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Study of generalized fractional drift-diffusion system in Besov-Morrey spaces

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Abstract. The main focus of this article is to investigate the generalized fractional drift-diffusion system with small initial data in Besov-Morrey spaces. Our goal is to establish the global well-posedness and asymptotic stability of mild solutions for this system. The results obtained in this study have broad applicability in the modeling of various types of fractional parabolic systems. In other words, the techniques developed here can be useful in studying similar types of systems in the future.

1. Introduction

Our article is based on the study of the global existence of mild solution of fractional drift-diffusion problem with initial data. we also show the asymptotic stability and analyticity of solutions with respect to spatial variables in Besov-Morrey spaces.
This is the fractional drift-diffusion system that we are interested in our study

$$\begin{cases} \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = -\Delta(v\nabla(-\Delta)^{-m}(w-v)) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w + (-\Delta)^{\frac{\alpha}{2}} v = \Delta(w\nabla(-\Delta)^{-m}(w-v)) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the variables v and w are used to represent the unknown densities of negatively and positively charged particles, respectively. $(-\Delta)^{-m}$ is the electric potential and is determined by the Poisson equation. They parameters are very important because they control how charges move in the semiconductor material. They determine the electrical and optical characteristics of the material, as well as the performance of electronic devices made from it. Generaly they coefficients are important because they allow for accurate modeling of charge behavior in the material, which can help with the design and optimization of electronic devices. The system (1) describes the transport of charged particles in a semiconductor material. It is an extension of the classical drift-diffusion model that takes into account the non-local and memory effects of the charge transport, the transport of charge carriers is described by a set of partial differential equations that incorporate fractional derivative terms.

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The fundamental solution of system (1) can be expressed in $\mathbb{R}^n$ for $(-\Delta)^{-m}$ by

$$(-\Delta)^{-m}(w-v)(x) = \begin{cases} \frac{1}{n(n-2)w_n} \displaystyle\int_{\mathbb{R}^n} \frac{w(y)-v(y)}{|x-y|^{n-2m}} dy \quad n \geq 3, \\ \frac{1}{2\pi} \int_{\mathbb{R}^n} (w(y)-v(y)) \log|x-y|^{-m} dy \quad n = 2. \end{cases}$$

By applying Duhamel's principle, we can transform the system (1) into a set of integral equations. This reduction allows us to analyze the system's behavior in a more manageable form and provides insights into its long-term dynamics. By studying the solutions to these integral equations, we can better understand how the system responds to different initial conditions and external stimuli. Overall, the use of Duhamel's principle represents a powerful tool for investigating the behavior of complex systems described by partial differential equations:

$$\begin{cases} v(t) = \mathcal{S}_\alpha(t)v_0 + B(v, w-v), \\ w(t) = \mathcal{S}_\alpha(t)w_0 + B(v, v-w), \end{cases} \tag{2}$$

with:

$$\begin{cases} \mathcal{S}_\alpha(t) = \exp^{-t(-\Delta)^{\frac{\alpha}{2}}}, \\ B(v, w-v) = -\int_0^t \mathcal{S}_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}(w-v))](\tau)d\tau. \end{cases}$$

In the case where $\alpha = 2$, the system (1) corresponds to the usual drift-diffusion system which has been well treated and developed in [1, 21], the authors showed existence results of mild solution, as well as time dependent convergence rate estimates of stationary solutions. Specifically, they investigate the asymptotic behavior of the solutions as time approaches infinity, and establish a convergence result under certain assumptions on the initial data. The authors also discuss the implications of their results for the modeling of semiconductor devices and other physical systems. Overall, the article provides insights into the behavior of solutions to the drift-diffusion Poisson system and its applications in the physical sciences.
In the case of $w = 0$, the system (1) reduces to the nonlinear system called the Keller-Segel model of chemotaxis [10], the author to prove that this system admits a global solution if $1 < \alpha < 2$. He also provides insights into the behavior of the solutions, including the formation of singularities, and discusses the implications of the model for the study of biological processes. Overall, the article contributes to the understanding of the fractional Keller-Segel model and its applications in mathematical biology. Same system was studied in [8, 9] where the authors showed the global existence of solution in case $\alpha = 2$.
In [27], the authors gave a stability and existence for the system (1) in critical Besov spaces.
Our study make a general study of (1) in Besov-Morrey spaces (see e.g. [2–6, 14, 15, 22], and reference therein).
Generaly, the fractional drift-diffusion system is used to study the behavior of charge carriers in various semiconductor devices, such as solar cells, photodetectors, and transistors. It can provide insights into the efficiency and performance of these devices, and help in the design of new and improved semiconductor materials and devices.
The study of the fractional drift-diffusion system is an active area of research in the fields of applied mathematics and semiconductor physics, and it has many practical applications in the development of new technologies.

The plan of this article is organized as follows. In Section 2, we recall the definitions of spaces; Morrey and Besov-Morrey via semigroup $\mathcal{S}_\alpha(t)$ and some important lemmas for the proofs of the theorems. Section 3 devoted to our main results, the proof is stated in the same section.

2. Preliminaries

To make the paper self contained, we recall the definitions of the spaces $\mathcal{N}^s_{p,q,u}\left(\mathbb{R}^{n+1}_+\right)$. For this we need to define a Morrey spaces and a Besov classical spaces.

Definition 2.1. *([23]) Let $1 \leq p \leq \infty$ and $0 \leq u < p$. The Morrey space $\mathcal{M}_u^p$ is defined to be the set of all u-locally Lebesgue-integrable functions $f \in \mathbb{R}^n$ with the norm:*

$$\|f\|_{\mathcal{M}_u^p} = \sup_{B_J} |B_J|^{\frac{1}{p}-\frac{1}{u}} \|f\|_{L^u(B_J)} < \infty,$$

where the supremum is taken over all balls B_J in $\mathbb{R}^n$.

Remark 2.2. *[24]. Obviously we have $\mathcal{M}_p^p = L^p$ and $\mathcal{M}_\infty^\infty = L^\infty$.*
As a consequence of Hölder's inequality we conclude monotonicity with respect to u, i.e.,
if $0 < u \leq w \leq p \leq \infty$, then $\mathcal{M}_w^p \hookrightarrow \mathcal{M}_u^p$.

Definition 2.3. *([17].)(Classical Besov spaces via semigroup $\mathcal{S}_\alpha(t)$)*
Let $s < 0$, $1 \leq p, q \leq \infty$. $n \in \mathbb{N}$ and $0 < \alpha < \infty$.
Then $v \in \mathcal{B}_{p,q}^s$ if and only if:

$$\begin{cases} \left(\int_0^\infty (t^{\frac{-s}{\alpha}} \|\mathcal{S}_\alpha(t)v\|_{L^p})^q dt\right)^{\frac{1}{q}} < \infty & if\ 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{-s}{\alpha}} \|\mathcal{S}_\alpha(t)v\|_{L^p} & if\ q = \infty. \end{cases}$$

Definition 2.4. *(Besov-Morrey spaces via semigroup $\mathcal{S}_\alpha(t)$)*
Let $s < 0$, $1 \leq u \leq p, q \leq +\infty$ and $0 < \alpha < \infty$. Then $v \in \mathcal{N}_{p,q,u}^s$ if and only if we have:

$$\begin{cases} \left(\int_0^\infty (t^{\frac{-s}{\alpha}} \|\mathcal{S}_\alpha(t)v\|_{\mathcal{M}_u^p})^q dt\right)^{\frac{1}{q}} < \infty & if\ 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{-s}{\alpha}} \|\mathcal{S}_\alpha(t)v\|_{\mathcal{M}_u^p} & if\ q = \infty. \end{cases}$$

Definition 2.5. *Let $n \geq 2$, $1 < \alpha < 2n$ and $\max\{1, \frac{n}{\alpha}\} < u < \min\{n, \frac{2n}{\alpha}\}$.*
We define the space $\mathcal{X}_u$ by:

$$\mathcal{X}_u = C_*\left([0,\infty), \mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n)\right) \cap \left\{v : v \in C\left((0,\infty), \mathcal{M}_u^p(\mathbb{R}^n)\right) \text{ and } \sup_{t>0} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|v\|_{\mathcal{M}_u^p(\mathbb{R}^n)} < \infty\right\},$$

with the follwing norm:

$$\|v\|_{\mathcal{X}_u} = \sup_{t>0} \|v\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} + \sup_{t>0} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|v\|_{\mathcal{M}_u^p}.$$

