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Blending type approximation of generalized Lupaş-Jain type operators via A-statistical convergence and power series method

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Abstract. In this paper, we introduce integral type modification of generalized Lupaş operators as presented by Goyal and Kajla [13]. In a weighted space, we explore how these operators approximate by employing both the A-statistical method and the power series method. We establish a generalization of the Lupaş-Jain type operators of order κ with the help of methodology that Kirov and Popova [23] used, and estimate the approximation error for a function of Lipschitz class. Using Maple software, we obtain error of approximation of newly defined operators to certain functions.

1. Introduction

For $\vartheta : [0,\infty) \to \mathbb{R}$, Lupaş [25] presented the operators

$$\mathcal{L}_m(\vartheta,x) = (1-\nu)^{mx}\sum_{j=0}^{\infty}\frac{(mx)_j}{j!}\vartheta\Big(\frac{j}{m}\Big)\nu^j, \quad |\nu|<1, m\in\mathbb{N} \text{ and } x\in[0,\infty), \tag{1}$$

with the identity $\frac{1}{(1-\nu)^{mx}} = \sum_{j=0}^{\infty}\frac{(mx)_j}{j!}\nu^j$ and $(mx)_j$ is rising factorial.
For $\nu = \frac{1}{2}$, Agratini [1] modified above operators as

$$\mathcal{A}_m(\vartheta,x) = 2^{-mx}\sum_{j=0}^{\infty}\frac{(mx)_j}{2^j j!}\vartheta\Big(\frac{j}{m}\Big).$$

He estimated quantitative approximation and Voronovskaja type theorem and refined them with the help of probabilistic method. Further, he defined integral modifications of above operators.

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Numerous modification of Lupaş operators have been suggested. Khan and Khan [22] introduced Stancu variant of Lupaş operators, also studied approximation properties using Korovkin theorem and obtained Voronovskaja type theorem for first and second order derivatives. Yadav et al. [49] proposed a bivariate extension of Lupaş-Durrmeyer operators involving Polya Distribution, and estimated convergence rate of these operators with the help of Peetre's K-functional and modulus of smoothness. Ostrovska and Turan [29] investigated the properties of block functions generating the limit q-Lupaş operators. Govil et al. [12] presented a modification in Lupaş operators and established direct approximation. Gupta and Yadav [15] introduced the integral modification of Lupaş operators with weights of Beta basis functions. In the following, Gupta et al. [16] proposed Lupaş-Beta operators that ensure the approximation of constant and linear functions. Kajla et al. [18] presented Baskakov-Jain type operators involving two parameters, and estimated weighted approximation, Voronovskaja type theorem and Grüss Voronovskaja type theorem. Başcanbaz-Tunca et al. [3] presented Jain modification of Lupaş operators, also obtained weighted approximation, monotonicity under convexity and presented preservative properties of each Lupaş-Jain operator. After that Patel and Bodur [34] continued [3] and estimated weighted approximation, local approximation in sense of Peetre's K-function and Lipschitz class, Voronovskaja type theorem. Further, defined that the operators can be estimated in sense of the Steklov means. Kajla [19] studied Szász operators based on Charlier polynomials and obtained the rates of A-statistical convergence, also obtained local approximation and Voronovskaja type theorem by means of statistical convergence. Özger and Ansari [32] introduced an extension of bivariate Bernstein type operators, incorporating multiple shape parameters. This extension utilizes four-dimensional infinite matrices in the field of approximation theory. They derived the statistical convergence rate and convergence rate with the help of power series method. Nasiruzzaman et al. [27] proposed α-Schurer-Kantorovich operators and bivariate form, also established approximation properties. Braha et al. [5] introduced a new class of Baskakov-Schurer-Szász-Stancu operators and derived approximation properties and some results related to weighted space. Özger et al. [33] proposed α-Bernstein-Schurer operators and estimated monotonicity and convexity, also established global approximation and Voronovskaja type theorem. Srivastava et al. [42] pressented Bernstein-Stancu operators with Bézier bases and parameter $\lambda \in [-1, 1]$ for one and two variables. Further, investigated approximation properties and Voronovskaja type theorem. Srivastava et al. [41] introduced Szász-Mirakjan-Beta operators. Also estimated approximation properties and derived Voronovskaja type theorem. Srivastava et al. [43] presented q-Szász-Mirakjan-Kantorovich operators which are generated by Dunkl's generalization of the exponential function for one and two variables. Further, established approximation properties.
Statistical convergence of operators are discussed in ([2], [4], [6]-[9], [11], [17], [20], [21], [24]-[26], [28], [31], [35]-[40], [44]-[46], [48]) and references therein.

Goyal and Kajla [13] introduced generalized Lupaş operators as

$$\mathcal{G}_m^{\nu}(\vartheta, x) = \sum_{j=0}^{\infty} l_{m,j}(x,\nu)\vartheta\left(\frac{j}{m}\right), \quad \nu \geq 0, \quad x \geq 0, \tag{2}$$

where $l_{m,j}(x,\nu) = \frac{e^{-\nu x}}{2^{mx}} \frac{p^{\nu}_{m,j}(x)x^j}{j!}$ such that $\sum_{j=0}^{\infty} l_{m,j}(x,\nu) = 1$ and $p^{\nu}_{m,j}(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{\nu^{j-i}(mx)_i}{(2x)^i}$.

We derive integral type generalization of the operators defined by (2) as follows:

$$\wp^{\varsigma}_{m,\nu}(\vartheta, x) = \sum_{j=0}^{\infty} l_{m,j}(x,\nu) \int_0^{\infty} q_{m,j}(\ell)\vartheta(\ell)d\ell, \tag{3}$$

where $q_{m,j}(\ell) = \frac{\varsigma}{B\left(j+1, \frac{m}{\varsigma}\right)} \frac{(\varsigma\ell)^j}{(1+\varsigma\ell)^{\frac{m}{\varsigma}+j+1}}$ and $l_{m,j}(x,\nu)$ is defined as above.

In this note, we establish the moments and central moments. Subsequently, we derive weighted approximations using both power series and direct approximation methods. We then delve into the realm of A-statistical convergence, discussing its implications. Through A-statistical convergence, we establish results regarding local approximation and a Voronovskaja-type theorem. Additionally, we extend the operators defined in equation (3) to the κ-th order.

