)}|(\tau_2^2-\tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta-1)}\wp \\ &+\frac{\chi_4 l^{(\zeta-1)}\,|(\tau_2-\tau_1)|^{(\zeta-1)}}{\Gamma(\zeta)}\wp+\frac{\left|(\tau_1^{(\zeta)}-\tau_2^{(\zeta)})+(\tau_2-\tau_1)^{(\zeta)}\right|}{\Gamma(\zeta+1)}\wp \\ &+\frac{\left|((l-\tau_2)^{(\zeta)}-(l-\tau_1)^{(\zeta)})+(\tau_2-\tau_1)^{(\zeta)}\right|}{\Gamma(\zeta+1)}\wp+\frac{\left|\tau_1^{(\zeta-\eta-1)}-\tau_2^{(\zeta-\eta-1)}\right|}{2\Gamma(\zeta-\eta)}\wp \\ &+\frac{\left|((l-\tau_1)^{(\zeta-\eta)}-(l-\tau_2)^{(\zeta-\eta)})+(\tau_2-\tau_1)^{(\zeta-\eta-1)}\right|}{\Gamma(\zeta)}\wp \\ &+\frac{\chi_3\wp}{2\Gamma(\zeta-1)\Gamma(3-\eta)}\left|(\tau_2-\tau_1)^{(\zeta-\eta)}+(l-\tau_1)^{(\zeta-\eta)}-(l-\tau_2)^{(\zeta-\eta)}\right| \end{aligned}$$

which implies that $\|(\mathcal{T}\varpi)(\tau_2)-(\mathcal{T}\varpi)(\tau_1)\|_\Lambda \to 0$ as $\tau_2 \to \tau_1$. We infer from Arzela-Ascoli theorem that $\mathcal{T}$ is a completely continuous operator.

Step 4: We demonstrate that the set Δ defined by

$$\Delta = \{\varpi \in \Lambda : \varpi = \varrho(\mathcal{T}\varpi),\ 0 < \varrho < 1\}$$

is bounded. Consider $\varpi \in \Delta$, for some $\varrho \in (0,1)$. Then for every $\tau \in \mathcal{J}$

$$\begin{aligned}\frac{1}{\varrho}|\varpi(\tau)| \leq & \frac{\chi_1 l^2+\chi_2 l\tau+\chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^l (l-\varphi)^{(\zeta-3)}\left|\mathfrak{I}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi \\ &+\frac{\chi_4\tau+\chi_5 l}{\Gamma(\zeta-1)}\int_0^l (l-\varphi)^{(\zeta-2)}\left|\mathfrak{I}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi \\ &+\frac{1}{\Gamma(\zeta)}\int_0^\tau (\tau-\varphi)^{(\zeta-1)}\left|\mathfrak{I}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi \\ &-\frac{1}{\Gamma(\zeta)}\int_\tau^l (\varphi-\tau)^{(\zeta-1)}\left|\mathfrak{I}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi \\ &\leq \frac{\chi_1 l^{\zeta}\wp}{\Gamma(\zeta-1)}+\frac{\chi_5 l^{\zeta}\wp}{\Gamma(\zeta)}+\frac{2l^{\zeta}\wp}{\Gamma(\zeta+1)}. \end{aligned}$$

Therefore,

$$\|\varpi\| \leq \varrho\Theta_1\wp \tag{3.8}$$

and

$$\begin{aligned}\frac{1}{\varrho}\left|{}_0^{RC}D_l^{\eta}\varpi(\tau)\right| \leq &\frac{1}{\Gamma(\zeta-\eta-1)}\int_0^{\tau}(\tau-\varphi)^{(\zeta-\eta-2)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ &+\frac{1}{\Gamma(\zeta-\eta-1)}\int_{\tau}^{l}(\varphi-\tau)^{(\zeta-\eta-2)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ &+\frac{\chi_3\left[\tau^{(2-\eta)}+(l-\tau)^{(2-\eta)}\right]}{2\Gamma(3-\eta)\Gamma(\zeta-2)}\int_0^{l}(l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ \leq &\frac{\chi_3 l^{(\zeta-\eta)}\wp}{2\Gamma(3-\eta)\Gamma(\zeta-1)}+\frac{2l^{(\zeta-\eta-1)}\wp}{\Gamma(\zeta-\eta)}.\end{aligned}$$