The following lemma provides a reminder of the $\mathcal{M}_u^p - \mathcal{M}_r^p$ estimates for the semigroup operator $\mathcal{S}_\alpha(t)$:

Lemma 2.6. *Let $\alpha > 1$ and $1 \leq u \leq r \leq p \leq \infty$. Then for any $f \in \mathcal{M}_u^p$ we have*

$$\|\mathcal{S}_\alpha(t)f\|_{\mathcal{M}_r^p} \leq C(n,\alpha) t^{-n\alpha(\frac{1}{u}-\frac{1}{r})} \|f\|_{\mathcal{M}_u^p},$$

and

$$\|(-\Delta)^{\frac{\alpha}{2}} \mathcal{S}_\alpha(t)f\|_{\mathcal{M}_r^p} \leq C(n,\alpha) t^{-\frac{1}{\alpha}-n\alpha(\frac{1}{u}-\frac{1}{r})} \|f\|_{\mathcal{M}_u^p}.$$

In the following lemma we recall the classical Hardy-Littlewood-sobolev inequality:

Lemma 2.7. *For any $1 < u \leq p < n$, the nonlocal operator $(-\Delta)^{-\frac{\alpha}{2}}$ is bounded from $\mathcal{M}_u^p(\mathbb{R}^n)$ to $\mathcal{M}_{nu(n-u)}^p(\mathbb{R}^n)$, i.e., for $f \in \mathcal{M}_u^p(\mathbb{R}^n)$ we have:*

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{\mathcal{M}_{nu(n-u)}^p(\mathbb{R}^n)} \leq C(n,u) \|f\|_{\mathcal{M}_u^p(\mathbb{R}^n)}.$$

Lemma 2.8. *Let $v_0 \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$. Then $\mathcal{S}_\alpha(t)v_0 \in \mathcal{X}_u$ and*

$$\|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{X}_u} \leq C(n,\alpha)\,\|v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}}.$$

Proof. Since $v_0 \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$, Definition 1.1 and Lemma 2.1 yield that

$$\begin{aligned}\|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} &= \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\,\|\mathcal{S}_\alpha(s)\mathcal{S}_\alpha(t)v_0\|_{\mathcal{M}^p_u}\\ &\leq C(n,\alpha)\sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\,\|\mathcal{S}_\alpha(s)v_0\|_{\mathcal{M}^p_u} = C(n,\alpha)\,\|v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}},\\ &\sup_{t>0} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\,\|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{M}^p_u} = \|v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}}.\end{aligned}$$

Therefore, $\mathcal{S}_\alpha(t)v_0 \in L^\infty\left([0,\infty),\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)\right)$ and $t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\mathcal{S}_\alpha(t)v_0 \in L^\infty((0,\infty),\ \mathcal{M}^p_u(\mathbb{R}^n))$. Moreover, by using the similar argument of [7] we see that the map $t \mapsto \mathcal{S}_\alpha(t)v_0$ from $[0,\infty)$ to $\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$ is continuous for $t>0$ and weakly continuous for $t=0$, so that $\mathcal{S}_\alpha(t)v_0 \in C_*\left([0,\infty),\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)\right)$. Besides, it is clear that $\mathcal{S}_\alpha(t)v_0 \in C\left((0,\infty),\mathcal{M}^p_u(\mathbb{R}^n)\right)$ due to $v_0 \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$, then also $t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\mathcal{S}_\alpha(t)v_0 \in C\left((0,\infty),\mathcal{M}^p_u(\mathbb{R}^n)\right)$.
Hence, $\mathcal{S}_\alpha(t)v_0 \in \mathcal{X}_u$. This completes the proof of Lemma 2.8.
□

Now, the following proposition needed to show the existence and uniqueness result for an abstract operator equation in a Banach space:

Proposition 2.9. *([7]) Let $\mathcal{X}$ be a Banach space and $B : \mathcal{X}\times\mathcal{X} \longrightarrow \mathcal{X}$ is a bilinear bounded operator, and $\|.\|_{\mathcal{X}}$ being the $\mathcal{X}$-norm. Suppose that for any $v_1, v_2 \in \mathcal{X}$, there exists a constant C such that*

$$\|B(v_1,v_2)\|_{\mathcal{X}} \leq C\|v_1\|_{\mathcal{X}}\|v_2\|_{\mathcal{X}}.$$

Then for any $y \in \mathcal{X}$, such that $\|y\|_{\mathcal{X}} \leq \varepsilon < \frac{1}{4C}$, the equation $v = y + B(v,v)$ admit a solution $v \in \mathcal{X}$. Moreover, this solution is the only one, such that $\|v\|_{\mathcal{X}} \leq 2\varepsilon$ and depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \leq \varepsilon$, $\tilde{v} = \tilde{y} + B(\tilde{v},\tilde{v})$, $\|\tilde{v}\|_{\mathcal{X}} \leq 2\varepsilon$. Then

$$\|v-\tilde{v}\|_{\mathcal{X}} \leq \frac{1}{1-4\varepsilon C}\|y-\tilde{y}\|_{\mathcal{X}}.$$

Lemma 2.10. *Let $(v,w) \in \mathcal{X}_u$. Then $B(u,v) \in \mathcal{X}_u$ and*

$$\|B(v,w)\|_{\mathcal{X}_u} \leq C(n,\alpha,u)\|(v,w)\|^2_{\mathcal{X}_u}.$$

Proof. We have:

$$\begin{aligned}\|B(v,w)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} &= \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|S_\alpha(s)B(v,w)\|_{\mathcal{M}^p_u}\\ &= \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|-S_\alpha(s)\int_0^t S_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}^p_u}\\ &\leq \int_0^t \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|S_\alpha(s)S_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}^p_u}.\end{aligned}$$

For $0 < s \leq t-\tau$, we use Lemmas 2.6 and 2.7, we obtain:

$$\begin{aligned}&\sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|S_\alpha(s)S_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}^p_u}\\ &\leq C(n,\alpha)(t-\tau)^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|S_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}^p_u}\\ &\leq C(n,\alpha)(t-\tau)^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}(t-\tau)^{\frac{-n}{\alpha u}}\|v\|_{\mathcal{M}^p_u}\|w\|_{\mathcal{M}^p_u}.\end{aligned}$$

For $s > t-\tau$

$$
\begin{aligned}
&\sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|S_\alpha(s)S_\alpha(t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}_u^p} \\
&= \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|S_\alpha(s+t-\tau)\nabla.[v\nabla((-\Delta)^{-m}w)](\tau)d\tau\|_{\mathcal{M}_u^p} \\
&\leq C(n,\alpha)\sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}(s+t-\tau)^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v\|_{\mathcal{M}_u^p}\|\nabla((-\Delta)^{-m}w)\|_{\mathcal{M}^p_{\frac{nu}{n-u}}} \\
&\leq C(n,\alpha,u)(t-\tau)^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v\|_{\mathcal{M}_u^p}\|w\|_{\mathcal{M}_u^p}\sup_{s>t-\tau}(1+\frac{s}{t-\tau})^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \\
&\leq C(n,\alpha,u)(t-\tau)^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v\|_{\mathcal{M}_u^p}\|w\|_{\mathcal{M}_u^p}.
\end{aligned}
$$

So we obtain:

$$
\begin{aligned}
\|B(v,w)\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} &\leq C(n,\alpha,u)\|v\|_{\mathcal{X}_u}\|w\|_{\mathcal{X}_u}\int_0^t (t-\tau)^{1-\frac{2n}{\alpha u}}\tau^{-2+\frac{2n}{\alpha u}}d\tau \\
&\leq C(n,\alpha,u)\|v\|_{\mathcal{X}_u}\|w\|_{\mathcal{X}_u} \\
&\leq C(n,\alpha,u)\|(v,w)\|^2_{\mathcal{X}_u}.
\end{aligned}
$$

By using the standart argument of [7], it remains the continuity of $B(v,w)(t)$ for $t>0$ and weak continuity for $t=0$, so we omit it here. Hence, we have proved that $B(v,w)\in C_*\left([0,\infty),\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n)\right)$ and

$$
\sup_{t>0}\|B(v,w)\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \leq C(n,\alpha,u)\|v\|_{\mathcal{X}_u}\|w\|_{\mathcal{X}_u} \leq C(n,\alpha,u)\|(v,w)\|^2_{\mathcal{X}_u}.
$$