Lemma 1.1. *For $\varsigma > 0$, we have*

(i) $\wp^{\varsigma}_{m,\nu}(e_0, x) = 1$;

(ii) $\wp^{\varsigma}_{m,\nu}(e_1, x) = \frac{x}{m-\varsigma}(\nu+m) + \frac{1}{m-\varsigma}$;

(iii) $\wp^{\varsigma}_{m,\nu}(e_2, x) = \frac{x^2}{(m-2\varsigma)(m-\varsigma)}(\nu+m)^2 + \frac{x}{(m-2\varsigma)(m-\varsigma)}(4\nu+5m) + \frac{2}{(m-2\varsigma)(m-\varsigma)}$;

(iv) $$\wp^{\varsigma}_{m,\nu}(e_3, x) = \frac{x^3}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(\nu+m)^3 + \frac{x^2}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(3(\nu+m)(3\nu+4m)) + \frac{x}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(18\nu+29m) + \frac{6}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)};$$

(v) $$\wp^{\varsigma}_{m,\nu}(e_4, x) = \frac{x^4}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(\nu+m)^4 + \frac{x^3}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(2(\nu+m)^2(8\nu+11m)) + \frac{x^2}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(72\nu^2+200\nu m+131m^2) + \frac{x}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(96\nu+206m) + \frac{24}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}.$$

Lemma 1.2. *For $x \in [0, \infty)$, we obtain*

1. $\wp^{\varsigma}_{m,\nu}((\ell - x), x) = \frac{x}{m-\varsigma}(\nu+\varsigma) + \frac{1}{m-\varsigma}$;
2. $$\wp^{\varsigma}_{m,\nu}((\ell - x)^2, x) = \frac{x^2}{(m-2\varsigma)(m-\varsigma)}(\nu^2+4\nu\varsigma+\varsigma(m+2\varsigma)) + \frac{x}{(m-2\varsigma)(m-\varsigma)}(4\nu+3m+4\varsigma) + \frac{2}{(m-2\varsigma)(m-\varsigma)} = \lambda^{\varsigma}_{m,\nu}(x);$$
3. $$\wp^{\varsigma}_{m,\nu}((\ell - x)^3, x) = \frac{x^3}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(\nu^3+9\nu^2\varsigma+3\nu m\varsigma+18\nu\varsigma^2+7m\varsigma^2+6\varsigma^3) + \frac{x^2}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(9\nu^2+9\nu m+36\nu\varsigma+30m\varsigma+18\varsigma^2) + \frac{x}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(18\nu+23m+18\varsigma) + \frac{6}{(m-3\varsigma)(m-2\varsigma)(m-\varsigma)};$$
4. $$\wp^{\varsigma}_{m,\nu}((\ell - x)^4, x) = \frac{x^4}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(\nu^4+16\nu^3\varsigma+6\nu^2m\varsigma+72\nu^2\varsigma^2+40\nu m\varsigma^2+3m^2\varsigma^2 + 96\nu\varsigma^3+46m\varsigma^3+24\varsigma^4) + \frac{x^3}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(16\nu^3+18\nu^2m+144\nu^2\varsigma + 168\nu m\varsigma+18m^2\varsigma+288\nu\varsigma^2+256m\varsigma^2+96\varsigma^3) + \frac{x^2}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(72\nu^2 + 128\nu m+27m^2+288\nu\varsigma+380m\varsigma+144\varsigma^2) + \frac{x}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}(96\nu+182m + 96\varsigma) + \frac{24}{(m-4\varsigma)(m-3\varsigma)(m-2\varsigma)(m-\varsigma)}.$$

Let $\varphi^{\varsigma,k}_{m,\nu} = \wp^{\varsigma}_{m,\nu}((\ell - x)^k, x)$, where $k = 1, 2, 3, 4$.

Lemma 1.3. *We have*

1. $\lim_{m\to\infty} m\varphi^{\varsigma,1}_{m,\nu} = x(\varsigma+\nu) + 1$;

2. $\lim_{m\to\infty} m\varphi_{m,\nu}^{\varsigma,2} = x^2\varsigma + 3x$;
3. $\lim_{m\to\infty} m^2\varphi_{m,\nu}^{\varsigma,3} = x^3(\varsigma^2 + 3\nu\varsigma) + x^2(30\varsigma + 9\nu) + 23x$;
4. $\lim_{m\to\infty} m^2\varphi_{m,\nu}^{\varsigma,4} = 3x^4\varsigma^2 + 18x^3\varsigma + 27x^2$.

2. Direct Approximation

Consider $C_B[0,\infty)$ the space formed by real valued bounded and continuous functions $\vartheta \in [0,\infty)$, with the norm $\|\vartheta\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} |\vartheta(x)|$.
For $\vartheta \in C_B[0,\infty)$ and $\hat{\delta} > 0$ the l^{th} ordered modulus of continuity is defined as

$$\omega_l(\vartheta,\hat{\delta}) = \sup_{0\le|\tau|\le\hat{\delta}} \sup_{x\in[0,\infty)} |\Delta_\tau^l \vartheta(x)|,$$

where Δ_τ^l is a forward difference given by [10].
When $l = 1$, the modulus of continuity is defined as

$$\omega(\vartheta,\hat{\delta}) = \sup_{0\le|\tau|\le\hat{\delta}} \sup_{x\in[0,\infty)} |\vartheta(x+\tau) - \vartheta(x)|.$$

Peetre's K−functional is given as

$$K_2(\vartheta,\hat{\delta}) = \inf_{\mu\in C_B^*[0,\infty)} \{\|\vartheta - \mu\| + \hat{\delta}\|\mu''\|\}, \quad \hat{\delta} > 0,$$

where $C_B^*[0,\infty) = \{\mu \in C_B[0,\infty) : \mu', \mu'' \in C_B[0,\infty)\}$ and the norm

$$\| \vartheta \|_{C_B^*[0,\infty)} = \| \vartheta \|_{C_B[0,\infty)} + \| \vartheta' \|_{C_B[0,\infty)} + \| \vartheta'' \|_{C_B[0,\infty)} .$$

For $\hat{\delta} > 0$, we have the inequality

$$K_2(\vartheta,\hat{\delta}) \le C\{\omega_2(\vartheta,\sqrt{\hat{\delta}}) + min(1,\hat{\delta})\|\vartheta\|_{C_{B[0,\infty)}}\}, \tag{4}$$

where C is positive constant. For $\vartheta \in C_B[0,\infty)$, the Steklov mean is given as

$$\vartheta_\tau(x) = \frac{4}{\tau^2}\int_0^{\frac{\tau}{2}}\int_0^{\frac{\tau}{2}} [2\vartheta(x+u+v) - \vartheta(x+2(u+v))]dudv, \tag{5}$$

and verify the following inequalities:

1. $\|\vartheta_\tau - \vartheta\| \le \omega_2(\vartheta,\tau)$,
2. $\vartheta', \vartheta'' \in C_B[0,\infty)$ and $\|\vartheta_\tau'\| \le \frac{5}{\tau}\omega(\vartheta,\tau), \|\vartheta_\tau''\| \le \frac{9}{\tau^2}\omega_2(\vartheta,\tau)$.