Therefore,

$$\left\|{}_0^{RC}D_l^{\eta}\varpi(\tau)\right\| \leq \varrho\Theta_2\wp. \tag{3.9}$$

From inequalities (3.8) and (3.9), we get

$$\|\varpi\|_{\Lambda} \leq \varrho\left(\Theta_1+\Theta_2\right)\wp.$$

Hence, $\|\varpi\|_{\Lambda} < \infty$. This leads to the fact that Δ is bounded and employing Schafer fixed point theorem we conclude that the equation (1.2) has at least one solution in $\mathcal{J}$. □

3.2. Second existence result

Theorem 3.4. *Let $\mathfrak{T}:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following:*

(H_2) There is a non-negative function $\Theta\in\mathfrak{C}(\mathcal{J},\mathbb{R}^+)$ satisfying

$$|\mathfrak{T}(\tau,\varpi,\eta)| \leq \Theta(\tau) \text{ for all } (\tau,\varpi,\eta)\in\mathcal{J}\times\mathbb{R}\times\mathbb{R},$$

(H_3) $(M_1+M_2)(N_1+N_2)<1$.

Then Λ contains at least one solution to the problem (1.2).

Proof. Define the operators $(\mathcal{T}_1\varpi)(\tau)$ and $(\mathcal{T}_2\varpi)(\tau)$ as

$$\begin{aligned}(\mathcal{T}_1\varpi)(\tau) = &\frac{\chi_1 l^2+\chi_2 l\tau+\chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^{l}(l-\varphi)^{(\zeta-3)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi\\ &+\frac{\chi_4\tau+\chi_5 l}{\Gamma(\zeta-1)}\int_0^{l}(l-\varphi)^{(\zeta-2)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi, \text{ and}\\ (\mathcal{T}_2\varpi)(\tau) = &-\frac{1}{\Gamma(\zeta)}\int_0^{\tau}(\tau-\varphi)^{(\zeta-1)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi\\ &-\frac{1}{\Gamma(\zeta)}\int_{\tau}^{l}(\varphi-\tau)^{(\zeta-1)}\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)d\varphi.\end{aligned}$$

Now we choose $d \geq (\Theta_1+\Theta_2)(M_1+M_2)\|\Theta\|$, and define $\mathcal{B}_s=\{\varpi\in\Lambda:\|\varpi\|_{\Lambda}\leq d\}$.

Step 1: We attest that $\mathcal{T}_1\varpi(\tau)+\mathcal{T}_2\varpi(\tau)\in\mathcal{B}_s$. For any $\varpi,\eta\in\mathcal{B}_s$ and for each $\tau\in\mathcal{J}$, we derive

$$\left|(\mathcal{T}_1\varpi)(\tau)+(\mathcal{T}_2\eta)(\tau)\right| \leq \frac{\chi_1 l^2+\chi_2 l\tau+\chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^{l}(l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi$$

$$
\begin{aligned}
&+ \frac{\chi_4\tau + \chi_5 l}{\Gamma(\zeta-1)}\int_0^l (l-\varphi)^{(\zeta-2)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
&+ \frac{1}{\Gamma(\zeta)}\int_0^{\tau} (\tau-\varphi)^{(\zeta-1)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
&- \frac{1}{\Gamma(\zeta)}\int_{\tau}^{l} (\varphi-\tau)^{(\zeta-1)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
\leq& \frac{\chi_1 l^{\zeta}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta-1)} + \frac{\chi_5 l^{\zeta}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta)} + \frac{2l^{\zeta}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta+1)}.
\end{aligned}
$$