By utilizing Lemma 2.6 and Lemma 2.8, we are able to obtain the following result

$$
\begin{aligned}
\|B(v,w)\|_{\mathcal{M}_u^p} &= \left\| -\int_0^t \mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}w\right)\right]d\tau \right\|_{\mathcal{M}_u^p} \\
&\leq C(n,\alpha)\int_0^t (t-\tau)^{-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v\|_{\mathcal{M}_u^p}\left\|\nabla\left((-\Delta)^{-m}w\right)\right\|_{\mathcal{M}^p_{nu/(n-u)}}d\tau \\
&\leq C(n,\alpha,u)\|v\|_{\mathcal{X}_u}\|w\|_{\mathcal{X}_u}\int_0^t (t-\tau)^{-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\tau^{-2+2n/(\alpha u)}d\tau \\
&\leq C(n,\alpha,u)t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v\|\mathcal{X}_u\|w\|_{\mathcal{X}_u} \\
&\leq C(n,\alpha,u)t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|(v,w)\|^2_{\mathcal{X}_u}.
\end{aligned}
$$

This implies that

$$
\sup_{t>0} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|B(v,w)\|_{\mathcal{M}_u^p} \leq C(n,\alpha,u)\|v\|_{\mathcal{X}_u}\|w\|_{\mathcal{X}_u} \leq C(n,\alpha,u)\|(v,w)\|^2_{\mathcal{X}_u}.
$$

Combining the two above estimates to get the result of Lemma 2.10. □

3. Main Results

In this section, we put the most essential results. First result concerning the global existence of mild soultion of system (1).

Theorem 3.1. *Let* $n\geq 2$, $1<\alpha<2n$ *and* $\max\{1,\frac{n}{\alpha}\}<u<\min\{n,\frac{2n}{\alpha}\}$.
Assume that $(v_0,w_0)\in\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n)$. *Then there exists* $\varepsilon>0$ *such that* $\|(v_0,w_0)\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n)}\leq\varepsilon$, *the system* (1) *admit a unique global mild solution* $(v,w)\in\mathcal{X}_u$ *such that* $\|(v,w)\|_{\mathcal{X}_u}\leq 2\varepsilon$.

Remark 3.2. • $C_*([0,\infty),\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n))$ *denotes the set of bounded maps from* $[0,\infty)$ *to* $\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}(\mathbb{R}^n)$, *which are continuous for* $t>0$ *and weakly continuous for* $t=0$.

• *The solution depends continuously on initial data in the following sense: Let* $(\tilde{v}, \tilde{w})$ *be the solution of* (1) *with initial data* $(\tilde{v}_0, \tilde{w}_0)$ *such that*

$$\|(\tilde{v}_0, \tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)} \leq \varepsilon.$$

Then there exists a constant C, such that

$$\|(v - \tilde{v}_0, w - \tilde{w}_0)\|_{\mathcal{X}_u} \leq C\|(v - \tilde{v}_0, w - \tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)}.$$

• *If* $(\tilde{v}, \tilde{w})$ *belong to* $\mathcal{S}(\bar{\mathbb{R}}^n)$ *in the space* $\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$, *then* $(v, w) \in C([0,\infty), \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n))$.
• *Under the hypotheses of Theorem 3.1, we assume further that* v_0 *and* w_0 *are homogeneous functions with degree* $-\alpha$, *then the unique global mild solution ensured by Theorem 3.1 is the so-called self-similar solution.*

Proof. The proof of Theorem 3.1 is based on Proposition 2.9 and the following lemma:

Lemma 3.3. *Let* $v_0 \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$. *Then* $\mathcal{S}_\alpha(t)v_0 \in \mathcal{X}_u$ *and*

$$\|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{X}_u} \leq C(n,\alpha)\|v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)}$$

.

Proof. Since $v_0 \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$, then by using Definition 2.4, and Lemma 2.6, we have:

$$\begin{aligned}\|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)} &= \sup_{s>0} s^{\frac{\alpha-\frac{n}{u}}{\alpha}} \|\mathcal{S}_\alpha(s)\mathcal{S}_\alpha(t)v_0\|_{\mathcal{M}^p_u} \\ &= \sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(s)\mathcal{S}_\alpha(t)v_0\|_{\mathcal{M}^p_u} \\ &\leq C(n,\alpha)\sup_{s>0} s^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(t)v_0\|_{\mathcal{M}^p_u} = C(n,\alpha)\|v_0\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)}.\end{aligned}$$

Since $\mathcal{S}_\alpha(t)v_0 \in L^\infty([0,\infty), \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n))$ and $t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\mathcal{S}_\alpha(t)v_0 \in L^\infty([0,\infty), \mathcal{M}^p_u(\mathbb{R}^n))$. Moreover, by using the similar argument of Proposition 3.11 of [7] we see that the map $t \longrightarrow \mathcal{S}_\alpha(t)v_0$ is continuous from $[0,\infty)$ to $\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$ for $t > 0$ and weakly continuous for $t = 0$, so $\mathcal{S}_\alpha(t)v_0 \in C_*([0,\infty), \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n))$. Besides, it is clear that $\mathcal{S}_\alpha(t)v_0 \in C((0,\infty), \mathcal{M}^p_u(\mathbb{R}^n))$ and $t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\mathcal{S}_\alpha(t)v_0 \in C((0,\infty), \mathcal{M}^p_u(\mathbb{R}^n))$. This completes the proof of Lemma 3.3.
Now, using Lemma 2.10, we have:

$$\|(v,w)\|_{\mathcal{X}_u} \leq \|(\mathcal{S}_\alpha(t)v_0, \mathcal{S}_\alpha(t)w_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)} + C(n,\alpha,u)\|(v,w)\|^2_{\mathcal{X}_u}.$$

Finally, Proposition 2.9 with Lemma 3.3 leads to Theorem 3.1. □
□

Theorem 3.4. *Under the hypotheses of Theorem 3.1. Suppose further that* (v, w) *and* $(\tilde{v}, \tilde{w})$ *are two mild solutions of system* (1) *corresponding to initial data conditions* (v_0, w_0) *and* $(\tilde{v}_0, \tilde{w}_0)$, *respectly. Then the two following conditions are equivalent:*

- $\lim_{t\longrightarrow\infty} \|\mathcal{S}_\alpha(t)(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} = 0$
- $\lim_{t\longrightarrow\infty} \left(\|(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} + t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)\|_{\mathcal{M}^p_u}\right) = 0$

Proof. Let (v, w) and $(\tilde{v}, \tilde{w})$, respectively be two mild solutions of system (1) constructed in Theorem 3.1 with initial conditions (v, w) and $(\tilde{v}, \tilde{w})$, respectively. From Theorem 3.1, we have:

$$\|(v,w)\|_{\mathcal{X}_u} \leq 2\varepsilon \tag{3.1}$$

and

$$\|(\tilde{v}, \tilde{w})\|_{\mathcal{X}_u} \leq 2\varepsilon.$$

Lemma 3.5. *Suppose that* $(v,w),(\tilde{v},\tilde{w}) \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}$. *If the condition*

$$\lim_{t\longrightarrow\infty} \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} = 0$$

holds, then

$$\lim_{t\longrightarrow\infty} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{\mathcal{M}^p_u}.$$

Proof. By Definition 2.1, we have:

$$t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{\mathcal{M}^p_u}$$
$$= \sup_B |\beta|^{\frac{1}{p}-\frac{1}{u}} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{L^u}$$
$$\leq \sup_B |\beta|^{\frac{1}{p}-\frac{1}{u}} \times 2^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \sup_{t>0} (\frac{t}{2})^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(\frac{t}{2})\mathcal{S}_\alpha(\frac{t}{2})(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{L^u}$$
$$\leq 2^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(\frac{t}{2})(v_0-\tilde{v}_0, w_0-\tilde{w}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}}$$

This proves Lemma 3.5. □

If we subtract the integral system for $(\tilde{v},\tilde{w})$, we have

$$\begin{cases} v-\tilde{v} = \mathcal{S}_\alpha(t)(v_0-\tilde{v}_0) + B(v-\tilde{v}, w-v) + B(\tilde{v}, (w-\tilde{w})-(v-\tilde{v})), \\ w-\tilde{w} = \mathcal{S}_\alpha(t)(w_0-\tilde{w}_0) + B(w-\tilde{w}, v-w) + B(\tilde{w}, (v-\tilde{v})-(w-\tilde{w})). \end{cases}$$

First we calculate the norm: $\|v-\tilde{v}\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}}$

$$\begin{aligned} \|v-\tilde{v}\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} &\leq \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} + \|B(v-\tilde{v}, w-v)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} \\ &\quad + \|B(\tilde{v}, (w-\tilde{w})-(v-\tilde{v}))\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} \\ &\leq \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} + L_1 + L_2. \end{aligned}$$