Theorem 2.1. *Let $\vartheta \in C_B[0,\infty)$. For any $x \ge 0$, the following inequality holds*

$$|\wp_{m,\nu}^{\varsigma}(\vartheta,x) - \vartheta(x)| \le 5\omega(\vartheta,\sqrt{\lambda_{m,\nu}^{\varsigma}(x)}) + \frac{13}{2}\omega_2(\vartheta,\sqrt{\lambda_{m,\nu}^{\varsigma}(x)}),$$

where $\lambda_{m,\nu}^{\varsigma}(x)$ is defined in Lemma 1.2.

Proof. By applying the Steklov mean ϑ_τ, which is defined in equation (5), we obtain

$$|\wp_{m,\nu}^{\varsigma}(\vartheta,x) - \vartheta(x)| \le \wp_{m,\nu}^{\varsigma}(|\vartheta - \vartheta_\tau|,x) + |\wp_{m,\nu}^{\varsigma}(\vartheta_\tau - \vartheta_\tau(x),x)| + |\vartheta_\tau(x) - \vartheta(x)|. \tag{6}$$

Also,

$$|\wp_{m,\nu}^{\varsigma}(\vartheta,x)| \le \|\vartheta\|.$$

Using the property (1) of Steklov mean and above inequality, we get

$$\wp_{m,\nu}^{\varsigma}(|\vartheta - \vartheta_\tau|, x) \leq \|\vartheta - \vartheta_\tau\| \leq \omega_2(\vartheta, \tau).$$

Using Cauchy-Schwarz inequality and Taylor's expansion, we get

$$|\wp_{m,\nu}^{\varsigma}(\vartheta_\tau - \vartheta_\tau(x), x)| \leq \|\vartheta_\tau'\| \sqrt{\wp_{m,\nu}^{\varsigma}((\ell - x)^2, x)} + \frac{1}{2}\|\vartheta_\tau''\|\wp_{m,\nu}^{\varsigma}((\ell - x)^2, x).$$

According to Lemma 1.2 and property (2) of Steklov mean, we obtain

$$|\wp_{m,\nu}^{\varsigma}(\vartheta_\tau - \vartheta_\tau(x), x)| \leq \frac{5}{\tau}\omega(\vartheta, \tau) \sqrt{\lambda_{m,\nu}^{\varsigma}(x)} + \frac{9}{2\tau^2}\omega_2(\vartheta, \tau)\lambda_{m,\nu}^{\varsigma}(x).$$

Now, by choosing $\tau = \sqrt{\lambda_{m,\nu}^{\varsigma}(x)}$ and substituting the above values in (6), we obtain the desired results. □

Lipschitz type space with two parameters $a_1 \geq 0$ and $a_2 > 0$ is given as [30]

$$Lip_M^{(a_1,a_2)}(\eta) = \left\{\vartheta \in C_B[0,\infty) : |\vartheta(\ell) - \vartheta(x)| \leq M\frac{|\ell - x|^\eta}{(\ell + a_1x^2 + a_2x)^{\frac{\eta}{2}}}; \quad x, \ell \in (0,\infty)\right\},$$

where $M > 0$ and $0 < \eta \leq 1$.

Theorem 2.2. *Let $\vartheta \in Lip_M^{(a_1,a_2)}(\eta)$. Then, we have*

$$|\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)| \leq M\left(\frac{\lambda_{m,\nu}^{\varsigma}(x)}{a_1x^2 + a_2x}\right)^{\frac{\eta}{2}}.$$

Proof. First, consider $\eta = 1$, we can say

$$|\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)| \leq \wp_{m,\nu}^{\varsigma}(|\vartheta(\ell) - \vartheta(x)|, x) \leq M\wp_{m,\nu}^{\varsigma}\left(\frac{|\ell - x|}{\sqrt{\ell + a_1x^2 + a_2x}}, x\right).$$

Using the fact that $\frac{1}{\sqrt{\ell+a_1x^2+a_2x}} \leq \frac{1}{\sqrt{a_1x^2+a_2x}}$, Cauchy-Schwarz inequality and applying Lemma 1.2, we get

$$\begin{aligned}
|\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)| &\leq M\frac{1}{\sqrt{a_1x^2 + a_2x}}\wp_{m,\nu}^{\varsigma}(|\ell - x|, x) \\
&\leq \frac{M}{\sqrt{a_1x^2 + a_2x}}\left(\wp_{m,\nu}^{\varsigma}((\ell - x)^2, x)\right)^{\frac{1}{2}} \\
&\leq M\left(\frac{\lambda_{m,\nu}^{\varsigma}(x)}{a_1x^2 + a_2x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, the result holds when $\eta = 1$.