Then,

$$
\left\|(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\eta)(\tau)\right\| \leq \Theta_1 (M_1+M_2)\|\Theta\|. \tag{3.10}
$$

Again,

$$
\begin{aligned}
&\left\|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau) + {}_0^{RC}D_l^{\eta}(\mathcal{T}_2\eta)(\tau)\right\|\\
\leq& \frac{1}{\Gamma(\zeta-\eta-1)}\int_0^{\tau} (\tau-\varphi)^{(\zeta-\eta-2)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
&+ \frac{1}{\Gamma(\zeta-\eta-1)}\int_{\tau}^{l} (\varphi-\tau)^{(\zeta-\eta-2)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
&+ \frac{\chi_3\left[\tau^{(2-\eta)} + (l-\tau)^{(2-\eta)}\right]}{2\Gamma(3-\eta)\Gamma(\zeta-2)}\int_0^{l} (l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
\leq& \frac{\chi_3 l^{(\zeta-\eta)}(M_1+M_2)\|\Theta\|}{2\Gamma(3-\eta)\Gamma(\zeta-1)} + \frac{2l^{(\zeta-\eta-1)}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta-\eta)}.
\end{aligned}
$$

Hence,

$$
\left\|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau) + {}_0^{RC}D_l^{\eta}(\mathcal{T}_2\eta)(\tau)\right\| \leq \Theta_2 (M_1+M_2)\|\Theta\|. \tag{3.11}
$$

It follows from (3.10) and (3.11) that

$$
\left\|(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\eta)(\tau)\right\|_{\Lambda} \leq (\Theta_1+\Theta_2)(M_1+M_2)\|\Theta\| \leq d,
$$

implying that $(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\varpi)(\tau) \in \mathcal{B}_s$.

Step 2: The continuity of $\mathfrak{T}$ implies the continuity of $\mathcal{T}_1$. Next, we prove that $\mathcal{T}_1$ is a self-mapping on the bounded set of Λ. For $\varpi \in \mathcal{B}_s$, and for every $\tau \in \mathcal{J}$, we get

$$
\begin{aligned}
|(\mathcal{T}_1\varpi)(\tau)| \leq& \frac{\chi_1 l^2 + \chi_2 l\tau + \chi_3\tau^2}{\Gamma(\zeta-2)}\int_0^l (l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
&+ \frac{\chi_4\tau + \chi_5 l}{\Gamma(\zeta-1)}\int_0^l (l-\varphi)^{(\zeta-2)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi\\
\leq& \frac{\chi_1 l^{\zeta}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta-1)} + \frac{\chi_5 l^{\zeta}(M_1+M_2)\|\Theta\|}{\Gamma(\zeta)}.
\end{aligned}
$$

Hence,

$$
\|(\mathcal{T}_1\varpi)(\tau)\| \leq \left(\frac{\chi_1 l^{\zeta}}{\Gamma(\zeta-1)} + \frac{\chi_5 l^{\zeta}}{\Gamma(\zeta)}\right)(M_1+M_2)\|\Theta\| \tag{3.12}
$$

and

$$
\left|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau)\right| \leq \frac{\chi_3\left[\tau^{(2-\eta)} + (l-\tau)^{(2-\eta)}\right]}{2\Gamma(3-\eta)\Gamma(\zeta-2)}\int_0^l (l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^{\eta}\varpi(\varphi)\right)\right| d\varphi
$$

$$\leq \frac{\chi_3 l^{(\zeta-\eta)}\left[(M_1+M_2)\|\Theta\|\right]}{2\Gamma(3-\eta)\Gamma(\zeta-1)}.$$