For estimate $L_i (i=1,2)$, we split the time integral $\int_0^t$ into $\int_0^{\delta_t} + \int_{\delta_t}^t$ where $\delta_t \in (0,1)$.
For L_1, we use the Definition 2.3, we have

$$\begin{aligned} L_1 &= \|B(v-\tilde{v}, w-v)\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} \\ &= \|-\int_0^t \mathcal{S}_\alpha(t-\tau)\nabla.[(v-\tilde{v})\nabla((-\Delta)^{-m}(w-v))](\tau)d\tau\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}} \\ &= \sup_{s>0} s^{\alpha-\frac{n}{\alpha u}-\frac{2}{\alpha}+\frac{2m}{\alpha}} \|\mathcal{S}_\alpha(s)\int_0^t \mathcal{S}_\alpha(t-\tau)\nabla.[(v-\tilde{v})\nabla((-\Delta)^{-m}(w-v))](\tau)d\tau\|_{\mathcal{M}^p_u} \\ &\leq \left[\int_0^{\sigma_t} + \int_{\sigma_t}^0\right] \sup_{s>0} s^{\alpha-\frac{n}{\alpha u}-\frac{2}{\alpha}+\frac{2m}{\alpha}} \|\mathcal{S}_\alpha(s)\int_0^t \mathcal{S}_\alpha(t-\tau)\nabla.[(v-\tilde{v})\nabla((-\Delta)^{-m}(w-v))](\tau)\|_{\mathcal{M}^p_u} d\tau \\ &= I_1 + I_2. \end{aligned}$$

For I_1, using Lemma 2.6 and Lemma 2.7, we have

$$
\begin{aligned}
I_1 &= \int_0^{\delta t} \sup_{s>0} s^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \left\| \mathcal{S}_\alpha(s)\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[(v-\tilde{v})\nabla\left((-\Delta)^{-m}(w-v)\right)\right](\tau)\right\|_{\mathcal{M}_u^p} d\tau \\
&\le C\int_0^{\delta t}(t-\tau)^{1-2n/(\alpha p)}\tau^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left(\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v-\tilde{v}\|_{\mathcal{M}_u^p}\right)\left(\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|w-v\|_{\mathcal{M}_u^p}\right)d\tau \\
&\le C\varepsilon\int_0^{\delta t}(t-\tau)^{1-2n/(\alpha p)}\tau^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left(\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v-\tilde{v}\|_{\mathcal{M}_u^p}\right)d\tau \\
&\le C\varepsilon\int_0^{\delta}(1-s)^{1-2n/(\alpha p)}s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(ts)-\tilde{v}(ts)\|_{\mathcal{M}_u^p}\right)ds.
\end{aligned}
$$

Here we made the change of variables $\tau = ts$ and used the boundedness of (v, w), we have:

$$
\begin{aligned}
I_2 &= \int_{\delta t}^{t} \sup_{s>0} s^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \left\| \mathcal{S}_\alpha(s)\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[(v-\tilde{v})\nabla\left((-\Delta)^{-m}(w-v)\right)\right](\tau)\right\|_{\mathcal{M}_u^p} d\tau \\
&\le C\varepsilon\int_{\delta t}^{t}(t-\tau)^{1-2n/(\alpha p)}\tau^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left(\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v-\tilde{v}\|_{\mathcal{M}_u^p}\right)d\tau \\
&\le C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)-\tilde{v}(\tau)\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

Hence, combining the two last inequalities, yield that

$$
\begin{aligned}
L_1 \le & C\varepsilon\int_0^{\delta}(1-s)^{1-2n/(\alpha p)}s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(ts)-\tilde{v}(ts)\|_{\mathcal{M}_u^p}\right)ds \\
&+ C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)-\tilde{v}(\tau)\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_2 \le & C\varepsilon\int_0^{\delta}(1-s)^{1-2n/(\alpha p)}s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(ts),(w-\tilde{w})(ts))\|_{\mathcal{M}_u^p}\right)ds \\
&+ C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(\tau),(w-\tilde{w})(\tau))\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

Next, we calculate the norm $\|v-\tilde{v}\|_{\mathcal{M}_u^p}$:

$$
\begin{aligned}
\|v-\tilde{v}\|_{\mathcal{M}_u^p} \le & \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0)\|_{\mathcal{M}_u^p} + \|B(v-\tilde{v},w-v)\|_{\mathcal{M}_u^p} \\
&+ \|B(\tilde{v},(w-\tilde{w})-(v-\tilde{v}))\|_{\mathcal{M}_u^p} = J_0+J_1+J_2.
\end{aligned}
$$

By Lemma 3.3, it is clear that

$$
t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} J_0 \le 2^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \|\mathcal{S}_\alpha(t)(v_0-\tilde{v}_0,w_0-\tilde{w}_0)\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} .
$$

Using a similar line of reasoning to the previous argument, we are able to obtain an estimate for the term J_1 as follows:

$$
\begin{aligned}
J_1 &= \left\| - \int_0^t \mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[(v-\tilde{v})\nabla\left((-\Delta)^{-m}(w-v)\right)\right](\tau)d\tau \right\|_{\mathcal{M}_u^p} \\
&\le C\left[\int_0^{\delta t} + \int_{\delta t}^t\right](t-\tau)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \|v-\tilde{v}\|_{\mathcal{M}_u^p}\|w-v\|_{\mathcal{M}_u^p} d\tau \\
&\le C\varepsilon\left[\int_0^{\delta t} + \int_{\delta t}^t\right](t-\tau)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \tau^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left(\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v-\tilde{v}\|_{\mathcal{M}_u^p}\right)d\tau \\
&\le C\varepsilon t^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\int_0^\delta (1-s)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(ts)-\tilde{v}(ts)\|_{\mathcal{M}_u^p}\right)ds \\
&\quad + C\varepsilon t^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)-\tilde{v}(\tau)\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

This means that

$$
\begin{aligned}
t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} J_1 \le & C\varepsilon\int_0^\delta (1-s)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(ts)-\tilde{v}(ts)\|_{\mathcal{M}_u^p}\right)ds \\
& + C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)-\tilde{v}(\tau)\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} J_2 \le & C\varepsilon\int_0^\delta (1-s)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(ts),(w-\tilde{w})(ts))\|_{\mathcal{M}_u^p}\right)ds \\
& + C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(\tau),(w-\tilde{w})(\tau))\|_{\mathcal{M}_u^p}\right].
\end{aligned}
$$

Therefore, by combining the two above inequality, we obtain

$$
\begin{aligned}
&\|v-\tilde{v}\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} + t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|v-\tilde{v}\|_{\mathcal{M}_u^p} \le C\left\|\mathcal{S}_\alpha(t)\left(v_0-\tilde{v}_0\right)\right\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \\
&+ C\varepsilon\int_0^\delta (1-s)^{1-2n/(\alpha p)} s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(ts),(w-\tilde{w})(ts))\|_{\mathcal{M}_u^p}\right)ds \\
&+ C\varepsilon\int_0^\delta (1-s)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}\left((ts)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(ts),(w-\tilde{w})(ts))\|_{\mathcal{M}_u^p}\right)ds \\
&+ C\varepsilon\left[\sup_{\delta t\le\tau\le t}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|((v-\tilde{v})(\tau),(w-\tilde{w})(\tau))\|_{\mathcal{M}_u^p}\right]. \qquad (3.2)
\end{aligned}
$$

To estimate $w-\tilde{w}$ we use the same thing as the estimation of $v-\tilde{v}$, we omit it here.
Now, To simplify the proof of Theorem 3.4, we introduce the two auxiliary functions

$$
\begin{aligned}
h(t) &= \|\mathcal{S}_\alpha(t)\left(v_0-\tilde{v}_0, w_0-\tilde{w}_0\right)\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \\
g(t) &= \|(v-\tilde{v}, w-\tilde{w})\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} + t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}}\|(v-\tilde{v}, w-\tilde{w})\|_{\mathcal{M}_u^p}.
\end{aligned}
$$

We first assume that the first term of equivalent in Theorem 3.4 holds. Then from Lemma 2.7 and (3.1) we can easily see that

$$
h(t) \in L^\infty([0,\infty)) \text{ and } \lim_{t\to\infty} h(t) = 0. \qquad (3.3)
$$