Take $0 < \eta < 1$. Then, by Hölder's inequality for $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$ and consider Lemma 1.2, we get

$$\begin{aligned}
|\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)| &\leq \sum_{j=0}^{\infty} l_{m,j}(x,\nu) \int_0^{\infty} q_{m,j}(\ell) |\vartheta(\ell) - \vartheta(x)| d\ell \\
&\leq \Big(\sum_{j=0}^{\infty} l_{m,j}(x,\nu) \int_0^{\infty} q_{m,j}(\ell) (|\vartheta(\ell) - \vartheta(x)| d\ell)^{\frac{2}{\eta}} \Big)^{\frac{\eta}{2}} \Big(\sum_{j=0}^{\infty} l_{m,j}(x,\nu) \int_0^{\infty} q_{m,j}(\ell) d\ell \Big)^{\frac{2-\eta}{2}} \\
&\leq M \Big(\sum_{j=0}^{\infty} l_{m,j}(x,\nu) \int_0^{\infty} q_{m,j}(\ell) \frac{(\ell - x)^2}{(\ell + a_1 x^2 + a_2 x)} d\ell \Big)^{\frac{\eta}{2}} \\
&\leq \frac{M}{(a_1 x^2 + a_2 x)^{\frac{\eta}{2}}} \Big(\wp_{m,\nu}^{\varsigma}((\ell - x)^2, x) \Big)^{\frac{\eta}{2}} \\
&\leq M \Big(\frac{\lambda_{m,\nu}^{\varsigma}}{a_1 x^2 + a_2 x} \Big)^{\frac{\eta}{2}}.
\end{aligned}$$

□

3. Weighted Approximation via Power Series ([37], [50], [51], [52])

Let B_w represents as the space of functions $\vartheta \in [0,\infty)$ which satisfy $|\vartheta(x)| \leq K_\vartheta w(x)$, where $K_\vartheta > 0$ and w is a continuous weight function on $\mathbb{R}$ that satisfy

$$w(x) \geq 1 \quad and \quad \lim_{x\to\infty} \frac{1}{w(x)} = 0.$$

Let C_w represents the subspace formed by all continuous functions in B_w and $C_w^* = \Big\{ \vartheta \in C_w : \lim_{x\to\infty} \frac{|\vartheta(x)|}{w(x)} \text{ is finite} \Big\}$. B_w and C_w are Banach spaces equipped with the norm given as $\|\vartheta\|_w = \sup_{x\geq 0} \frac{|\vartheta(x)|}{w(x)}$, for all $\vartheta \in B_w$. If w_1 and w_2 are two weight functions, then they satisfy

$$\lim_{x\to\infty} \frac{w_1(x)}{w_2(x)} = 0, \tag{7}$$

as we can see that $B_{w_1} \subset B_{w_2}$, hence $C_{w_1} \subset C_{w_2}$. If $J : C_{w_1} \to B_{w_2}$ is linear positive operator, then it's norm is given as

$$\|J\| = \sup_{\|\vartheta\|_{w_1}=1} \|J\vartheta\|_{w_2} = \|J(w_1)\|_{w_2}.$$

Let (β_m) be consider as a sequence of real numbers that are non-negative, with $\beta_1 > 0$, then the power series

$$l(x) = \sum_{m=1}^{\infty} \beta_m x^{m-1},$$

has radius of convergence $\mathring{r} \in (0,\infty]$ and $x \in (0,\mathring{r})$. In terms of power series method a sequence of numbers from the real domain (γ_m) is convergent to a if for all $x \in (0,\mathring{r})$

$$\lim_{x\to\mathring{r}^-} \frac{1}{l(x)} \sum_{m=1}^{\infty} \beta_m \gamma_m x^{m-1} = a.$$

The power series method is considered regular if and only if for every $m \in \mathbb{N}$

$$\lim_{x\to\mathring{r}^-} \frac{\beta_m x^{m-1}}{l(x)} = 0.$$

We will use these test functions

$$\vartheta_i(x) = \frac{x^i}{1+x^2} w_1(x), \quad i = 0, 1, 2.$$

Suppose that the linear positive operator $\wp_{m,\nu}^{\varsigma} : C_{w_1} \to B_{w_2}$ satisfy

$$\sup_{0<x<\mathring{r}^-} \frac{1}{l(x)} \sum_{m=1}^{\infty} \|\wp_{m,\nu}^{\varsigma}\vartheta\|_{w_1} \beta_m x^{m-1} < \infty, \quad \text{for every } \vartheta \in C_{w_1}$$

Then from [47]

$$\mathcal{K}(\vartheta, x) = \frac{1}{l(x)} \sum_{m=1}^{\infty} \wp_{m,\nu}^{\varsigma}(\vartheta, x) \beta_m x^{m-1}, \quad x \in (0, \mathring{r}),$$

is also a linear positive operator from C_{w_1} to B_{w_2}.

Theorem 3.1. *([47]) If a linear positive operator $\mathcal{L} \in C_{w_1}$ satisfy*

$$\lim_{x \to \mathring{r}^-} \|\mathcal{L}(\vartheta_i) - \vartheta_i\|_{w_1} = 0, \quad i = 0, 1, 2,$$

then for every $\vartheta \in C_{w_1}$, we have

$$\lim_{x \to \mathring{r}^-} \|\mathcal{L}(\vartheta) - \vartheta\|_{w_1} = 0.$$

Theorem 3.2. *For every $\vartheta \in C_{w_1}$, we have*

$$\lim_{x \to \mathring{r}^-} \|\mathcal{K}(\vartheta) - \vartheta\|_{w_1} = 0,$$

where $w_1(x) = 1 + x^2$.

Proof. From Lemma 1.1, we have

$$\|\wp_{m,\nu}^{\varsigma}(\vartheta_0) - \vartheta_0\|_{w_1} = 0,$$

hence,

$$\|\mathcal{K}(\vartheta_0) - \vartheta_0\|_{w_1} = 0.$$

Now,

$$\begin{aligned}
\|\wp_{m,\nu}^{\varsigma}(\vartheta_1) - \vartheta_1\|_{w_1} &= \sup_{x\geq 0} \frac{1}{1+x^2} \left\{ \frac{x(\nu+m)}{m-\varsigma} + \frac{1}{m-\varsigma} - x \right\} \\
&= \sup_{x\geq 0} \left\{ \frac{x}{1+x^2} \left(\frac{\nu+\varsigma}{m-\varsigma} \right) + \frac{1}{(1+x^2)(m-\varsigma)} \right\} \\
&\leq \frac{\nu+\varsigma}{m-\varsigma} + \frac{1}{m-\varsigma}.
\end{aligned}$$

We can see that the sequence $\left\{ \frac{\nu+\varsigma}{m-\varsigma} + \frac{1}{m-\varsigma} \right\}$ converges to zero as $m \to \infty$. Let us show that in terms of power series method, this sequence is also converges to zero as $m \to \infty$.