Hence,

$$\left\|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau)\right\| \leq \frac{\chi_3 l^{(\zeta-\eta)}\left[(M_1+M_2)\|\Theta\|\right]}{2\Gamma(3-\eta)\Gamma(\zeta-1)}. \tag{3.13}$$

By Combining (3.12) and (3.13), we get

$$\begin{aligned}\|(\mathcal{T}_1\varpi)(\tau)\|_\Lambda &\leq \left(\frac{\chi_1 l^{\zeta}}{\Gamma(\zeta-1)}+\frac{\chi_5 l^{\zeta}}{\Gamma(\zeta)}+\frac{\chi_3 l^{(\zeta-\eta)}}{2\Gamma(3-\eta)\Gamma(\zeta-1)}\right)\left[(M_1+M_2)\|\Theta\|\right]\\ &\Rightarrow \|\mathcal{T}_1\varpi(\tau)\|_\Lambda<\infty.\end{aligned}$$

As a result of the previous inequality, $\mathcal{T}_1$ is uniformly bounded. Now we prove that $\mathcal{T}_1$ maps bounded set into equi-continuous set of Λ. Consider $\tau_1,\tau_2\in\mathcal{J},\tau_1<\tau_2,\varpi\in\mathcal{B}_s$. Therefore,

$$\begin{aligned}&|(\mathcal{T}_1\varpi)(\tau_2)-(\mathcal{T}_1\varpi)(\tau_1)|\\ &\leq \frac{\chi_2 l(\tau_2-\tau_1)+\chi_3(\tau_2^2-\tau_1^2)}{\Gamma(\zeta-2)}\int_0^l (l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ &\quad+\frac{\chi_4(\tau_2-\tau_1)}{\Gamma(\zeta-1)}\int_0^l (l-\varphi)^{(\zeta-2)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ &\leq \frac{\chi_2 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}+\chi * l^{(\zeta-2)}|(\tau_2^2-\tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta-1)}\left[(M_1+M_2)\|\varpi\|\right]\\ &\quad+\frac{\chi_4 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}}{\Gamma(\zeta)}\left[(M_1+M_2)\|\varpi\|\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\|(\mathcal{T}_1\varpi)(\tau_2)-(\mathcal{T}_1\varpi)(\tau_1)\|\\ &\leq\left[\frac{\chi_2 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}+\chi * l^{(\zeta-2)}|(\tau_2^2-\tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta-1)}+\frac{\chi_4 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}}{\Gamma(\zeta)}\right]\left[(M_1+M_2)\|\Theta\|\right].\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}&\left|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau_2)-{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau_1)\right|\\ &\leq \frac{\chi_3\left[\tau^{(2-\eta)}+(l-\tau)^{(2-\eta)}\right]}{2\Gamma(3-\eta)\Gamma(\zeta-2)}\int_0^l (l-\varphi)^{(\zeta-3)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^{\eta}\varpi(\varphi)\right)\right|d\varphi\\ &\leq \frac{\chi_3 l^{(\zeta-\eta)}(M_1+M_2)\|\Theta\|}{2\Gamma(3-\eta)\Gamma(\zeta-1)}.\end{aligned}$$

Hence,

$$\left\|{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau_2)-{}_0^{RC}D_l^{\eta}(\mathcal{T}_1\varpi)(\tau_1)\right\| \leq \frac{\chi_3 l^{(\zeta-\eta)}(M_1+M_2)\|\Theta\|}{2\Gamma(3-\eta)\Gamma(\zeta-1)}. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned}&\|(\mathcal{T}_1\varpi)(\tau_2)-(\mathcal{T}_1\varpi)(\tau_1)\|_\Lambda\\ &\leq\left[\frac{\chi_2 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}+\chi * l^{(\zeta-2)}|(\tau_2^2-\tau_1^2)|^{(\zeta-2)}}{\Gamma(\zeta-1)}+\frac{\chi_4 l^{(\zeta-1)}|(\tau_2-\tau_1)|^{(\zeta-1)}}{\Gamma(\zeta)}\right]\left[(M_1+M_2)\|\Theta\|\right]\end{aligned}$$

$$+\left[\frac{\chi_3 l^{(\zeta-\eta)}}{2\Gamma(3-\eta)\Gamma(\zeta-1)}\right][(M_1+M_2)\|\Theta\|].$$