To prove the second term of equivalent in Theorem 3.4, we set

$$
M = \limsup_{t\to\infty} g(t) = \lim_{k\in\mathbb{N}, k\to\infty}\sup_{t\ge k} g(t)
$$

It is obviously sufficient to prove that $M = 0$. Note that M is nonnegative and finite according to (3.1).
By combining the estimate (3.2) with a similar estimate for $w - \tilde{w}$, we can obtain the following result. Through the use of appropriate mathematical techniques and the application of the Lebesgue dominated convergence theorem, we can establish a relationship between the estimates and gain insights into the long-term behavior of the system. Additionally, the fact that $M = 0$ as given in (3.3) plays a key role in the deduction of this result

$$M \leq C(n, \alpha, u)\varepsilon(F(\delta) + 1)M, \qquad (3.4)$$

where $F(\delta)$ is defined by

$$F(\delta) = \int_0^\delta (1-s)^{1-\frac{2n}{\alpha u}+\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}+\frac{1-2m}{\alpha}} ds + \int_0^\delta (1-s)^{-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} s^{-2+\frac{2n}{\alpha u}+\frac{1-2m}{\alpha}} ds.$$

Note that, under the hypotheses of Theorem 3.4, all integrals in $F(\delta)$ are convergent, and $lim_{\delta\longrightarrow 0}F(\delta) = 0$. Hence, if ε is sufficiently small such that $C(n, \alpha, u)\varepsilon < 1$, then we can choose δ small enough, such that $C(n, \alpha, u)\varepsilon(F(\delta) + 1) < 1$. This means that $M = 0$ by (3.4). The proof of second term of equivalent in Theorem 3.4 is complete. Conversely, we assume that the second term of equivalent in Theorem 3.4 holds. Observe that from (3.1),

$$g(t) \in \mathcal{M}^p_\infty([0, \infty]) \quad \text{and} \quad \lim_{t\longrightarrow\infty} g(t) = 0. \qquad (3.5)$$

We need to prove the first term of equivalent in Theorem 3.4. It suffices to repeat the above calculations to obtain that

$$h(t) \leq g(t) + C(n, \alpha, u)\varepsilon F((\delta) + 1)g(t). \qquad (3.6)$$

Since $C(n, \alpha, u)\varepsilon(F(\delta) + 1)$ is bounded and independent of t, the first term of equivalent in Theorem 3.4 follows immediately from (3.5) and (3.6).
That complete the proof of Theorem 3.4. □

Remark 3.6. • *Interpretation of Theorem 3.4 reveals that it can be viewed as an asymptotic stability result under certain conditions. Specifically, if the initial data $(\tilde{v}0, \tilde{w}0)$ is restricted to a neighborhood of $(\tilde{v}, \tilde{w})$ and satisfies the first term of equivalence as given in Theorem 3.4, then the system can be considered asymptotically stable.*
• *The condition the first term of equivalent in Theorem 3.4 holds true if the difference of the initial data $(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)$ is not too singular. In fact, it is easy to verify that the condition of the first term of equivalent in Theorem 3.4 is satisfied if $(v_0 - \tilde{v}_0, w_0 - \tilde{w}_0)$ belongs to the closure of $\mathcal{S}(\mathbb{R}^n)$ in the space $\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$.*

Theorem 3.7. *Let $n \geq 2$, $1 < \alpha < 2n$ and $\max\{1, \frac{1}{\alpha}\} < p < \min\{n, \frac{2n}{\alpha}\}$. Assume that $(v_0, w_0) \in \mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)$ and (v, w) be a mild solution of the system (1) with the initial data (v_0, w_0). Assume the further there exist two positive constants K_1 and K_2 such that*

$$\sup_{0\leq t<T} \|(v(t), w(t))\|_{\mathcal{N}^{-\alpha+\frac{n}{u}+2-2m}_{p,\infty,u}(\mathbb{R}^n)} < K_1,$$

and

$$\sup_{0\leq t<T} t^{1-\frac{n}{\alpha u}+\frac{1-2m}{\alpha}} \|(v(t), w(t))\|_{\mathcal{M}^p_u} < K_2.$$

Then tere exists two positive constants N_1 and N_2 depending of K_1, K_2, n, α and u such that

$$\|(\partial^\beta_x v(t), \partial^\beta_x w(t))\|_{\mathcal{M}^p_u} \leq N_1(N_2|\beta|)^{2|\beta|} t^{\frac{-|\beta|}{\alpha-1}+\frac{n}{\alpha u}},$$

for all $u < p \leq q \leq \infty$, $t \in (0, T)$ and $\beta \in \mathbb{N}^n$.

Proof. Firstly we prove an equivalent one:

$$\|\partial^\beta_x v(t), \partial^\beta_x w(t)\|_{\mathcal{M}^p_r} \leq K_1(K_2|\beta|)^{2|\beta|-\delta} t^{\frac{-|\beta|}{\alpha}-\frac{1+n}{\alpha r}},$$

with $\delta \in (1, 2]$ and K_1, K_2 are constants sufficiently large.
Firstly let us prepare the refined $\mathcal{M}^p_u - \mathcal{M}^p_r$ estimates for the semigroup operator $\mathcal{S}_\alpha(t)$.

Lemma 3.8. *Let $1 \leq r \leq u \leq p \leq q \leq \infty$. Then for any $f \in \mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}$ we have the following inequality*

$$\|\partial_x^\beta \mathcal{S}_\alpha(t) f\|_{\mathcal{M}_r^p} \leq C_0^{|\beta|} |\beta|^{\frac{|\beta|}{\alpha}} t^{\frac{-|\beta|}{\alpha-1}+\frac{n}{\alpha r}} \|f\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \quad \forall t > 0,\ \beta \in \mathbb{N},$$

and C_0 a constant depending in n and α.

Proof. Since $\mathcal{S}_\alpha(t)$ is the convolution operator with kernel $K_t(x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^\alpha}\right)$, by scaling we see that

$$K_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi = t^{-n/\alpha} K\left(\frac{x}{t^{1/\alpha}}\right),$$

where $K(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^\alpha} d\xi$.
We need only to establish that $\nabla K(x), \nabla K_t(x) \in L^1(\mathbb{R}^n)$ for any $0 < t < \infty$ and $1 \leq p \leq \infty$.
It is evident to see that $e^{-|\xi|^{2\alpha}} \in L^1(\mathbb{R}^n)$, so $K(x) \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,
and $\lim_{|x|\longrightarrow\infty} K(x) = 0$ (using the Riemann-Lesbesgue theorem).
In the same way, we have $i|\xi|^\nu e^{-|\xi|^{2\alpha}} \in (L^1(\mathbb{R}^n))^n$, then

$$\nabla K(x) \in L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n),$$

with $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) / \lim_{|x|\longrightarrow\infty} f(x) = 0\}$.
Similary we can obtain that $\nabla K_t(x) \in L^1(\mathbb{R}^n)$, (see [17], Lemma 2.7).
Thus the Young's inequality implies that

$$\left\|\partial_x \mathcal{S}_\alpha(t) f\right\|_{\mathcal{M}_r^p} \leq \|\partial_x K_t(x)\|_{L^1} \|f\|_{\mathcal{M}_r^p} \leq C(n,\alpha) t^{-1/\alpha} \|f\|_{\mathcal{M}_r^p}. \qquad (3.7)$$

By using the commutativity between semigroup $\mathcal{S}_\alpha(t)$ and differential, we get

$$\partial_x^\beta \mathcal{S}_\alpha(t) f = \prod_{i=1}^n \left(\partial_{x_i} \mathcal{S}_\alpha\left(\frac{t}{2|\beta|}\right)\right)^{\beta_i} \mathcal{S}_\alpha\left(\frac{t}{2}\right) f. \qquad (3.8)$$