For a given $\epsilon > 0$, there exist a $m_0 \in \mathbb{N}$ such that

$$\frac{\nu+\varsigma}{m-\varsigma} + \frac{1}{m-\varsigma} < \frac{\epsilon}{2} \quad \text{for all } m > m_0.$$

Hence

$$\begin{aligned}\|\mathcal{K}(\vartheta_1)-\vartheta_1\|_{w_1} &\le \frac{1}{l(s)}\sum_{m=1}^{\infty}\|\wp^{\varsigma}_{m,\nu}(\vartheta_1)-\vartheta_1\|_{w_1}\beta_m s^{m-1}\\ &= \frac{1}{l(s)}\sum_{m=1}^{m_0}\|\wp^{\varsigma}_{m,\nu}(\vartheta_1)-\vartheta_1\|_{w_1}\beta_m s^{m-1}+\frac{1}{l(s)}\sum_{m=m_0+1}^{\infty}\|\wp^{\varsigma}_{m,\nu}(\vartheta_1)-\vartheta_1\|_{w_1}\beta_m s^{m-1}\\ &\le \frac{1}{l(s)}\sum_{m=1}^{m_0}\|\wp^{\varsigma}_{m,\nu}(\vartheta_1)-\vartheta_1\|_{w_1}\beta_m s^{m-1}+\frac{\epsilon}{2}.\end{aligned}$$

Let $N=\max_{1\le i\le m_0}\|\wp^{\varsigma}_{i,\nu}(\vartheta_1)-\vartheta_1\|_{w_1}$, then

$$\|\mathcal{K}_s(\vartheta_1)-\vartheta_1\|_{w_1}\le N\frac{1}{l(x)}\sum_{m=1}^{m_0}\beta_m x^{m-1}+\frac{\epsilon}{2}. \tag{8}$$

Since, the power series method is considered regular if and only if $\lim_{x\to\mathring{r}^-}\frac{\beta_m x^{m-1}}{l(x)}=0$, hence for a given $\epsilon>0$ and for every $1\le m\le m_0$ there exist $\hat{\delta}_m>0$ such that

$$\frac{\beta_m x^{m-1}}{l(x)}<\frac{\epsilon}{2Nm_0},\quad \mathring{r}-\hat{\delta}_m<u<\mathring{r}.$$

Taking $\hat{\delta}=\min_{1\le m\le m_0}\hat{\delta}_m$, we have

$$\frac{1}{l(x)}\sum_{m=1}^{m_0}\beta_m x^{m-1}<\frac{\epsilon}{2N},\quad \mathring{r}-\hat{\delta}<u<\mathring{r}.$$

From (8), we can say that

$$\lim_{x\to\mathring{r}^-}\|\mathcal{K}(\vartheta_1)-\vartheta_1\|_{w_1}=0.$$

Now,

$$\begin{aligned}\|\wp^{\varsigma}_{i,\nu}(\vartheta_2)-\vartheta_2\|_{w_1} &= \sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{x^2(\nu+m)^2}{(m-2\varsigma)(m-\varsigma)}+\frac{x(4\nu+5m)}{(m-2\varsigma)(m-\varsigma)}+\frac{2}{(m-2\varsigma)(m-\varsigma)}-x^2\right|\\ &= \left|\frac{x^2}{1+x^2}\left(\frac{\nu^2+2\nu n+3m\varsigma-2\varsigma^2}{(m-2\varsigma)(n-\varsigma)}\right)+\frac{x}{1+x^2}\left(\frac{4\nu+5m}{(m-2\varsigma)(m-\varsigma)}\right)+\frac{2}{(1+x^2)(m-2\varsigma)(m-\varsigma)}\right|\\ &\le \frac{\nu^2+2\nu n+3m\varsigma-2\varsigma^2}{(m-2\varsigma)(n-\varsigma)}+\frac{4\nu+5m}{(m-2\varsigma)(m-\varsigma)}+\frac{2}{(m-2\varsigma)(m-\varsigma)}.\end{aligned}$$

Based on above arguments for the function ϑ_1, we can deduce that

$$\lim_{x\to\mathring{r}^-}\|\mathcal{K}(\vartheta_2)-\vartheta_2\|_{w_1}=0.$$

Hence, according to Theorem 3.1, for every $\vartheta\in C_{w_1}$, we have

$$\lim_{x\to\mathring{r}^-}\|\mathcal{K}(\vartheta)-\vartheta\|_{w_1}=0.$$

□

4. A-Statistical Convergence

Let us begin by providing fundamental definitions and symbols regarding the concept of A-statistical convergence. Suppose we have a non-negative infinite summability matrix denoted by $B = (b_{mj}), (m, j \in \mathbb{N})$. Let $x = (x_j)$ be a given sequence. The A-transform of x, denoted by Bx, is defined as follows:

$$(Bx)_m = \sum_{j=1}^{\infty} b_{mj} x_j,$$

and the series is convergent for every m.
B is considered regular if $\lim_m (Bx)_m = l$ where $\lim_m x_m = l$. The sequence $x = (x_m)$ is said to be A-statistical convergent to l i.e. $st_A - \lim_m x_m = K$ if for a given $\epsilon > 0$, $\lim_m \sum_{j:|x_j - L| \geq \epsilon} b_{mj} = 0$.

Theorem 4.1. *Consider a non-negative regular summability matrix denoted by $B = (b_{mj})$ and let $x \in [0, \infty)$. For any $\vartheta \in C_w^*[0, \infty)$, it can be obtain that*

$$st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)\|_{w_\beta} = 0$$

where $w_\beta(x) = 1 + x^{2+\beta}$, $\beta > 0$.

Proof. It is sufficient to show that $st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}(e_i) - e_i\|_w = 0$ where $e_i = x^i$, $i = 0, 1, 2$.
It is obvious from Lemma 1.1, that

$$st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}(e_0) - e_0\|_w = 0. \tag{9}$$

Now, from Lemma 1.1, we have

$$\|\wp_{m,\nu}^{\varsigma}(e_1) - e_1\|_w = \sup_{x \geq 0} \frac{|\frac{x(\nu+m)}{m-\varsigma} + \frac{1}{m-\varsigma} - x|}{1 + x^2} \leq \frac{\nu + \varsigma}{m - \varsigma} + \frac{1}{m - \varsigma}.$$

For any given $\epsilon > 0$, we can define

$$F := \left\{ m \in \mathbb{N} : \|\wp_{m,\nu}^{\varsigma}(e_1) - e_1\|_w \geq \epsilon \right\},$$

$$F_1 := \left\{ m \in \mathbb{N} : \frac{\nu + \varsigma}{m - \varsigma} \geq \frac{\epsilon}{2} \right\},$$

and

$$F_2 := \left\{ m \in \mathbb{N} : \frac{1}{m - \varsigma} \geq \frac{\epsilon}{2} \right\}.$$