As a result of these procedures, we deduce that $\mathcal{T}_1$ is continuous and compact.
Step 3: We demonstrate that $\mathcal{T}_2$ is a contraction mapping. Assume $\varpi, \eta \in \Lambda$. Thus, for every $\tau \in \mathcal{J}$, we derive

$$\begin{aligned}
&\left|(\mathcal{T}_2\varpi)(\tau)-(\mathcal{T}_2\eta)(\tau)\right| \\
&\le \frac{1}{\Gamma(\zeta)}\int_0^\tau (\tau-\varphi)^{(\zeta-1)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^\eta\varpi(\varphi)\right)-\mathfrak{T}\left(\varphi,\eta(\varphi),{}^{RC}D^\eta\eta(\varphi)\right)\right|d\varphi \\
&+\frac{1}{\Gamma(\zeta)}\int_\tau^l (\varphi-\tau)^{(\zeta-1)}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^\eta\varpi(\varphi)\right)-\mathfrak{T}\left(\varphi,\eta(\varphi),{}^{RC}D^\eta\eta(\varphi)\right)\right|d\varphi \\
&\le \frac{2l^\zeta}{\Gamma(\zeta+1)}(M_1+M_2)\left(\|\varpi-\eta\|+\left\|{}^{RC}D^\eta\varpi-{}^{RC}D^\eta\eta\right\|\right).
\end{aligned}$$

Therefore, we obtain

$$\left\|(\mathcal{T}_2\varpi)(\tau)-(\mathcal{T}_2\eta)(\tau)\right\| \le N_1(M_1+M_2)\left(\|\varpi-\eta\|+\left\|{}^{RC}D^\eta\varpi-{}^{RC}D^\eta\eta\right\|\right). \tag{3.16}$$

and

$$\begin{aligned}
&|\,{}_0^{RC}D_l^\eta(\mathcal{T}_2\varpi)(\tau)-{}_0^{RC}D_l^\eta(\mathcal{T}_2\eta)(\tau)\,| \\
&\le \frac{1}{\Gamma(\zeta-\eta-1)}\int_0^\tau (\tau-\varphi)^{\zeta-\eta-2}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^\eta\varpi(\varphi)\right)-\mathfrak{T}\left(\varphi,\eta(\varphi),{}^{RC}D^\eta\eta(\varphi)\right)\right|d\varphi \\
&+\frac{1}{\Gamma(\zeta-\eta-1)}\int_\tau^l (\varphi-\tau)^{\zeta-\eta-2}\left|\mathfrak{T}\left(\varphi,\varpi(\varphi),{}^{RC}D^\eta\varpi(\varphi)\right)-\mathfrak{T}\left(\varphi,\eta(\varphi),{}^{RC}D^\eta\eta(\varphi)\right)\right|d\varphi \\
&\le \frac{2l^{(\zeta-\eta-1)}}{\Gamma(\zeta-\eta)}(M_1+M_2)\left(\|\varpi-\eta\|+\left\|{}^{RC}D^\eta\varpi-{}^{RC}D^\eta\eta\right\|\right)
\end{aligned}$$

and

$$\left\|{}_0^{RC}D_l^\eta(\mathcal{T}_2\varpi)(\tau)-{}_0^{RC}D_l^\eta(\mathcal{T}_2\eta)(\tau)\right\| \le N_2(M_1+M_2)\left(\|\varpi-\eta\|+\left\|{}^{RC}D^\eta\varpi-{}^{RC}D^\eta\eta\right\|\right). \tag{3.17}$$