Using (3.7) and (3.8), and using Definition 2.3, we have

$$\begin{aligned}
\left\|\partial_x^\beta \mathcal{S}_\alpha(t) f\right\|_{\mathcal{M}_r^p} &\leq \prod_{i=1}^n \left\|\partial_{x_i} \mathcal{S}_\alpha\left(\frac{t}{2|\beta|}\right)\right\|_{\mathcal{L}(\mathcal{M}_r^p,\mathcal{M}_r^p)}^{\beta_i} \left\|\mathcal{S}_\alpha\left(\frac{t}{2}\right) f\right\|_{\mathcal{M}_r^p} \\
&\leq \left[C(n,\alpha)\left(\frac{t}{2|\beta|}\right)^{-1/\alpha}\right]^{|\beta|} \left(\frac{t}{4}\right)^{-n(1/p-1/r)/\alpha} \left\|\mathcal{S}_\alpha\left(\frac{t}{4}\right) f\right\|_{\mathcal{M}_u^p} \\
&\leq C(n,\alpha)^{|\beta|} |\beta|^{|\beta|/\alpha} t^{\frac{-|\beta|}{\alpha}-1+\frac{n}{\alpha r}+\frac{-2+2m}{\alpha}} \sup_{\frac{t}{4}>0} \left(\frac{t}{4}\right)^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \left\|\mathcal{S}_\alpha\left(\frac{t}{4}\right) f\right\|_{\mathcal{M}_u^p} \\
&\leq C(n,\alpha)^{|\beta|} |\beta|^{|\beta|/\alpha} t^{\frac{-|\beta|}{\alpha}-1+\frac{n}{\alpha r}+\frac{-2+2m}{\alpha}} \|f\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}},
\end{aligned}$$

where $\|\mathbf{T}\|_{\mathcal{L}(\mathcal{M}_u^p,\mathcal{M}_r^p)}$ denotes the operator norm of linear operator $\mathbf{T}$ from $\mathcal{M}_u^p(\mathbb{R}^n)$ to $\mathcal{M}_r^p(\mathbb{R}^n)$. This proves Lemma 3.8. Next we recall the following result due to Kahane [25]:

Lemma 3.9. *Let $\delta > \frac{1}{2}$. Then there exists a positive constant C depending only on δ, such that*

$$\sum_{\alpha\leq\beta} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} |\alpha|^{|\alpha|-\delta} |\beta-\alpha|^{|\beta-\alpha|-\delta} \leq C(\delta) |\beta|^{|\beta|-\delta} \text{ for all } \beta \in \mathbb{N}_0^n. \qquad (3.9)$$

Note that the notation $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for all $i = 1, 2, \cdots, n$, and $\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \prod_{i=1}^{n} \frac{\beta_i!}{\alpha_i!(\beta_i - \alpha_i)!}$, and that the dependence of $C(\delta)$ on δ is simply of the form $\sum_{j=1}^{\infty} j^{-\delta-1/2}$.

At last, we recall the following lemma, whose proof can be found in [12]:

Lemma 3.10. *[12]. Let φ_0 be a measurable and locally bounded function in $(0,\infty)$. Let $\left\{\varphi_j\right\}_{j=1}^{\infty}$ be a sequence of measurable functions in $(0,\infty)$. Assume that $\alpha \in \mathbb{R}$ and $\mu, v > 0$ satisfying $\mu + v = 1$. Let $B_\eta > 0$ be a number depending on $\eta \in (0,1)$, and assume that B_η is nonincreasing with respect to η. Assume that there is a positive constant σ, such that*

$$0 \leq \varphi_0(t) \leq B_\eta t^{-\alpha} + \sigma \int_{(1-\eta)t}^{t} (t-\tau)^{-\mu} \tau^{-v} \varphi_0(\tau) d\tau$$

and

$$0 \leq \varphi_{j+1}(t) \leq B_\eta t^{-\alpha} + \sigma \int_{(1-\eta)t}^{t} (t-\tau)^{-\mu} \tau^{-v} \varphi_j(\tau) d\tau$$

for all $j \geq 0, t > 0$ and $\eta \in (0,1)$. Let η_0 be a unique positive number such that $I(\eta_0) = \min\left\{\frac{1}{2\sigma}, I(1)\right\}$ with $I(\eta) = \int_{1-\eta}^{1} (1-\tau)^{-\mu} \tau^{-\alpha-v} d\tau$. Then for any $0 < \eta \leq \eta_0$, we have

$$\varphi_j(t) \leq 2B_\eta t^{-\alpha} \text{ for all } j \geq 0 \text{ and } t > 0.$$

Now we prove the Theorem 3.7.
Follows the idea of [12], we first prove a variant of Theorem 3.7 under extra regularity hypothes.

Proposition 3.11. *Under the same hypotheses of Theorem 3.7. Suppose further that*

$$\left(\partial_x^\beta v(t), \partial_x^\beta w(t)\right) \in C\left((0,T), \mathcal{M}_r^p(\mathbb{R}^n)\right) \tag{3.10}$$

for all $p \leq q \leq \infty$ and $\beta \in \mathbb{N}_0^n$. Then for any $\delta \in (1,2]$, there exist two positive constants K_1 and K_2 depending only on $K_1, K_2, n, \alpha, p, \varepsilon$ and δ, such that

$$\left\|\left(\partial_x^\beta v(t), \partial_x^\beta w(t)\right)\right\|_{\mathcal{M}_r^p} \leq K_1 \left(K_2|\beta|\right)^{2|\beta|-\delta} t^{\frac{-|\beta|}{\alpha} - 1 + \frac{n}{\alpha r} + \frac{-2+2m}{\alpha}} \tag{3.11}$$

for all $p \leq q \leq \infty, t \in (0,T)$ and $\beta \in \mathbb{N}_0^n$.

Proof. We divide the proof into two steps by induction $|beta| = m$.

Step 1: In this step, we aim to demonstrate the validity of (3.11) for $m = 0$. It is worth noting that, given the conditions stipulated in Theorem 3.7, the assertion of (3.11) is evident if $q = p$. Consequently, our focus is primarily on the case where q falls within the range $(p, \infty]$. Let $\eta \in (0,1)$ be a constant to be determined later, we take $\mathcal{M}_r^p$-norm of both sides of the first equation of (1.3) and split the time integral into two parts as follows:

$$\begin{aligned} \|v(t)\|_{\mathcal{M}_r^p} &\leq \|\mathcal{S}_\alpha(t) v_0\|_{\mathcal{M}_r^p} + \left(\int_0^{t(1-\eta)} + \int_{t(1-\eta)}^{t}\right) . \left\|\mathcal{S}_\alpha(t-\tau)\nabla \cdot \left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau) d\tau \\ &= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3. \end{aligned} \tag{3.12}$$

We will estimate the terms one by one. For $\mathcal{E}_1$, by Lemma 3.8 and the conditions of Theorem 3.7,we can easily see that

$$\mathcal{E}_1 \leq C_1(n,\alpha) t^{-1+\frac{n}{\alpha u} - \frac{1-2m}{\alpha}} \|v_0\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \leq C_1(n,\alpha,K_1) t^{-1+\frac{n}{\alpha u} - \frac{1-2m}{\alpha}} \tag{3.13}$$

For $\mathcal{E}_2$, by Lemma 2.6, Lemma 2.7 and the condition of Theorem 3.7, we have

$$\mathcal{E}_2 = \int_0^{t(1-\eta)} \left\|\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau$$
$$\leq C_2(n,\alpha,p)\int_0^{t(1-\eta)}(t-\tau)^{-n(2/p-1/r)/\alpha}\|v(\tau)\|_{\mathcal{M}_u^p}\|(v(\tau),w(\tau))\|_{\mathcal{M}_u^p}d\tau$$
$$\leq C_2(n,\alpha,p,K_2)\,\eta^{-2}t^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}. \qquad (3.14)$$

By using Lemma 2.6, Lemma 2.7 and the condition of Theorem 3.7 again, we estimate $\mathcal{E}_3$ as:

$$\begin{aligned}\mathcal{E}_3 &= \int_{t(1-\eta)}^{t} \left\|\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau\\ &\leq C_3(n,\alpha,p)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)\|_{\mathcal{M}_r^p}\|(v(\tau),w(\tau))\|_{\mathcal{M}_u^p}d\tau\\ &\leq C_3(n,\alpha,p,K_2)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)\|_{\mathcal{M}_r^p}d\tau. \qquad (3.15)\end{aligned}$$

Using (3.13)-(3.15), and setting $\bar{B}_\eta = C_1(n,\alpha,K_1) + C_2(n,\alpha,p,K_2)\,\eta^{-2}$, the inequality (3.12) yields:

$$\|v(t)\|_{\mathcal{M}_r^p} \leq \bar{B}_\eta t^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} + C_3(n,\alpha,p,K_2)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)\|_{\mathcal{M}_r^p}d\tau. \qquad (3.16)$$

Analogously as (3.16) we can estimate $w(t)$. Hence, we have

$$\begin{aligned}\|(v(t),w(t))\|_{\mathcal{M}_r^p} \leq& B_\eta t^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} + C_4\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\\ &\times\|(v(\tau),w(\tau))\|_{\mathcal{M}_r^p}d\tau, \qquad (3.17)\end{aligned}$$

where $B_\eta = 2\bar{B}_\eta$ and $C_4 = 2C_3(n,\alpha,p,K_2)$. By applying Lemma 3.10, we get the desired estimate (3.11) for $|\beta| = m = 0$ with $K_1 = 2B_{\eta_0}$ for some $\eta_0 = \eta_0(n,\alpha,p,K_1,K_2) \in (0,1)$.