We obtain $F \subseteq F_1 \cup F_2$, which implies $\sum_{j \in F} b_{mj} \leq \sum_{j \in F_1} b_{mj} + \sum_{j \in F_2} b_{mj}$, hence

$$st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}(e_1) - e_1\|_w = 0. \tag{10}$$

Finally, we can say that

$$\begin{aligned}\|\wp_{m,\nu}^{\varsigma}(e_2) - e_2\|_w &= \sup_{x \geq 0} \frac{1}{1 + x^2} \left| \frac{x^2(\nu + m)^2}{(m - 2\varsigma)(m - \varsigma)} + \frac{x(4\nu + 5m)}{(m - 2\varsigma)(m - \varsigma)} + \frac{2}{(m - 2\varsigma)(m - \varsigma)} - x^2 \right| \\ &\leq \frac{\nu^2 + 2\nu m + 3m\varsigma - 2\varsigma^2}{(m - 2\varsigma)(m - \varsigma)} + \frac{(4\nu + 5m)}{(m - 2\varsigma)(m - \varsigma)} + \frac{2}{(m - 2\varsigma)(m - \varsigma)}.\end{aligned}$$

Now, we define

$$F := \Big\{ m \in \mathbb{N} : \|\wp_{m,\nu}^{\varsigma}(e_2, ., \nu) - e_2\|_w \geq \epsilon \Big\},$$

$$F_1 := \Big\{ m \in \mathbb{N} : \frac{\nu^2 + 2\nu m + 3m\varsigma - 2\varsigma^2}{(m-2\varsigma)(m-\varsigma)} \geq \frac{\epsilon}{3} \Big\},$$

$$F_2 := \Big\{ m \in \mathbb{N} : \frac{(4\nu + 5m)}{(m-2\varsigma)(m-\varsigma)} \geq \frac{\epsilon}{3} \Big\},$$

and

$$F_3 := \Big\{ m \in \mathbb{N} : \frac{2}{(m-2\varsigma)(m-\varsigma)} \geq \frac{\epsilon}{3} \Big\}.$$

We obtain $F \subseteq F_1 \cup F_2 \cup F_3$, which implies $\sum_{j\in F} b_{mj} \leq \sum_{j\in F_1} b_{mj} + \sum_{j\in F_2} b_{mj} + \sum_{j\in F_3} b_{mj}$, hence

$$st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}(e_2) - e_2\|_w = 0. \tag{11}$$

Similarly, from Lemma 1.2, we obtain

$$st_A - \lim_m \|\wp_{m,\nu}^{\varsigma}((e_1 - xe_0)^i)\|_w = 0, \quad i = 0, 1, 2, 3, 4. \tag{12}$$

□

Now, we prove Voronovskaja type theorem.

Theorem 4.2. *Let $B = (b_{mj})$ be a non-negative regular summability matrix. Then, for every $\vartheta \in C_w^*[0,\infty)$ such that $\vartheta', \vartheta'' \in C_w^*[0,\infty)$, we have*

$$st_A - \lim_{m\to\infty} m(\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)) = \vartheta'(x) + \frac{1}{2}(x^2\varsigma + 3x)\vartheta''(x),$$

uniformly with respect to $x \in [0, a]$.

Proof. Consider $\vartheta, \vartheta', \vartheta'' \in C_w^*[0,\infty)$. For every $x \geq 0$, we define a function

$$\Phi(\ell, x) = \begin{cases} \frac{\vartheta(\ell)-\vartheta(x)-(\ell-x)\vartheta'(x)-\frac{1}{2}(\ell-x)^2\vartheta''(x)}{(\ell-x)^2}, & \text{if } \ell \neq x \\ 0, & \text{if } \ell = x. \end{cases}$$

Then

$$\Phi(x, x) = 0 \text{ and } \Phi(., x) \in C_w[0,\infty).$$

Thus, we obtain

$$\vartheta(\ell) = \vartheta(x) + (\ell - x)\vartheta'(x) + \frac{1}{2}(\ell - x)^2\vartheta''(x) + (\ell - x)^2\Phi(\ell, x).$$

Operating by $\wp_{m,\nu}^{\varsigma}$ on above inequality, we get

$$\begin{aligned} m(\wp_{m,\nu}^{\varsigma}(\vartheta, x) - \vartheta(x)) = \; & \vartheta'(x) m\wp_{m,\nu}^{\varsigma}((\ell - x), x) \\ & + \frac{1}{2}\vartheta''(x) m\wp_{m,\nu}^{\varsigma}((\ell - x)^2, x) \\ & + m\wp_{m,\nu}^{\varsigma}((\ell - x)^2\Phi(\ell, x), x, \vartheta) \end{aligned}$$

According to Lemma 1.2, we obtain

$$st_A - \lim_{n\to\infty} m\wp_{m,\nu}^{\varsigma}((\ell - x), x) = x(\nu + \varsigma) + 1, \tag{13}$$

$$st_A - \lim_{m\to\infty} m\wp^{\varsigma}_{m,\nu}((\ell - x)^2, x) = x^2\varsigma + 3x, \tag{14}$$

and

$$st_A - \lim_{m\to\infty} m^2\wp^{\varsigma}_{m,\nu}((\ell - x)^4, x) = 3x^4\varsigma^2 + 18x^3\varsigma + 27x^2, \tag{15}$$

is uniformly with respect to x in the interval $[0, a]$.
By applying the Cauchy-Schwarz inequality, we obtain

$$m\wp^{\varsigma}_{m,\nu}((\ell - x)^2\Phi(\ell, x), x) \leq \sqrt{m^2\wp^{\varsigma}_{m,\nu}((\ell - x)^4, x)}\sqrt{\wp^{\varsigma}_{m,\nu}(\Phi^2(\ell, x), x)}$$

Let $\kappa(\ell, x) = \Phi^2(\ell, x)$, we see that $\kappa(x, x) = 0$ and $\kappa(., x) \in C^*_w[0, \infty)$. From Theorem 4.1, it follows that

$$st_A - \lim_{m\to\infty} \wp^{\varsigma}_{m,\nu}(\Phi^2(\ell, x), x) = st_A - \lim_{m\to\infty} \wp^{\varsigma}_{m,\nu}(\kappa(\ell, x), x) = \kappa(x, x) = 0,$$

is uniformly with respect to x in the interval $[0, a]$.
According to equation (15), we get

$$st_A - \lim_{m\to\infty} m\wp^{\varsigma}_{m,\nu}((\ell - x)^2\Phi(\ell, x), x) = 0. \tag{16}$$