Adding (3.16) and (3.17), we obtain

$$\left\|(\mathcal{T}_2\varpi)(\tau)-(\mathcal{T}_2\eta)(\tau)\right\|_\Lambda \le (N_1+N_2)(M_1+M_2)\left(\|\varpi-\eta\|+\left\|{}^{RC}D^\eta\varpi-{}^{RC}D^\eta\eta\right\|\right).$$

Using condition (H_3), we conclude that $\mathcal{T}_2$ is a contraction mapping. Employing Krasnoselskii fixed point theorem, $\mathcal{T}$ owns a fixed point, which is the solution of (1.2). □

4. Distinctiveness result

Theorem 4.1. *Consider $\mathfrak{T}:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose*

(H_4) there are non-negative real numbers M_1, M_2 satisfying

$$\left|\mathfrak{T}(\tau,\varphi,\zeta)-\mathfrak{T}(\tau,\varphi',\zeta')\right| \le M_1\left|\varphi-\varphi'\right|+M_2|\zeta-\zeta'|,$$

for all $(\varphi,\zeta),(\varphi',\zeta')\in\mathbb{R}^2$ and $\tau\in\mathcal{J}$ if, $(\Theta_1+\Theta_2)(M_1+M_2)<1$.

Then there is a unique solution to the problem (1.2) on $\mathcal{J}$.

Proof. The operator $\mathcal{T}$ has already been defined in Theorem 3.3 and also it is well-defined. One can note that the function $\varpi(\tau)$ is a solution of APBVP (1.2) if and only if ϖ is a fixed point of the operator $\mathcal{T}$. Therefore, we demonstrate that $\mathcal{T}$ is a contraction. Consider $\varpi, \eta \in \Lambda$. Therefore, for every $\tau \in \mathcal{J}$

$$\begin{aligned}
&|(\mathcal{T}\varpi)(\tau) - (\mathcal{T}\eta)(\tau)| \\
\leq& \frac{\chi_1 l^2 + \chi_2 l\tau + \chi_3\tau^2}{\Gamma(\zeta-2)} \int_0^l (l-\varphi)^{(\zeta-3)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
&+ \frac{\chi_4\tau + \chi_5 l}{\Gamma(\zeta-1)} \int_0^l (l-\varphi)^{(\zeta-2)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
&+ \frac{1}{\Gamma(\zeta)} \int_0^\tau (\tau-\varphi)^{(\zeta-1)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
&+ \frac{1}{\Gamma(\zeta)} \int_\tau^l (\varphi-\tau)^{(\zeta-1)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
\leq& \frac{\chi_1 l^\zeta}{\Gamma(\zeta-1)}\left(M_1\|\varpi-\eta\| + M_2\left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right) + \frac{\chi_5 l^\zeta}{\Gamma(\zeta)}\left(M_1\|\varpi-\eta\| + M_2\left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right) \\
&+ \frac{2l^\zeta}{\Gamma(\zeta+1)}\left(M_1\|\varpi-\eta\| + M_2\left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right) \\
\leq& \left[\frac{\chi_1 l^\zeta}{\Gamma(\zeta-1)} + \frac{\chi_5 l^\zeta}{\Gamma(\zeta)} + \frac{2l^\zeta}{\Gamma(\zeta+1)}\right](M_1+M_2)\left(\|\varpi-\eta\| + \left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right).
\end{aligned}$$

Therefore, we obtain

$$\|(\mathcal{T}\varpi)(\tau) - (\mathcal{T}\eta)(\tau)\| \leq \Theta_1(M_1+M_2)\left(\|\varpi-\eta\| + \left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right). \tag{4.1}$$