Step 2: Next we will prove (3.11) for $|\beta| = m \geq 1$. Due to the appearance of nonlocal function ϕ, we shall use a different argument to prove (3.11) for $p \leq q < n$ and $n \leq q \leq \infty$, thus we devide the step into the following two cases.

Case 1 $(p \leq q < n)$ We differentiate the first equation of (1.3) to obtain the equality

$$\partial_x^\beta v(t) = \partial_x^\beta\mathcal{S}_\alpha(t)v_0 - \int_0^t \partial_x^\beta\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right](\tau)d\tau.$$

Now we take the $\mathcal{M}_r^p$-norm of $\partial_x^\beta v$, for some $\eta \in (0,1)$ to be chosen later, we split the time integral into the following two parts:

$$\begin{aligned}\left\|\partial_x^\beta v(t)\right\|_{\mathcal{M}_r^p} \leq& \left\|\partial_x^\beta\mathcal{S}_\alpha(t)v_0\right\|_{\mathcal{M}_r^p}\\ &+\left(\int_0^{t(1-\eta)}+\int_{t(1-\eta)}^{t}\right)\left\|\partial_x^\beta\mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau\\ =&\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3. \qquad (3.18)\end{aligned}$$

We will estimate $\mathcal{F}_i (i = 1,2,3)$ as follows: For $\mathcal{F}_1$, Lemma 3.8 implies that

$$\mathcal{F}_1 \leq C_0^m m^{m/\alpha} t^{\frac{-m}{\alpha}-1+\frac{n}{\alpha r}+\frac{2-2m}{\alpha}}\|v_0\|_{\mathcal{N}_{p,\infty,u}^{-\alpha+\frac{n}{u}+2-2m}} \leq K_1 C_0^m m^{m/\alpha} t^{\frac{-m}{\alpha}-1+\frac{n}{\alpha r}+\frac{2-2m}{\alpha}}. \qquad (3.19)$$

Go to $\mathcal{F}_2$, using Lemma 3.8, Lemma 2.7 and the conditions of Theorem 3.7, we have

$$\begin{aligned}
\mathcal{F}_2 &= \int_0^{t(1-\eta)} \left\| \partial_x^\beta \mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau \\
&\leq C_5(n,\alpha,p)\int_0^{t(1-\eta)} \left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\nabla\right\|_{\mathcal{L}(\mathcal{M}_r^p,\mathcal{M}_r^p)} \left\|\partial_x^\beta\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\right\|_{\mathcal{L}\left(\mathcal{M}^p_{nu/2n-u},\mathcal{M}_r^p\right)} \\
&\quad \times \|(v(\tau),w(\tau))\|^2_{\mathcal{M}_u^p}d\tau \\
&\leq C_5(n,\alpha,p)C_0^m m^{m/\alpha}\int_0^{t(1-\eta)}\left(\frac{t-\tau}{2}\right)^{m/\alpha-n(2/u-1/r)/\alpha+2/\alpha}\|(v(\tau),w(\tau))\|^2_{\mathcal{M}_u^p}d\tau \\
&\leq C_5(n,\alpha,p,K_2)C_0^m m^{m/\alpha}\int_0^{t(1-\eta)}\left(\frac{t-\tau}{2}\right)^{m/\alpha-n(2/u-1/r)/\alpha+2/\alpha}\tau^{-2+\frac{2n}{\alpha u}-\frac{2-4m}{\alpha}}d\tau \\
&\leq C_5(n,\alpha,p,K_2)C_0^m(2m)^{m/\alpha}\eta^{-m/\alpha-2}t^{m/\alpha-1+n/(\alpha r)+2/\alpha}. \qquad (3.20)
\end{aligned}$$

Using Leibniz's rule, we split $\mathcal{F}_3$ into the following three parts:

$$\begin{aligned}
\mathcal{F}_3 &= \int_{t(1-\eta)}^{t} \left\| \partial_x^\beta \mathcal{S}_\alpha(t-\tau)\nabla\cdot\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau \\
&\leq \int_{t(1-\eta)}^{t}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\nabla\right\|_{\mathcal{L}(\mathcal{M}_r^p,\mathcal{M}_r^p)}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\partial_x^\beta\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau \\
&\leq C_6(n,\alpha)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-1/\alpha}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\partial_x^\beta\left[v\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau \\
&\leq C_6(n,\alpha)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-1/\alpha}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\left[\left(\partial_x^\beta v\right)\nabla\left((-\Delta)^{-m}(w-v)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau + C_6(n,\alpha) \\
&\quad\times\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-1/\alpha}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\left[\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\left(\partial_x^\gamma v\right)\left(\partial_x^{\beta-\gamma}\nabla\left((-\Delta)^{-m}(w-v)\right)\right)\right]\right\|_{\mathcal{M}_r^p}d\tau \\
&\quad + C_6(n,\alpha)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-1/\alpha}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\left[v\partial_x^\beta\left(\nabla\left((-\Delta)^{-m}(w-v)\right)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau \\
&= \mathcal{F}_{31}+\mathcal{F}_{32}+\mathcal{F}_{33}. \qquad (3.21)
\end{aligned}$$

Here, the notation $\gamma<\beta$ means that $\gamma\leq\beta$ and $|\gamma|<|\beta|$. Now we shall establish the estimate for $\mathcal{F}_{3j}(j=1,2,3)$. For $\mathcal{F}_{31}$, Lemma 3.9 implies that

$$\begin{aligned}
\mathcal{F}_{31} &\leq C_7(n,\alpha,p)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left\|\partial_x^\beta v(\tau)\right\|_{\mathcal{M}_r^p}\|(v(\tau),w(\tau))\|_{\mathcal{M}_u^p}d\tau \\
&\leq C_7(n,\alpha,p,K_2)\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left\|\partial_x^\beta v(\tau)\right\|_{\mathcal{M}_r^p}d\tau. \qquad (3.22)
\end{aligned}$$

Go to $\mathcal{F}_{32}$, by our induction assumption and Lemma 3.9, we get

$$\mathcal{F}_{32} \leq C_8(n,\alpha,p)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\left\|\partial_x^{\gamma}v(\tau)\right\|_{\mathcal{M}_r^p}\left\|\left(\partial_x^{\beta-\gamma}v(\tau),\partial_x^{\beta-\gamma}w(\tau)\right)\right\|_{\mathcal{M}_u^p}d\tau$$
$$\leq C_8(n,\alpha,p)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left(\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}K_1\left(K_2|\gamma|\right)^{2|\gamma|-\delta}\tau^{-\gamma/\alpha-1+n/(\alpha q)}\right.$$
$$\left.\times K_1\left(K_2|\beta-\gamma|\right)^{2|\beta-\gamma|-\delta}\tau^{-|\beta-\gamma|/\alpha-1+n/(\alpha p)}\right)d\tau$$
$$\leq C_8(n,\alpha,p)K_1^2K_2^{2m-2\delta}\left(\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta/2}|\beta-\gamma|^{|\beta-\gamma|-\delta/2}\right)^2$$
$$\times\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-m/\alpha-2+n(1/p+1/r)/\alpha}d\tau$$
$$\leq C_8(n,\alpha,p,\delta)K_1^2K_2^{2m-2\delta}m^{2m-\delta}I(\eta)t^{m/\alpha-1+n/(\alpha r)+2/\alpha}, \tag{3.23}$$

where $I(\eta)=\int_{1-\eta}^{1}(1-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-m/\alpha-2+n(1/p+1/r)/\alpha}d\tau$. For $\mathcal{F}_{33}$, using Lemma 3.8, the result of **Step 1** and our induction assumption, we obtain

$$\mathcal{F}_{33} = C_6(n,\alpha)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-1/\alpha}\left\|\mathcal{S}_\alpha\left(\frac{t-\tau}{2}\right)\left[v\partial_x^{\beta}\left(\nabla\left((-\Delta)^{-m}(w-v)\right)\right)\right]\right\|_{\mathcal{M}_r^p}(\tau)d\tau$$
$$\leq C_9(n,\alpha,p)\int_{t(1-\eta)}^{t}\left(\frac{t-\tau}{2}\right)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\|v(\tau)\|_{\mathcal{M}^p_{nu/2n-u}}\left\|\left(\partial_x^{\beta-1}v(\tau),\partial_x^{\beta-1}w(\tau)\right)\right\|_{\mathcal{M}_r^p}d\tau$$
$$\leq C_9(n,\alpha,p,\delta)K_1^2K_2^{2m-2\delta}m^{2m-\delta}I(\eta)t^{m/\alpha-1+n/(\alpha r)+2/\alpha}. \tag{3.24}$$