Combine equation (13),(14) and (16), we get desired result. □

5. Rate of A-Statistical Convergence

Theorem 5.1. *Let $\vartheta \in C^*_B[0, \infty)$. Then, we have*

$$st_A - \lim_{m} \|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} = 0$$

Proof. By Taylor's expansion, we obtain

$$\wp^{\varsigma}_{m,\nu}(\vartheta, x) - \vartheta(x) = \vartheta'(x)\wp^{\varsigma}_{m,\nu}((\ell - x), x) + \frac{1}{2}\vartheta''(\chi)\wp^{\varsigma}_{m,\nu}((\ell - x)^2, x),$$

where χ is a value that lies between ℓ and x.
As a result, we obtain

$$\begin{aligned}\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} &\leq \|\vartheta'\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot), \cdot)\|_{C_B[0,\infty]} \\ &+ \|\vartheta''\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot)^2, \cdot)\|_{C_B[0,\infty]}\end{aligned} \tag{17}$$

According to equation (12), for a given $\epsilon > 0$, we get

$$\lim_{m} \sum_{\|\vartheta'\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot), \cdot)\|_{C_B[0,\infty]} \geq \frac{\epsilon}{2}} b_{mj} = 0,$$

$$\lim_{m} \sum_{\|\vartheta''\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot)^2, \cdot)\|_{C_B[0,\infty]} \geq \frac{\epsilon}{2}} b_{mj} = 0.$$

From (17), we can say that

$$\begin{aligned}&\sum_{j\in\mathbb{N}:\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} \geq \epsilon} b_{mj} \\ &\leq \sum_{j\in\mathbb{N}:\|\vartheta'\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot), \cdot)\|_{C_B[0,\infty]} \geq \frac{\epsilon}{2}} b_{mj} \\ &+ \sum_{\|\vartheta''\|_{C_B[0,\infty)}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot)^2, \cdot)\|_{C_B[0,\infty]} \geq \frac{\epsilon}{2}} b_{mj}.\end{aligned}$$

Taking the limit as $m \to \infty$, we obtain the desired result. □

Theorem 5.2. *Let* $\vartheta \in C_B[0,\infty)$*, we obtain*

$$\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} \leq C\omega_2(\vartheta, \sqrt{\hat{\delta}_m}),$$

where $\hat{\delta}_m = \|\wp^{\varsigma}_{m,\nu}((\ell - \cdot), \cdot)\|_{C_B[0,\infty)} + \|\wp^{\varsigma}_{m,\nu}((\ell - \cdot)^2, \cdot)\|_{C_B[0,\infty)}$.

Proof. Let $\mu \in C^*_B[0,\infty)$, by (17)

$$\begin{aligned}\|\wp^{\varsigma}_{m,\nu}(\mu) - \mu\|_{C_B[0,\infty)} &\leq \|\wp^{\varsigma}_{m,\nu}((\ell - \cdot), \cdot)\|_{C_B[0,\infty)}\|\mu'\|_{C_B[0,\infty)} \\ &+ \frac{1}{2}\|\wp^{\varsigma}_{m,\nu}((\ell - \cdot)^2, \cdot)\|_{C_B[0,\infty)}\|\mu''\|_{C_B[0,\infty)} \\ &\leq \hat{\delta}_m\|\mu\|_{C^*_B[0,\infty)}.\end{aligned}$$

For every $\vartheta \in C_B[0,\infty)$ and $\mu \in C^*_B[0,\infty)$, we obtain

$$\begin{aligned}\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} &\leq \|\wp^{\varsigma}_{m,\nu}(\vartheta) - \wp^{\varsigma}_{m,\nu}(\mu)\|_{C_B[0,\infty)} \\ &+ \|\wp^{\varsigma}_{m,\nu}(\mu) - \mu\|_{C_B[0,\infty)} + \|\mu - \vartheta\|_{C_B[0,\infty)} \\ &\leq 2\|\mu - \vartheta\|_{C_B[0,\infty)} + \|\wp^{\varsigma}_{m,\nu}(\mu) - \mu\|_{C_B[0,\infty)} \\ &\leq 2\|\mu - \vartheta\|_{C_B[0,\infty)} + \hat{\delta}_m\|\mu\|_{C^*_B[0,\infty)}\end{aligned}$$

Take infimum on right hand side over all $\mu \in C^*_B[0,\infty)$, we get

$$\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} \leq 2K_2(\vartheta, \hat{\delta}_m).$$

From (4), we get

$$\|\wp^{\varsigma}_{m,\nu}(\vartheta) - \vartheta\|_{C_B[0,\infty)} \leq C\{\omega_2(\vartheta, \sqrt{\hat{\delta}}) + min(1, \hat{\delta})\|\vartheta\|_{C_B[0,\infty)}\}.$$

From (12), we get $st_A - \lim_m \hat{\delta} = 0$, hence, $st_A - \omega_2(\vartheta, \sqrt{\hat{\delta}}) = 0$. Therefore, we obtain the A-statistical convergence rate of the operators $\wp^{\varsigma}_{m,\nu}(\vartheta, x)$ to $\vartheta(x)$ in the space $C_B[0,\infty)$. □

6. κ^{th} order generalization of operators $\wp^{\varsigma}_{m,\nu}$

Let $C^{\kappa}_s[0,\infty)$ denote the subspace of all function $\vartheta \in C_s[0,\infty) = \{\vartheta \in C[0,\infty) : \vartheta(u) = o(u^s)$, as $u \to \infty\}$ whose κ^{th} derivative exists and $\vartheta^{(\kappa)} \in C_s[0,\infty)$ for all $\kappa \in \mathbb{N}$. Then for any $\vartheta \in C^{\kappa}_s[0,\infty)$, the κ^{th} order generalization of operators $\wp^{\varsigma}_{m,\nu}$ is given as

$$\wp^{\kappa}_{m,\nu}(\vartheta, x) = \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)\Big(\sum_{r=0}^{\kappa} \vartheta^{(r)}(\ell)\frac{(x-\ell)^r}{r!}\Big)d\ell.$$

For $\kappa = 0$, we have $\vartheta^{(0)} = \vartheta$ and hence the operators $\wp^{\kappa}_{m,\nu}$ reduces to $\wp^{\varsigma}_{m,\nu}$.