Again, we obtain

$$\begin{aligned}
&\left|{}^{RC}_0D^\eta_l(\mathcal{T}\varpi)(\tau) - {}^{RC}_0D^\eta_l(\mathcal{T}\eta)(\tau)\right| \\
\leq& \frac{1}{\Gamma(\zeta-\eta-1)} \int_0^\tau (\tau-\varphi)^{(\zeta-\eta-2)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
&+ \frac{1}{\Gamma(\zeta-\eta-1)} \int_\tau^l (\varphi-\tau)^{(\zeta-\eta-2)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
&+ \frac{\chi_3\left[\tau^{(2-\eta)} + (l-\tau)^{(2-\eta)}\right]}{2\Gamma(3-\eta)\Gamma(\zeta-2)} \int_0^l (l-\varphi)^{(\zeta-3)} \left|\mathfrak{T}\left(\varphi, \varpi(\varphi), {}^{RC}D^\eta\varpi(\varphi)\right) - \mathfrak{T}\left(\varphi, \eta(\varphi), {}^{RC}D^\eta\eta(\varphi)\right)\right| d\varphi \\
\leq& \frac{\chi_3 l^{(\zeta-\eta)}}{2\Gamma(3-\eta)\Gamma(\zeta-1)}\left(M_1\|\varpi-\eta\| + M_2\left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right) \\
&+ \frac{2l^{(\zeta-\eta-1)}}{\Gamma(\zeta-\eta)}\left(M_1\|\varpi-\eta\| + M_2\left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right).
\end{aligned}$$

Thus,

$$\left\|{}^{RC}_0D^\eta_l(\mathcal{T}\varpi)(\tau) - {}^{RC}_0D^\eta_l(\mathcal{T}\eta)(\tau)\right\| \leq \Theta_2(M_1+M_2)\left(\|\varpi-\eta\| + \left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right). \tag{4.2}$$

From (4.1) and (4.2), we get

$$\|(\mathcal{T}\varpi)(\tau) - (\mathcal{T}\eta)(\tau)\|_\Lambda \leq (\Theta_1+\Theta_2)(M_1+M_2)\left(\|\varpi-\eta\| + \left\|{}^{RC}D^\eta\varpi - {}^{RC}D^\eta\eta\right\|\right).$$

This confirms that $\mathcal{T}$ is a contraction, and according to the Banach fixed point theorem, $\mathcal{T}$ owns a unique fixed point which implies that the problem (1.2) has a unique solution on $\mathcal{J}$. □

5. Illustrations

Example 5.1. *Consider the fractional APBVP*

$$\begin{cases} {}^{RC}_{0}D^{\frac{12}{5}}_{l}\varpi(\tau) + \frac{(\sqrt{\pi}+2)|\varpi(\tau)|}{\sqrt{\tau^2+121}(1+|\varpi(\tau)|)} + \frac{1}{(4+e^{\pi})^2}\cos\left({}^{RC}D^{\frac{5}{4}}\varpi(\tau)\right) = 0, \tau \in [0,1], \\ 2\varpi(0) + \frac{1}{2}\varpi(1) = 0, \quad 2\varpi'(0) + \frac{1}{2}\varpi'(1) = 0, \quad \frac{1}{3}\varpi''(0) + 3\varpi''(1) = 0. \end{cases} \tag{5.1}$$

Here $\zeta = \frac{12}{5}, \eta = \frac{5}{4}, a_1 = 2, a_2 = 2, a_3 = \frac{1}{3}, b_1 = \frac{1}{2}, b_2 = \frac{1}{2}, b_3 = 3$ *and,*

$$\mathfrak{T}\left(\tau, \varpi(\tau), {}^{RC}D^{\eta}\varpi(\tau)\right) = \frac{(\sqrt{\pi}+2)|\varpi|}{\sqrt{\tau^2+121}(1+|\varpi|)} + \frac{1}{(4+e^{\pi})^2}\cos\left({}^{RC}D^{\frac{5}{4}}\varpi(\tau)\right).$$