Using the above estimates (3.19)-(3.24) and setting $\bar{B}_\eta$ by

$$\bar{B}_\eta = K_1C_0^m m^{m/\alpha} + C_5C_0^m(2m)^{m/\alpha}\eta^{-m/\alpha-2} + C_{10}K_1^2K_2^{2m-2\delta}m^{2m-\delta}I(\eta),$$

where $C_{10}=C_8(n,\alpha,p,\delta)+C_9(n,\alpha,p,\delta)$, we have

$$\left\|\partial_x^{\beta}v(t)\right\|_{\mathcal{M}_r^p} \leq \bar{B}_\eta t^{m/\alpha-1+n/(\alpha r)+2/\alpha} + C_7\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left\|\partial_x^{\beta}v(\tau)\right\|_{\mathcal{M}_r^p}d\tau. \tag{3.25}$$

In the same way, we can proof with $\partial_x^{\beta}w(t)$. We conclude that

$$\left\|\left(\partial_x^{\beta}v(t),\partial_x^{\beta}w(t)\right)\right\|_{\mathcal{M}_r^p} \leq B_\eta t^{m/\alpha-1+n/(\alpha r)+2/\alpha} + C_{11}\int_{t(1-\eta)}^{t}(t-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\tau^{-1+\frac{n}{\alpha u}-\frac{1-2m}{\alpha}}\left\|\left(\partial_x^{\beta}v(\tau),\partial_x^{\beta}w(\tau)\right)\right\|_{\mathcal{M}_r^p}d\tau \tag{3.26}$$

Here $B_\eta = 2\bar{B}_\eta$ and $C_{11}=2C_7(n,\alpha,p,K_2)$. Let $\eta_m=\frac{1}{m}$. It is clear that $I(\eta_m)$ is strictly monotone decreasing in m and $I(\eta_m)\to 0$ as $m\to\infty$. Choosing m_0 sufficiently large, such that $I\left(\frac{1}{m}\right)\leq\frac{1}{2C_{11}}$ for all $m\geq m_0$ and applying Lemma 3.10 , we get

$$\left\|\left(\partial_x^{\beta}v(t),\partial_x^{\beta}w(t)\right)\right\|_{\mathcal{M}_r^p} \leq 2B_{1/m}t^{m/\alpha-1+n/(\alpha r)+2/\alpha} \quad \text{for all } t>0 \text{ and } |\beta|=m. \tag{3.27}$$

Let us notice that from (3.27), we can choose K_1 and K_2 sufficiently large such that (3.11) holds for all β satisfying $|\beta| \leq m_0$. Hence, it suffices to prove that it is possible to choose K_1 and K_2 such that $2B_{1/m} \leq K_1 (K_2 m)^{2m-\delta}$ for all $m > m_0$. Since

$$I\left(\frac{1}{m}\right) = \int_{1-1/m}^{1} (1-\tau)^{-\frac{n}{\alpha u}-\frac{1-2m}{\alpha}} \tau^{-m/\alpha-2+n(1/p+1/r)/\alpha} d\tau$$
$$\leq \left(1-\frac{1}{m}\right)^{-m/\alpha-2} \leq \left(1-\frac{1}{m}\right)^{-2} e^{1/\alpha} \leq 16 \quad \text{for all } m \geq 1,$$

we can calculate $2B_{1/m}$ as follows:

$$2B_{1/m} \leq 4\left[K_1 C_0^m m^{m/\alpha} + C_5 C_0^m (2m)^{m/\alpha} m^{m/\alpha+2} + 16 C_{10} K_1^2 K_2^{2m-2\delta} m^{2m-\delta}\right]$$
$$\leq 4\left[K_1 C_0^m + C_5 (2C_0)^m m^{\delta+2} + 16 C_{10} K_1^2 K_2^{2m-2\delta}\right] m^{2m-\delta}.$$

Obviously, there exists a constant $C_{12} > C_0$ such that $C_0^m + (2C_0)^m m^{\delta+2} \leq C_{12}^{2m-\delta}$. Hence,

$$2B_{1/m} \leq 4\left[(K_1 + C_5) C_{12}^{2m-\delta} + 16 C_{10} K_1^2 K_2^{2m-2\delta}\right] m^{2m-\delta}.$$

Now if we choose

$$K_1 := 8(K_1 + C_5) \text{ and } K_2 := \max\{C_{12}, 32 C_{10} K_1\}$$

then we obtain (3.11). This completes the proof of Proposition 3.11 for $u \leq r < n$.

Case 2 $(n \leq u \leq r \leq \infty)$ Now we are in a position to establish the estimate of $\left\|\partial_x^\beta v(t)\right\|_{\mathcal{M}_r^p}$ for $n \leq q \leq \infty$. For any given $\max\left\{1, \frac{n}{\alpha}\right\} < p < \min\left\{n, \frac{2n}{\alpha}\right\}$, using the same idea of Gagliardo-Nirenberg inequality [18], we see that

$$\left\|\partial_x^\beta v(t)\right\|_{\mathcal{M}_r^p} \leq C(n,p) \left\|\partial_x^\beta v(t)\right\|_{\mathcal{M}_u^p}^{\theta} \left\|\partial_x^2 \partial_x^\beta v(t)\right\|_{\mathcal{M}_u^p}^{1-\theta}, \qquad (3.28)$$

with

$$\theta = 1 - \frac{n}{2u} + \frac{n}{2r}. \qquad (3.29)$$

Now from (3.28), (3.29) and the result of Case 1 we see that

$$\left\|\partial_x^\beta v(t)\right\|_{\mathcal{M}_r^p} \leq C(n,p) \left(K_1 (K_2|\beta|)^{2|\beta|-\delta} t^{-|\beta|/\alpha-1+n/(\alpha u)}\right)^{\theta}$$
$$\times \left(K_1 (K_2(|\beta|+2))^{2|\beta|+4-\delta} t^{-(|\beta|+2)/\alpha-1+n/(\alpha u)}\right)^{1-\theta}$$
$$\leq C(n,p) K_1 (K_2(|\beta|+2))^{2|\beta|+4-\delta} t^{\frac{-|\beta|}{\alpha}-1+\frac{n}{\alpha r}+\frac{-2+2m}{\alpha}}. \qquad (3.30)$$

It is clear that there exists a constant $C_{13} \geq 2$ such that $|\beta|^4 \leq C_{13}^{2|\beta|-\delta}$, thus we have $(K_2(|\beta|+2))^{2|\beta|+4-\delta} = K_2^4 |\beta|^4 \left(1+\frac{2}{|\beta|}\right)^{2|\beta|+4} (K_2|\beta|)^{2|\beta|-\delta} \leq 81 e^4 K_2^4 (C_{13} K_2 |\beta|)^{2|\beta|-\delta}$. Hence, we can choose K_1 and K_2 sufficiently large such that (3.11) holds for all $p \leq q \leq \infty$. This completes the proof of Proposition 3.11. □

Finally, let us proof that under the hypotheses of Theorem 3.7, the mild solution $(v(t), w(t))$ of (1) always satisfies the regularity condition (3.10).

Proposition 3.12. *([12, 16, 20].) Under the hypotheses of Theorem 3.7, the mild solution $(v(t), w(t))$ satisfies*

$$t^{\frac{-|\beta|}{\alpha}-1+\frac{n}{\alpha r}+\frac{-2+2m}{\alpha}} \left\|\left(\partial_x^\beta v(t), \partial_x^\beta w(t)\right)\right\|_{\mathcal{M}_r^p} \leq \tilde{K}_1 \left(\tilde{K}_2 |\beta|\right)^{2|\beta|-\delta}, \qquad (3.31)$$

for all $p \leq q \leq \infty$, $t \in (0,T)$ and $\beta \in \mathbb{N}_0^n$, where $\tilde{K}_1$ and $\tilde{K}_2$ are constants depending only on $K_1, K_2, n, \alpha, p, \varepsilon$ and δ.

Now Theorem 3.7 follows immediately from Proposition 3.11 and Proposition 3.12. We complete the proof of Theorem 3.7. □

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