We obtain approximation error by the operators $\wp^{\kappa}_{m,\nu}$ for the functions $\vartheta \in C^{\kappa}_s[0,\infty)$ such that $\vartheta^{(\kappa)} \in Lip_M a$, where

$$Lip_M a = \Big\{\vartheta \in C[0,\infty) : |\vartheta(x) - \vartheta(\ell)| \leq M|x-\ell|^a, 0 < a \leq 1\Big\},$$

where $M > 0$.

Theorem 6.1. *Let* $\vartheta \in C^{\kappa}_s[0,\infty)$ *be such that* $\vartheta^{(\kappa)} \in Lip_M a$ *then*

$$|\wp^{\kappa}_{m,\nu}(\vartheta, x) - \vartheta(x)| \leq \frac{M\Gamma(a+1)}{\Gamma(\kappa+a+1)}\wp^{\varsigma}_{m,\nu}(|x-\ell|^{\kappa+a}, x).$$

Proof. For $0 \leq x < \infty$, we have $\vartheta(x) = \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)\vartheta(x)d\ell$. By Taylor's expansion's remainder in integral form to the function ϑ, we get

$$\vartheta(x) - \wp_{m,\nu}^{\varsigma}(\vartheta, x) = \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)\Big(\vartheta(x) - \sum_{r=0}^{\kappa} \vartheta^{(r)}(\ell)\frac{(x-\ell)^r}{r!}\Big)d\ell.$$

Using Taylor's formula

$$\begin{aligned}
|\wp_{m,\nu}^{\kappa}(\vartheta, x) - \vartheta(x)| &\leq \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)\frac{|x-\ell|^{\kappa}}{(\kappa-1)!}\Big(\int_0^1 (1-v)^{\kappa-1}|\vartheta^{(\kappa)}(\ell + v(x-\ell)) - \vartheta^{(\kappa)}(\ell)|dv\Big)d\ell \\
&\leq M \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)\frac{|x-\ell|^{\kappa+a}}{(\kappa-1)!}\Big(\int_0^1 (1-v)^{\kappa-1}v^a dv\Big)d\ell \\
&= \frac{M\beta(\kappa, a+1)}{(\kappa-1)!} \sum_{k=0}^{\infty} l_{m,k}(x,\nu) \int_0^{\infty} q_{m,k}(\ell)|x-\ell|^{\kappa+a}d\ell \\
&= \frac{M\Gamma(a+1)}{\Gamma(\kappa+a+1)} \wp_{m,\nu}^{\varsigma}(|x-\ell|^{\kappa+a}, x).
\end{aligned}$$

□

In the following Table 1, we compute the error of approximation of our operators $\wp_{m,\nu}^{\varsigma}(\vartheta, x)$ for $m = 10, 12, 15$ and choosing $\nu = 1, \alpha = 0.5$ and $\vartheta(x) = x^2(1+x)^2$. It is clear that as the value of m increase, the error in the approximation decreases.

Table 1
Error of Approximation

x	$m = 10$	$m = 12$	$m = 15$
0.1	0.06910063747	0.03918186759	0.02052247221
0.15	0.1282942102	0.07550670834	0.04146611521
0.2	0.2101069316	0.1271666102	0.07231430813
0.25	0.3183540423	0.1970013299	0.1151021232
0.3	0.4570143800	0.2879763341	0.1719576538
0.35	0.6302303828	0.4031827976	0.2451020157
0.4	0.8423080843	0.5458376057	0.3368493453
0.45	1.097717117	0.7192833528	0.4496068012
0.5	1.401090714	0.9269883425	0.5858745638

In the following Table 2, we compare the error of approximation of our operators $\wp_{m,\nu}^{\varsigma}(\vartheta, x)$ by $\vartheta(x) = x^4 + \sqrt{5}x^3 + \frac{7}{2}x^2 + 5x$ for $m = 10, 20, 30$ and choosing $\nu = 0.5, \alpha = 0.5$. It is clear that as the value of m increase, the error in the approximation decreases.

Table 2
Error of Approximation

x	$m = 10$	$m = 20$	$m = 30$
0.1	1.104920975	0.4496516902	0.2826960154
0.15	1.408951289	0.5589382816	0.3486714516
0.2	1.763760174	0.6859784982	0.4252149272
0.25	2.174157162	0.8325742019	0.5134240989
0.3	2.645115371	1.000591967	0.6144367706
0.35	3.181771536	1.191963096	0.7294308715
0.4	3.789425976	1.408683610	0.8596244865
0.45	4.473542605	1.652814252	1.006275812
0.5	5.239748972	1.926480486	1.170683217

In the following Table 3, we compare an estimate of the error in the approximation of $\vartheta(x) = x(3-5x)^3$ by the operators $\wp^{\varsigma}_{m,\nu}(\vartheta, x)$ and Lupaş-Durrmeyer operators [14], for $m = 10$ $\nu = 0.5$,and $\alpha = 0.5$. It is evident that the error in the approximation of ϑ by $\wp^{\varsigma}_{m,\nu}(\vartheta, x)$ is much less than the error by Lupaş-Durrmeyer operators.

Table 3
Convergence of $\wp^{\varsigma}_{10,\nu}(\vartheta, x)$ and Lupaş-Durrmeyer operators for $m = 10$

x	$\wp^{\varsigma}_{10,\nu}(\vartheta, x)$	Lupaş-Durrmeyer operators for $m = 10$
0.1	0.06797850018	0.08437526040
0.15	0.1124421440	0.1496356445
0.2	0.1672335742	0.2316895833
0.25	0.2327974503	0.3315958661
0.3	0.3095784313	0.4504132810
0.35	0.3980211775	0.5892006186
0.4	0.4985703477	0.7490166669
0.45	0.6116706009	0.930920214
0.5	0.7377665983	1.135970053

7. Conclusion

In this work, we proposed Lupaş-Jain type operators and investigated the approximation capabilities of these operators using both the A-statistical method and the power series method. We also studied the Lupaş-Jain type operators of order κ and obtained the approximation error for functions belonging to the Lipschitz class.

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