We have,

$$\left|\mathfrak{T}(\varpi,\eta) - \mathfrak{T}(\varpi',\eta')\right| \leq \frac{\sqrt{\pi}+2}{11}\left\|\varpi - \eta'\right\| + \frac{1}{(4+e^{\pi})^2}\left\|\varpi - \eta'\right\|.$$

Then, the assumption (H_1) *is satisfied with* $M_1 = \frac{\sqrt{\pi}+2}{11}, M_2 = \frac{1}{(4+e^{\pi})^2}$. *Using the MATLAB program, we obtain* Θ_1*=0.0309,* Θ_2*=0.0276. Therefore,* $(M_1 + M_2)(\Theta_1 + \Theta_2) = 0.0201 < 1$. *Hence in view of Theorem 4.1, the problem (5.1) owns a unique solution on* $[0,1]$.

Example 5.2. *Consider the fractional APBVP*

$$\begin{cases} {}^{RC}_{0}D^{\frac{9}{4}}_{l}\varpi(\tau) + \frac{\tau}{100}(4\cos\varpi(\tau) + \sin\frac{\varpi(\tau)}{2}) + \frac{4+\tau}{\sqrt{(\tau^2+121)}}\cos\frac{\left|{}^{RC}D^{\frac{8}{7}}\varpi(\tau)\right|}{\left(\left|{}^{RC}D^{\frac{8}{7}}\varpi(\tau)\right|+1\right)} = 0, \tau \in [0,1], \\ 2\varpi(0) + 2\varpi(1) = 0, \quad \varpi'(0) + 2\varpi'(1) = 0, \quad \frac{1}{2}\varpi''(0) + 3\varpi''(1) = 0. \end{cases} \tag{5.2}$$

Here $\zeta = \frac{9}{4}, \eta = \frac{8}{7}, a_1 = 2, a_2 = 1, a_3 = \frac{1}{2}, b_1 = 2, b_2 = 2, b_3 = 3$, $M_1 = \frac{4}{100}, M_2 = \frac{5}{122}$, *and*

$$\mathfrak{T}\left(\tau, \varpi(\tau), {}^{RC}D^{\eta}\varpi(\tau)\right) = \frac{1}{100}\left(4\cos\varpi(\tau) + \sin\frac{\varpi(\tau)}{2}\right) + \frac{4+\tau}{\sqrt{(\tau^2+121)}}\cos\frac{\left|{}^{RC}D^{\frac{8}{7}}\varpi(\tau)\right|}{\left(\left|{}^{RC}D^{\frac{8}{7}}\varpi(\tau)\right|+1\right)}.$$

Moreover,

$$\left|\mathfrak{T}(\varpi,\eta) - \mathfrak{T}(\varpi',\eta')\right| \leq \frac{4}{100}\left\|\varpi - \eta'\right\| + \frac{5}{122}\left\|\varpi - \eta'\right\|.$$

Therefore,

$$\left|\mathfrak{T}\left(\tau, \varpi(\tau), {}^{RC}D^{\eta}\varpi(\tau)\right)\right| \leq \frac{4}{100} + \frac{5}{122} = \Theta(\tau).$$

On calculation, we get $N_1 = 0.3553, N_2 = 1.9576$. *Then* $(N_1 + N_2)(M_1 + M_2) = 0.1850 < 1$, *which implies that* (H_4) *is satisfied. Therefore, by using the Theorem 3.4, the problem (5.2) possesses at least one solution in* $[0,1]$.

6. Final remarks

Our study investigates the existence and uniqueness of solutions of fractional APBVP in the RCD. The results are extended to non-linear 3^{rd}-order ODE with more general APBVP for a certain fractional order value. Besides, two examples are provided to demonstrate the findings. Moreover, future applications of this method of analysis to more general ψ-Riesz-Caputo fractional differential equations, coupled system [23] and convex combined Caputo fractional differential equations [30] are possible.

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Declaration of Competing Interest

The authors state that there is no pertinent financial regards or individual associations which can impact the findings studied in this article.

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